SUSY SIGMA MODELS

AS GROSS-NEVEU MODELS

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2011

A.A.S.
will remain
a great source
of inspiration
and an example
of both personal
and scientific
integrity

SUMMARY OF THE TALK

[Gross-Neveu '1974]
[Witten '1978]
[Andrei-Lowenstein '1979]
[Destri-de Vega '1989]

- Class of integrable sigma models explicitly equivalent to Bose/Fermi Gross-Neveu models

[DB 2020+]

Related: [LeClair, Ludwig, '2000
Saleur, Schomerus -2010]

$$\boxed{\text{GN-model is the "}\varphi^4\text{-theory" of 2D!}}$$

MAIN EXAMPLE: $\mathbb{CP}^{n-1}$ - MODEL

$$S = \int d^2 z \sum_{A,B} G_{A\bar{B}}\, \partial_\alpha U^A\, \partial_\alpha \overline{U^B} + \text{Top. term}$$

$\uparrow$ Fubini-Study metric

GLSM: $\mathbb{CP}^{n-1} = \mathbb{C}^n - \{0\} / \mathbb{C}^* \mapsto \left\{ \sum_{A=1}^{n} |U^A|^2 = 1 \right\} / U(1)$

$$S_{GLSM} = \int d^2 z \sum_{A=1}^{n} D_\alpha U^A\, \overline{D_\alpha U^A}$$

$$\boxed{D_\alpha U^A = \partial_\alpha U^A - i A_\alpha U^A}$$

$$\mathbb{CP}^{n-1} = \mathbb{C}^n - \{0\} \,/\, \mathbb{C}^*$$ "GIT-quotient"

$U, V \in \mathbb{C}^n$ $\quad \Psi = \begin{pmatrix} U \\ \overline{V} \end{pmatrix}$ "Dirac boson"

Chiral gauged GN-model

$$\mathcal{L} = \overline{\Psi}_a \not{D} \Psi_a + \varkappa \left(\overline{\Psi}_a \frac{1+\sigma_3}{2} \Psi_a \right) \cdot \left(\overline{\Psi}_b \frac{1-\sigma_3}{2} \Psi_b \right)$$

$U(1)$ $(\mathbb{C}^*)$ covariant derivative

$$\mathcal{L} = V \overline{D} U + \overline{U} \cdot D \overline{V} + \varkappa (\overline{U} U)(\overline{V} V)$$

Eliminate $V, \overline{V}$ $\mapsto$ $\mathcal{L} = \frac{1}{\varkappa} \frac{D\overline{U} \cdot \overline{D} U}{\overline{U} \cdot U}$ $= \mathbb{CP}^{n-1}$ model

THE SUSY MODEL

Main point: $\mathbb{CP}^{n-1}$-model with worldsheet SUSY

= gauged version of a model with target space $GL(1|1)$ SUSY

Let $U, V, B, C \in \mathbb{C}^n$. Start with the free theory

$$\mathcal{L}_{\beta\gamma} := V \cdot \bar{\partial} U + \bar{U} \cdot \partial \bar{V} + B \cdot \bar{\partial} C + \bar{C} \cdot \partial \bar{B}$$

("$\beta\gamma$-system")

[Witten, Nekrasov '2005]

Bosons (commuting)

Fermions (anti-commuting)

(SUPER) SYMMETRIES

- TARGET-SPACE SUSY

One can rotate the doublets

$$\begin{pmatrix} U \\ C \end{pmatrix} \mapsto g \cdot \begin{pmatrix} U \\ C \end{pmatrix}, \quad \begin{pmatrix} V & B \end{pmatrix} \mapsto \begin{pmatrix} V & B \end{pmatrix} \cdot g^{-1} \qquad g \in GL(n|n)$$

