Generalized Eigenvalue Models

based on

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Matrix models

Random Hermitian $N \times N$ matrix:

$$Z = \int dA \; e^{-N \operatorname{tr} V(A)}$$

polynomial

/ Wigner's /

Wilson loop:

$$W(x) = \left\langle \frac{1}{N} \operatorname{tr} e^{xA} \right\rangle$$

$$\boxed{N \to \infty}$$

or

$$W(x) = \left\langle \frac{1}{N} \operatorname{tr} e^{ixA} \right\rangle$$

Eigenvalues:

$$A = U^{-1}\,\mathrm{diag}(a_1 \ldots a_N)\,U$$

$$Z = \int d^N a \;\; \Delta^2(a)\; e^{-N\sum_i V(a_i)}$$

Vandermonde: $\Delta(a) = \prod_{i<j} (a_i - a_j)$

- solvable at $N \to \infty$

/ Brezin, Itzykson, Parisi, Zuber '78 /

- and recursively to all orders in $1/N$

/ Ambjørn, Chekhov, Kristjansen, Makeenko '96 /

Generalized eigenvalue models

$$\boxed{Z = \int d^N a \prod_{i<j} \mu(a_i - a_j)\, e^{-N \sum_i V(a_i)}}$$

- $\mu(a) = a^2$: ordinary matrix model

- $\mu(a) = a^{2\beta}$: β-ensemble / Chekhov, Eynard '06 /

- $\mu(a) = \dfrac{a^2}{a^2 + M^2}$: Hoppe model / Hoppe '89 Kazakov, Kostov Nekrasov '97 /

- $\mu(a) = \dfrac{H^2(a)}{H(a+M)\,H(a-M)}$: $\mathcal{N}=2^*$ super-Yang-Mills

/ Pestun '07
Chen-Lin, Gordon, Z '14 /

$$H(a) = \prod_{n=1}^{\infty} \left(1 + \frac{a^2}{n^2}\right)^n e^{-\frac{a^2}{n}}$$

Holographic duality

$$Z = \int d^N a \prod_{i<j} \mu(a_i - a_j)\, e^{-\frac{8\pi^2 N}{\lambda} \sum_j a_j^2}$$

Weak coupling:

$$\mathbb{W}(\lambda) = \sum_{n=0}^{\infty} w_n \lambda^n$$

$1 + \lambda$  $+ \lambda^2 \Big($  $+$  $+$  $\Big) + \dots$

Strong coupling:

$$\mathbb{W}(\lambda) = C \lambda^{-3/4}\, e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}} \sum_{n=0}^{\infty} \tilde{w}_n \left(\frac{2\pi}{\sqrt{\lambda}}\right)^n$$

 $+$  $+$  $+ \dots$

$$\boxed{Z = \int d^N a \prod_{i<j} \mu(a_i - a_j)\; e^{-N \sum_i V(a_i)}}$$

"arbitrary" function

Goal:

- solve at large-N

  - and at strong coupling ← this talk

Loop equation

$$W(\varkappa) = \left\langle \frac{1}{N} \sum_j e^{i\varkappa a_j} \right\rangle$$

Schwinger - Dyson eqs, closed at large-N:

$$\boxed{\frac{i}{2} V'\left(-i\frac{\partial}{\partial\varkappa}\right) W(\varkappa) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} R(\omega)\, W(\omega)\, W(\varkappa-\omega)}$$

$$R(\omega) = \int_{-\infty}^{+\infty} da\, e^{i\omega a} \frac{\mu'(a)}{2\mu(a)}$$

Ordinary matrix model:

$$\mu(a) = a^2 \quad \Longrightarrow \quad R(\omega) = -\pi \operatorname{sign}\omega$$

ν — model:

$$R(\omega) = -\pi \operatorname{sign}\omega \; |\omega|^{2\nu-2}$$

Loop eqn:

$$-i V'\left(-i \frac{\partial}{\partial x}\right) = \int_0^\infty d\omega \, \omega^{2\nu-2} \, W(\omega) \Big(W(x-\omega) - W(x+\omega) \Big)$$

Th Suppose Fourier image of $f(x)$ has a compact support*, then:

$$\int_0^\infty d\omega\, \omega^{2\nu-2} \frac{J_{\nu+n}(\omega)}{\omega^{\nu+n}} \Big(f(x-\omega) - f(x+\omega) \Big) = \mathcal{D}_n^{(\nu)} f(x)$$

differential operator of degree $2n+1$

$$\mathcal{D}_n^{(\nu)} = (-1)^n\, 2^{\nu-n-1}\, \Gamma(\nu-n-1)\, \mathbf{C}_{2n+1}^{(\nu-n-1)}\left(-i \frac{\partial}{\partial x}\right)$$