- WORLDSHEET SUSY

$$Q_+^2 = \overline{Q}_+^2 = 0 \qquad \{Q_+, \overline{Q}_+\} = \partial$$

$$\delta U = \varepsilon_1 C \qquad \delta B = -\varepsilon_1 V \qquad \delta C = -\varepsilon_2 \partial U \qquad \delta V = \varepsilon_2 \partial B$$

Q_+ $\qquad$ $\overline{Q}_+$

[Kapustin '2005]

[Grassi, Policastro, Scheidegger '2007]

GAUGING

$$\mathcal{U} = \begin{pmatrix} U \\ C \end{pmatrix} \qquad \mathcal{V} = \begin{pmatrix} V & B \end{pmatrix}$$

$$\bar{A}_{super} = \begin{pmatrix} \bar{A} & 0 \\ \bar{W} & \bar{A} \end{pmatrix}$$

Rewrite the Lagrangian:

$$\mathcal{L}_{CP} = \mathcal{V} \cdot \bar{\mathcal{D}} \mathcal{U} + \bar{\mathcal{U}} \cdot \mathcal{D} \bar{\mathcal{V}}, \qquad \bar{\mathcal{D}} = \bar{\partial} + i \bar{A}_{super}$$

Phase space: $T^* \mathbb{C}^{n|n} /\!/ G_\Delta$

$$G_\Delta := \left\{ g = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda \end{pmatrix} \in SL(1|1) \right\}$$

Bosonic $\mathbb{C}^*$ Fermionic $\mathbb{C}_F$

Supersymplectic reduction!

INTERACTIONS

Consider the Kac-Moody current

$$J = U \otimes V - C \otimes B \in gl_n$$

and the Lagrangian $\mathcal{L}_{int} = \mathcal{L}_{CP} + \varkappa \, \mathrm{Tr}(J\bar{J})$

coupling constant $\left(\varkappa = \frac{1}{R^2}\right)$

Level $k=0$ Kac-Moody current: $\langle J(z) J(w) \rangle = 0$

RIEMANN SURFACES

Amusingly, in gauged GN models the best gauge is (on $\mathbb{R}^2$)

$A = \bar{A} = 0$ (Gauge transfos: $\bar{A} \mapsto \bar{A} - i\bar{\partial}\chi = 0 \Rightarrow \chi(z,\bar{z}) = \frac{i}{\pi}\int \frac{d^2w}{z-w}\bar{A}(w,\bar{w})$)

↑ Gauge field analogue of conformal gauge in string theory!

On Riemann surfaces, moduli of gauge fields remain

[Narasimhan–Seshadri '1965]
[Atiyah–Bott '1983]
[Hori '1996]

(Like moduli of complex structures in string theory)

SUSY $\mathbb{CP}^{n-1}$ MODEL AS A LEVEL-ZERO GN MODEL

Gauge group of the SUSY model: G_Δ

$$i\bar{A} \mapsto i\bar{A} + \bar{\partial}\zeta$$

$$i\bar{W} \mapsto i\bar{W} + e^{-\zeta}(\bar{\partial}\xi - \bar{\partial}\zeta \cdot \xi)$$

Set $\boxed{A = \bar{A} = 0 \quad \text{and} \quad W = \bar{W} = 0}$

$J = \mathcal{V}\otimes\mathcal{V} - \mathcal{C}\otimes\mathcal{B}$

$$\mathcal{L} = \underbrace{\mathcal{V}\cdot\bar{\partial}\mathcal{U} + \bar{\mathcal{U}}\cdot\partial\bar{\mathcal{V}}}_{\mathcal{L}_{CFT}} + \varkappa\,\mathrm{Tr}(J\bar{J})$$

[DB, 2022]

Level-zero $(k=0)$ Gross-Neveu model!

ALL-LOOP β-FUNCTIONS?

$$S = S_{CFT} + \int d^2z \, d_{a\bar{b}} J^a \bar{J}^b$$

coupling constants

CFT with Kac-Moody symmetry

$\exists$ all-loop β-function proposal

Homogeneous case $(d_{a\bar{b}} = \varkappa \, \delta_{a\bar{b}})$:

$$\boxed{\beta_\varkappa = \frac{1}{2} \frac{C_2 \varkappa^2}{\left(1 + \frac{1}{2} k \varkappa\right)^2}}$$

level of Kac-Moody algebra

[Kutasov '1989
Gerganov, LeClair, Moriconi '2001]

SUSY: $k = 0 \Rightarrow$ β-function is one-loop exact!

[Morozov
Perelomov '1984
Shifman]

To compare:

4-loop β-function of the pure GN-model:

$$\beta(g) = (d-2)g - (N-1)\frac{g^2}{\pi} + (N-1)\frac{g^3}{2\pi^2} + (N-1)(2N-7)\frac{g^4}{16\pi^3} + (N-1)[-2N^2 - 19N + 24 - 6\zeta_3(11N-17)]\frac{g^5}{48\pi^4} + \mathcal{O}(g^6) \qquad (4.9)$$

[Gracey
Luthe
Schröder
'2016]