Gegenbauer

*) Vanishes outside the $[-1,1]$ interval

$$\mathcal{D}_0^{(\nu)}(a) = 2^\nu \Gamma(\nu)\, a$$

$$\mathcal{D}_1^{(\nu)}(a) = \frac{2^\nu \Gamma(\nu)}{6} (3a - 2\nu a^3)$$

$$\mathcal{D}_2^{(\nu)}(a) = \frac{2^\nu \Gamma(\nu)}{120} [15a - 20\nu a^3 + 4\nu(\nu+1) a^5]$$

$$\boxed{W(x) = \sum_{n\geq 0} A_n \frac{J_{\nu+n}(\mu x)}{(\mu x)^{\nu+n}}}$$

$$-i\,V'\left(-i\frac{\partial}{\partial x}\right) W(x) = \mu^{1-2\nu} \sum_n A_n\, \mathcal{D}_n^{(\nu)}\left(-\frac{i}{\mu}\frac{\partial}{\partial x}\right) W(x)$$

$$\rightsquigarrow$$

$$V'(\mu a) = \mu^{1-2\nu} \sum_n A_n\, \mathcal{D}_n^{(\nu)}(a)$$

- Need to expand a given polynomial $V'(\mu a)$ in the prescribed basis $\mathcal{D}_n^{(\nu)}(a)$

Solution (from orthogonality of Gegenbauers):

$$A_n = \frac{\mu^{2\nu-1}(2n+1)!}{4\pi\,\Gamma(2\nu-1)} \int_{-1}^{1} da\,(1-a^2)^{\nu-n-\frac{3}{2}}\,\mathcal{D}_n^{(\nu)}(a)\left(a\frac{\partial}{\partial a}+2\nu-1\right)V'(\mu a)$$

Wilson loop:

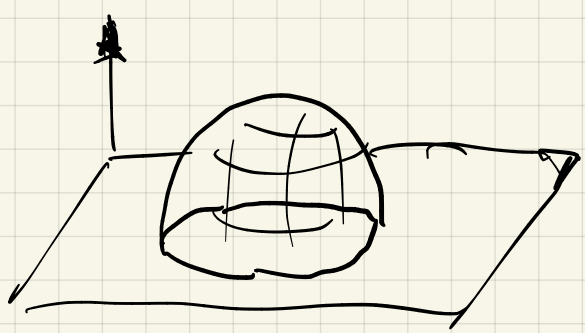
$$\boxed{W(x) = \sum_n A_n \frac{I_{\nu+n}(\mu x)}{(\mu x)^{\nu+n}}}$$

Normalization condition: $\mathbb{W}(0) = 1$

$$\sum_n \frac{A_n}{2^{\nu+n}\,\Gamma(\nu+n+1)} = 1 \quad \text{fixes } \mu$$

$$\mathbb{W} \overset{x\to\infty}{\sim} e^{\mu x}$$

translates into area law in holography: $\mu \sim \sqrt{\lambda}$



tension $\sim \sqrt{\lambda}$

area $\sim -x$

$$\mathbb{W}(x) \sim e^{-\text{tension} \times \text{Area}}$$

Strong coupling

$$Z = \int d^N a \prod_{i<j} \mu(a_i - a_j)\, e^{-\frac{N}{g^2}\sum_j a_j^2}$$

- Need to expand in $\frac{1}{g}$

More general (non-Gaussian) potential:

$$V(a) = \hbar^{2\nu-2}\, U(\hbar a)$$

$$\hbar = \frac{1}{g} \quad \text{for } U(a) = \frac{a^2}{2}, \ \nu = 1$$

- How to solve the model at $\hbar \to 0$?

Rescale $a_i \to \frac{a_i}{\hbar} \iff$ Typical $a_i \sim \frac{1}{\hbar} \gg 1$.

Caveat: $a_i - a_j$ can be ~ 1 even if each $a_j \sim \frac{1}{\hbar}$

$$\mathbb{W}(x) = \left\langle \frac{1}{N} \sum_i e^{x a_i} \right\rangle$$

Rescaling applies to <u>short</u> <u>loops</u>: $x = \mathcal{O}(\hbar)$

$$\mu(a) \overset{a\to\infty}{\simeq} a^{2\beta} \implies R(\omega) \overset{\omega\to 0}{\simeq} -\pi\beta \operatorname{sign}\omega$$

More generally: $R(\omega) \overset{\omega\to 0}{\simeq} -\pi\beta \operatorname{sign}\omega \, |\omega|^{2\beta-2}$