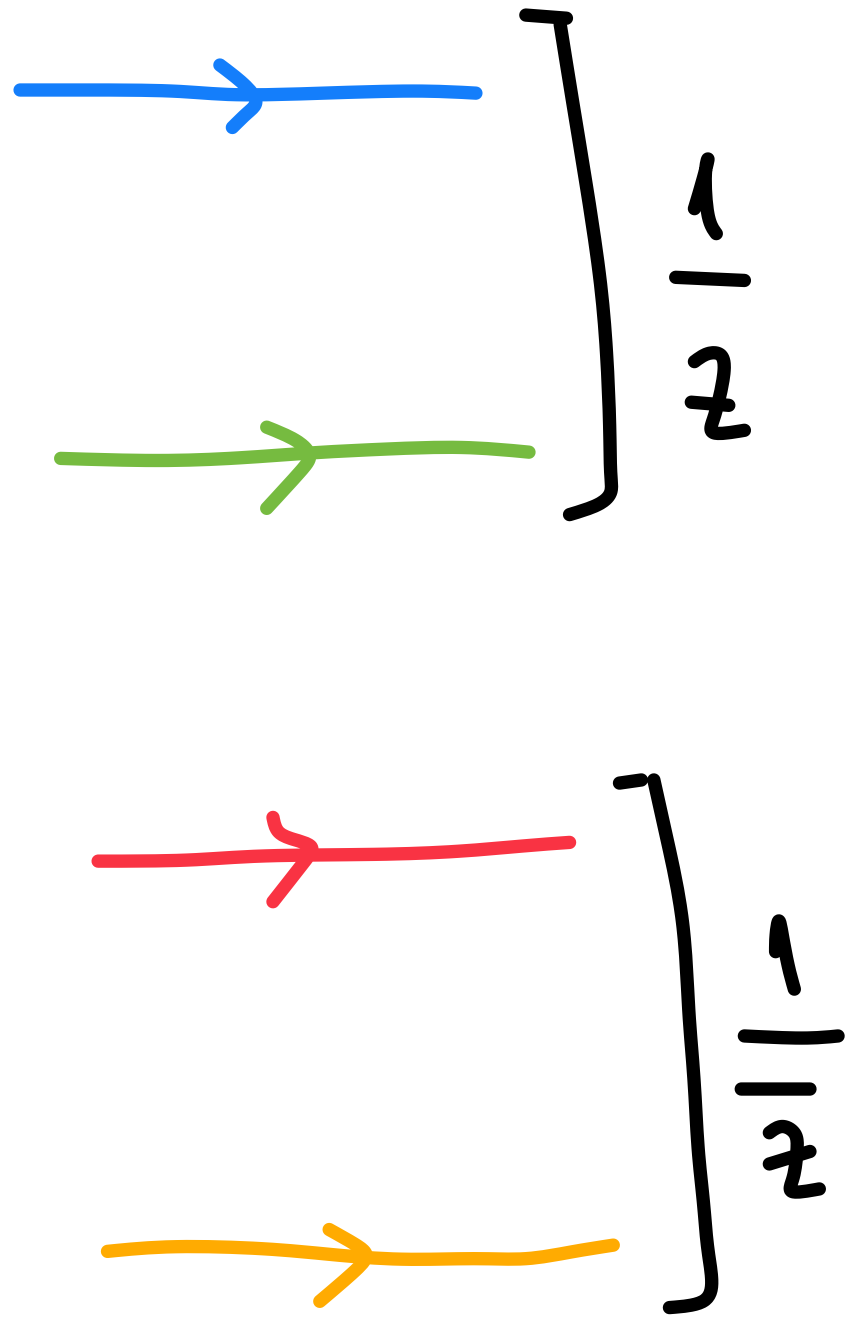
FEYNMAN RULES

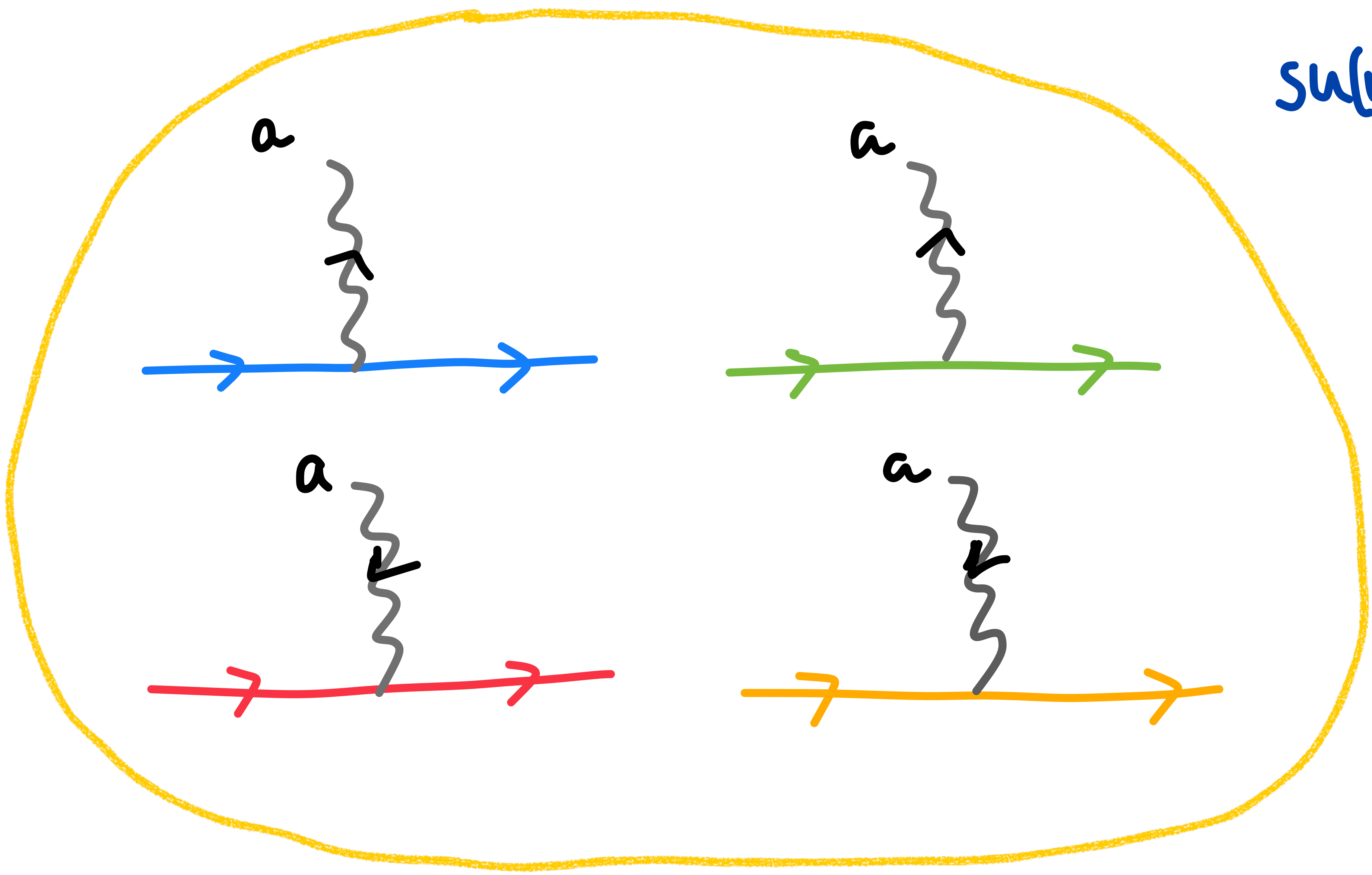
$\langle U V \rangle$

$\langle B C \rangle$

$\langle \bar{U} \bar{V} \rangle$

$\langle \bar{B} \bar{C} \rangle$





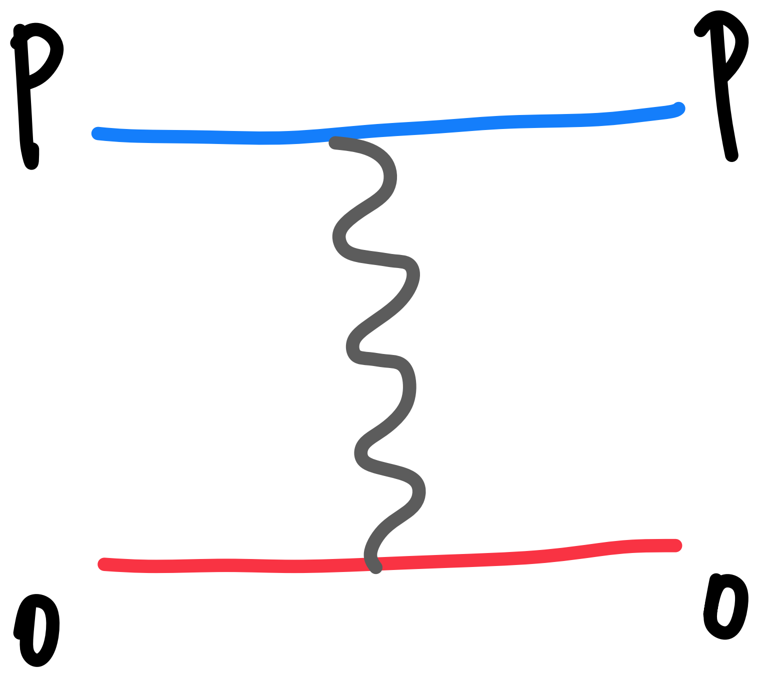
su(n) generators $\downarrow$ $\sqrt{x}\, T^a$