Solution for short loops

$$W(x) = \sum_n A_n \frac{I_{\nu+n}\left(\frac{\mu x}{\hbar}\right)}{\left(\frac{\mu x}{\hbar}\right)^{\nu+n}}$$

$$A_n = \frac{\mu^{2\nu-1}(2n+1)!}{4\pi\beta\,\Gamma(2\nu-1)} \int_{-1}^{1} da\,(1-a^2)^{\nu-n-\frac{3}{2}} \mathcal{D}_n^{(\nu)}(a)\left(a\frac{\partial}{\partial a} + 2\nu - 1\right) U'(\mu a)$$

$$\sum_n{}' \frac{A_n}{2^{\nu+n}\,\Gamma(\nu+n+1)} = 1$$

Matching

- Really interested in long loops: $x = O(1)$

Intermediate regime:

$$1 \gg x \gg \hbar$$

$$W(x) \simeq C \frac{\hbar^{\nu+\frac{1}{2}}}{x^{\nu+\frac{1}{2}}} e^{\sqrt{\frac{\mu x}{\hbar}}} \qquad C = \frac{A_0}{\sqrt{2\pi}\,\mu^{\nu+\frac{3}{2}}}$$

General form of solution:

$$W(x) = C\,\hbar^{\nu+\frac{3}{2}}\, e^{\sqrt{\frac{\mu x}{\hbar}}}\, f(x)$$

$$f(x) \overset{x \to 0}{\simeq} x^{-\frac{1}{2}-\nu} \equiv f_\infty(x)$$

Long loops

At strong coupling $\hbar \to 0$, $x \sim O(1)$

the Wilson loop acquires a scaling form:

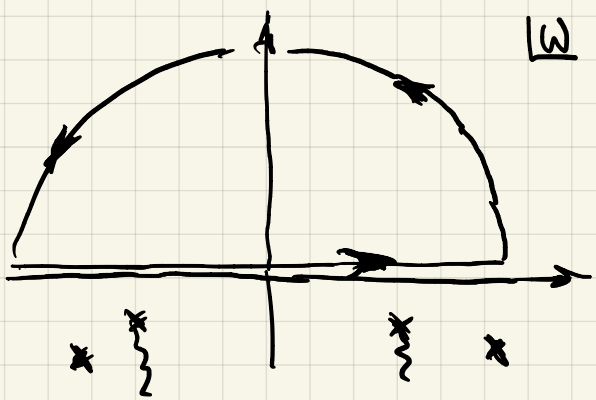
$$W(x) = C\,\hbar^{\nu+\frac{1}{2}}\, f(x)\, e^{\frac{i\mu x}{\hbar}}$$

Equation for the scaling function:

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, f(x-\omega)\Big(R(\omega) f(\omega) - R_\infty(\omega) f_\infty(\omega) \Big) = 0$$

Wiener-Hopf solution

- $\int_{-\infty}^{+\infty} d\omega\, \mathcal{F}(\omega) = 0$ if $\mathcal{F}(\omega)$ is analytic in the upper half-plane



Riemann - Hilbert factorization:

$$R(\omega) = \frac{G_+(\omega)}{G_-(\omega)}$$

$G_\pm$ - analytic in upper / lower half-plane

$$f(x) = \frac{i^{\frac{1}{2}-\nu}}{x - i\varepsilon}\, G_-(x)$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, f(x-\omega)\left[\frac{G_+(\omega)}{\omega - i\varepsilon} + \pi\beta\, \frac{(\omega+i\varepsilon)^{\nu-\frac{3}{2}}}{\omega - i\varepsilon} \right] = 0$$

because $\quad G_+(\omega) \overset{\omega\to\infty}{\simeq} -\pi\beta\,(\omega+i\varepsilon)^{\nu-\frac{3}{2}}$

$$\boxed{\Psi(x) = \frac{i^{\frac{1}{2}-\nu}\, C\, \hbar^{\nu+\frac{3}{2}}}{x}\, G_-(-ix)\, e^{\frac{\mu x}{\hbar}}}$$

Applications

- Localization in $\mathcal{N}=2^*$ super-Yang-Mills:

$$\mathcal{L}_{N=2^*} = \mathcal{L}_{N=4} + M^2 \Phi^2 + M \bar{\Psi}\Psi$$

$$W(2\pi) \overset{\lambda\to\infty}{=} C(M)\, \lambda^{-3/4}\, e^{\sqrt{\lambda} M}$$

String fluctuations

area law

in Pilch-Warner background

/ Buchel, Russo, Z.'13 /

/ Chen-Lin, Medina-Rincon, Z.'17 at $M=\infty$

Gautason, Puletti, Z.' to appear: any M /

- Hoppe model w. quartic potential:

$$V(a) = \frac{1}{g^2}\left(a^2 + \alpha a^4\right)$$

- solvable exactly at $\alpha = 0$
- our method works at $g \to \infty$ but any α.