Vertices

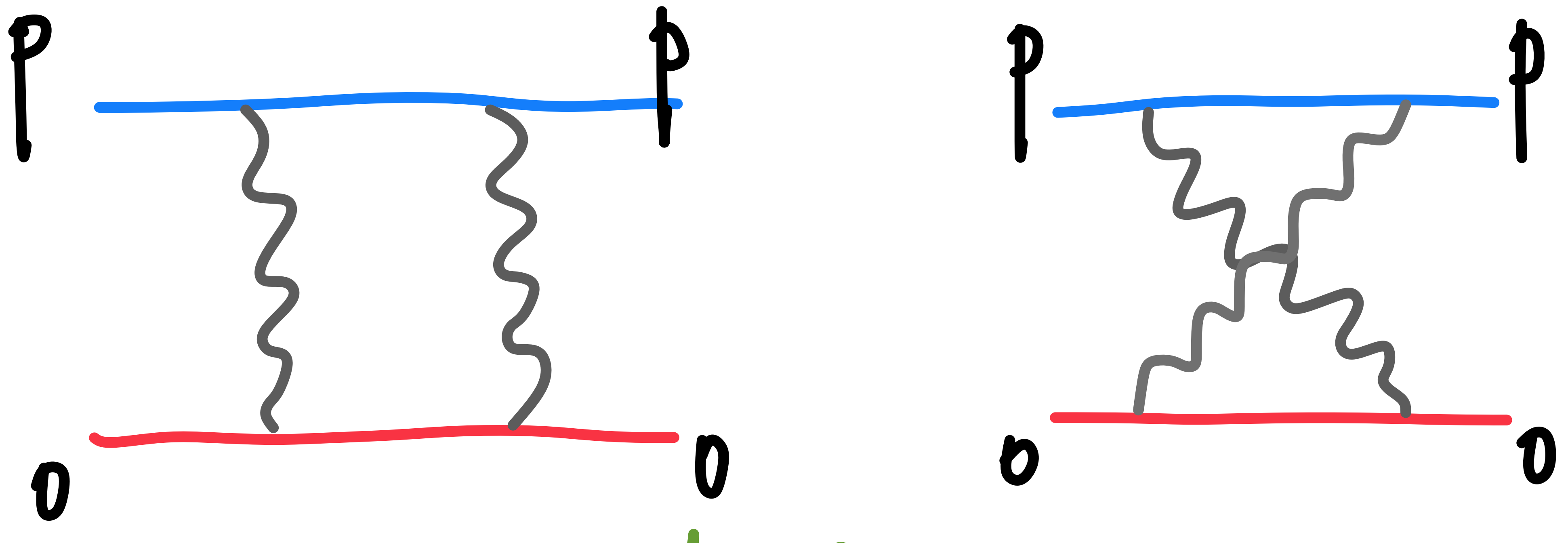
$\mathrm{Tr}(J\bar{J}) \mapsto$ Hubbard-Stratonovich

Propagators

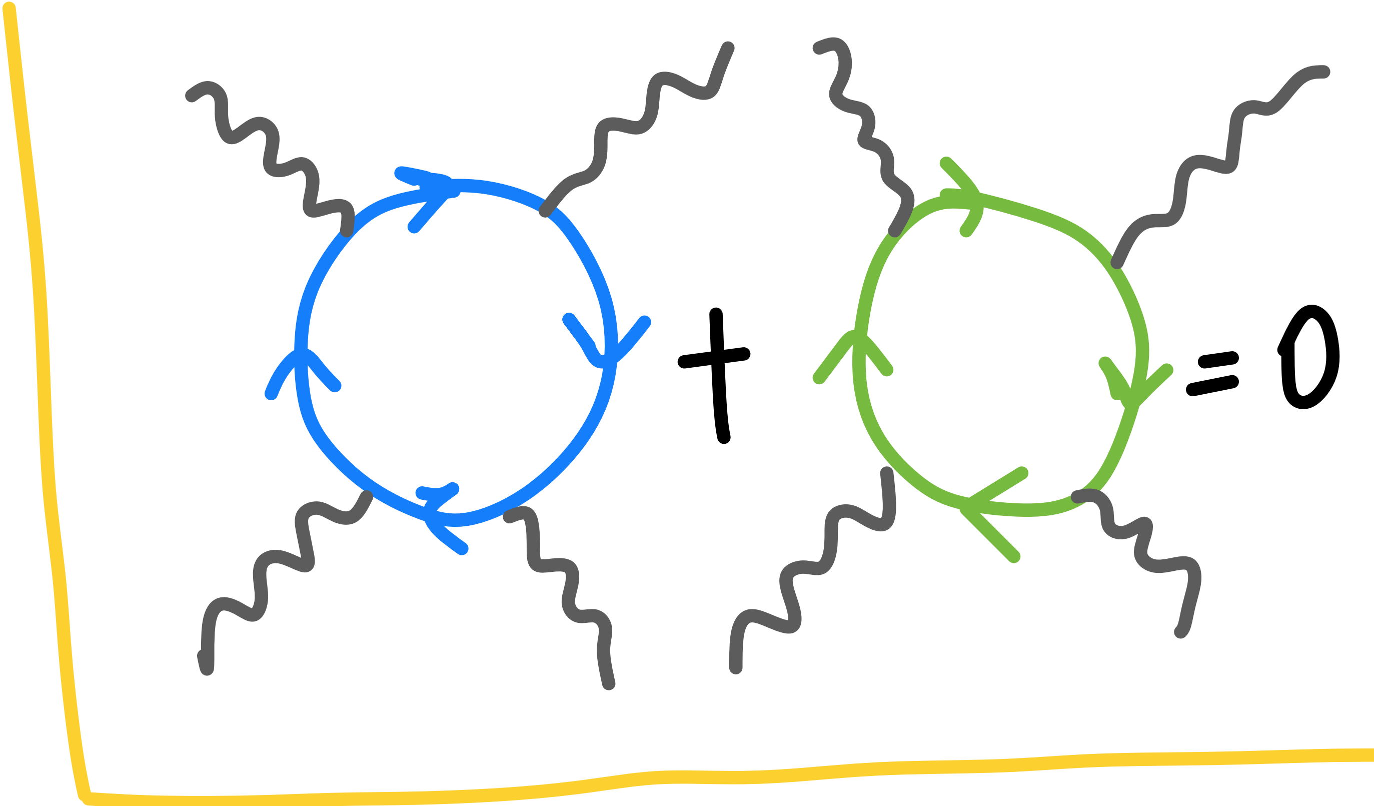
β-FUNCTION (k=0)



Tree

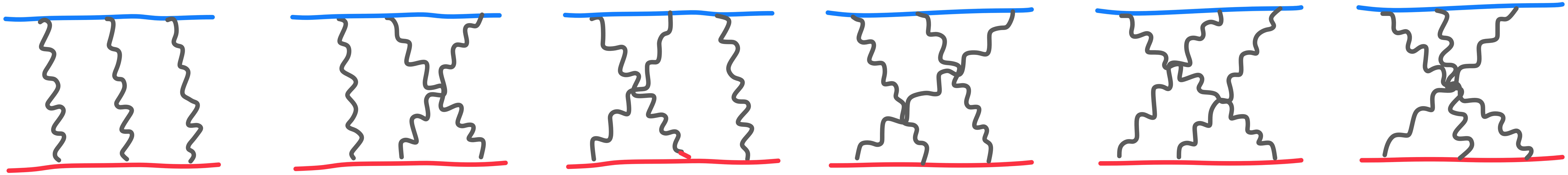


1-loop



1-loop β-function

$$\sim \varkappa\, T_a \otimes T_a \qquad \sim \varkappa^2 \int \frac{d^2 z}{|z|^2}\, e^{i(p,z)}\, \underbrace{T_a T_b \otimes \left(T_a T_b - T_b T_a\right)}_{\sim\, C_2\, T_a \otimes T_a} \sim C_2\, T_a \otimes T_a \ln\left(\frac{|p|}{\Lambda}\right)$$



2-loop (in general $(\ell+1)!$)

4-LOOP β-FUNCTION

[DB, 2022]

1 loop: $\beta^{(1)} = -n\,\varkappa^2$

2 loops: $\beta^{(2)} = 0$

3 loops: $\beta^{(3)} = 0$

4 loops: $\beta^{(4)} = -\frac{9}{2} n^2 \zeta(3)\, \varkappa^5$

MOM scheme

$G_4(p^2 = \mu^2) \equiv \varkappa$

- $\overline{MS}$ scheme: very complicated

[Vasil'ev-Derkachov-Kivel '1995]

$\zeta(3)$ also appears in 4-loop sigma model β-functions

[Grisaru-van de Ven-Zanon '1986]

[Jack-Jones-Mohammedi '1989]

QUANTUM MECHANICS

$$\mathcal{L} = V \cdot \frac{\overline{D}U}{dt} + \overline{U} \cdot \frac{D\overline{V}}{dt} + \varkappa\,(\overline{U} \cdot U)(\overline{V} \cdot V)$$

$$\frac{\overline{D}U}{dt} = \frac{dU}{dt} - i\overline{A}U$$

Canonical quant. $\quad V_j = -\dfrac{\partial}{\partial U_j}$

$$V \cdot U = \overline{U} \cdot \overline{V} = 0 \quad \longmapsto \quad \sum_{i=1}^{n} U_i \frac{\partial \psi}{\partial U^i} = \sum_{i=1}^{n} \overline{U}_i \frac{\partial \psi}{\partial \overline{U}^i} = 0$$

ψ is homogeneous

Hamiltonian = Laplacian

$$\mathbf{H}\,\psi = \left(\sum_{i=1}^{n} \overline{U}_i U_i\right) \sum_{j=1}^{n} \frac{\partial^2 \psi}{\partial U_j \partial \overline{U}_j}$$

Eigenfunctions

$$|M\rangle := \frac{1}{|U|^{2M}} \sum \psi_{i_1 \dots i_M | j_1 \dots j_M} U_{i_1} \dots U_{i_M} \overline{U_{j_1}} \dots \overline{U_{j_M}}$$

Traceless

$$\mathbf{H}\, |M\rangle = -M(n+M-1)\, |M\rangle$$

$n = 2$ (sphere S^2): $\quad \cos\theta := \dfrac{|U_1|^2 - |U_2|^2}{|U_1|^2 + |U_2|^2}$

$|M\rangle \mapsto$ Associated Legendre polynomials P_ℓ^m

MONOPOLE HARMONICS

[Tamm '1931, Wu-Yang '1976]

Particle on $\mathbb{CP}^{n-1}$ in external gauge field A: $dA = q\,\omega_{FS}$

$$|M,\tilde{M}\rangle = \frac{1}{|U|^{M+\tilde{M}}} \sum \Psi_{i_1 \dots i_M | j_1 \dots j_{\tilde{M}}} U^{i_1} \dots U^{i_M} \overline{U^{j_1}} \dots \overline{U^{j_{\tilde{M}}}}$$

Traceless

$M - \tilde{M} = q$ = Monopole charge

[Kuwabara '1988]

$n = 2$ $(\cos\theta)$

[Joint work with A. Smilga, to appear]

$|M,\tilde{M}\rangle \longmapsto$ Jacobi polynomials $(\cos\theta)$

SUSY MONOPOLE HARMONICS

[Joint work with A. Smilga, to appear]

$q \neq 0 \mapsto \mathcal{N}=2$ SUSY

(dim. red. of $\mathcal{N}=(0,2)$ SUSY in 2D $\mapsto$ Dolbeault complex)

$$Q = -i \sum_{j=1}^{n} c^j \frac{\partial}{\partial u^j}\,, \quad Q^\dagger = -i \sum_{j=1}^{n} |U^j|^2 \times \sum_{k=1}^{n} \frac{\partial^2}{\partial \bar{u}^k \partial c^k}$$

Supercharges

$$\sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} = 0\,, \quad \sum_{i=1}^{n} \bar{u}^i \frac{\partial}{\partial \bar{u}^i} = q\,, \quad \sum_{i=1}^{n} u^i \frac{\partial}{\partial c^i} = 0$$

Constraints

Hamiltonian: $H = \{Q, Q^\dagger\}$

One can explicitly compute the spectrum & wave functions!

SUMMARY & OUTLOOK

- Sigma models = Chiral Gross-Neveu models
- Integrability / Lax pairs
- Quiver supervarieties
- All-loop β-functions
- Riemann surface worldsheets
- Exact quantum theory ?