Comments on 4-derivative scalar theory

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• A. A. Slavnov, “Invariant regularization of nonlinear chiral theories,” Nucl. Phys. B **31** (1971), 301-315

$$L = \mathrm{tr}\big[J_m J^m + \Lambda^{-4}\partial_n J^n(\Box + \mu)\partial_m J^m\big], \quad J_m = g^{-1}\partial_m g$$

unitarity at low energies; Ostrogradsky method

• L.D. Faddeev and A.A. Slavnov, “Gauge fields. An introduction to quantum theory", Front. Phys. **50** (1980), 1-232

$$L = \mathrm{tr}\big(F_{mn}F^{mn} + \Lambda^{-4}\Box F_{mn}\Box F^{mn}\big)$$

$$S = \int d^4x \, \phi \Box^2 \phi \, , \qquad \langle \phi(x)\phi(x')\rangle \sim \log(x-x')^2$$

• “dipole ghost": interest from 1970’s despite apparent non-unitarity
motivations: improved UV, model of confinement, R^2-gravity, etc.

N. Nakanishi, “Remarks on the dipole-ghost scattering states,”(1971);
H. Narnhofer and W. E. Thirring, “The taming of the dipole ghost,” (1978);
D. Zwanziger, “The lesson of a soluble model of Quantum Electrodynamics,”(1978);
E. D’Emilio and M. Mintchev, “A gauge model with confinement in 4 dimensions,” (1980);
W. Heidenreich, “Group theory of the dipole ghost” (1984); B. T. Binegar, “On the state space of the dipole ghost,"; M. Flato and C. Fronsdal, “Singleton field theory,” (1987)

• Euclidean theory: free 4d CFT for dim 0 scalar ϕ

non-unitary $SO(2,4)$ representation (but may apply projection?)

correlators of $\partial_m ... \partial_n \phi$ are well defined (decay at $|x| \to \infty$)

analogy with scalar $\phi \partial^2 \phi$ in 2d: primary operators $e^{q\phi}$, $\partial_m ... \partial_n \phi$

relation to partially massless fields in AdS_5

[C. Brust, "Free $\Box^k$ scalar conformal field theory," (2017); M. Safari et al, "Scale and conformal invariance in higher derivative shift symmetric theories," (2022); A. Stergiou, "Weyl covariance and energy momentum of higher-derivative free CFTs," (2022); A. Chalabi et al, "Boundaries in Free Higher Derivative CFTs," (2022).]

• Minkowski theory: 4-th time derivative, non-positivity of energy but ok if fixing some initial conditions in cosmological background?
[cf. G. W. Gibbons, C. N. Pope and S. Solodukhin,
"Higher Derivative Scalar Quantum Field Theory in Curved Spacetime," (2019)]

• other speculations: relevance for scale-invariant cosmology, etc.
[cf. L. Boyle and N. Turok, "Cancelling vacuum energy and Weyl anomaly in standard model with dimension-zero scalars," (2021)]

• beyond free theory: interactions and observables?

• which interactions allowed? to avoid IR ∞ at loop level require $\phi \to \phi + c$ shift symmetry – derivative interactions only
cf. 2d: massless S-matrix for $L = (\partial\phi)^2 + V(\phi)$ "does not exist" but is well defined for $(\partial\phi)^2 + g(\partial\phi)^4 + g'(\partial\phi)^6 + ...$
[e.g. $\sqrt{\det(\delta_{ij} + \partial_i\phi^a\partial_j\phi^a)} = 1 + \frac{1}{2}\partial^i\phi^a\partial_i\phi^a + \frac{1}{8}\left[(\partial^i\phi^a\partial_i\phi^a)^2 - 2(\partial_i\phi^a\partial_j\phi^a)^2\right] + ...$
Dubovsky, Flauger, Gorbenko (2012)]

• preserve classical scale invariance in 4d: dimensionless coupling g

$$L = \partial^2\phi\partial^2\phi + g(\partial\phi)^4$$

• similar Euclidean action: energy of crystalline membranes

[F. David and E. Guitter, "Crumpling Transition in Elastic Membranes," (1988);
J. Aronovitz and T. Lubensky, "Fluctuations of Solid Membranes," (1988);
M. Bowick and A. Travesset, "The Statistical mechanics of membranes", (2001)]

$$\int d^dx \left[T(\partial X^a)^2 + \kappa(\partial^2 X^a)^2 + \lambda(\partial_m X^a \partial_m X^a)^2 + \mu(\partial_m X^a \partial_n X^a)^2\right]$$

X^a $(a = 1, ..., N)$; $d = 2$ and $N = 3$ for 2d membrane in 3 dim
T tension; λ, μ elastic constants (Lame coeffs);
κ bending rigidity (extrinsic curvature coupling);
transition: crumpled elastic phase $\to$ crystalline phase $T \to 0$;
$X^a = (x^n + u^n(x), X^\alpha(x))$; limit of $N = 1,\ d = 4$: above action

• another instance: $\mathcal{N}=4$ conformal supergravity

complex dim 0 scalar – “superpartner" of Weyl graviton

[Bergshoeff, de Roo, de Wit (1981); Fradkin, AT (1982)]

also induced from Maxwell (or SYM) coupled to $\tau = C + ie^{-\phi}$

[Osborn (2003); Buchbinder, Pletnev, AT (2012)]

$$e^{-\phi(x)} F_{mn} F^{mn} + iC(x) F^*_{mn} F^{mn} \quad \to \quad \Gamma \sim \log \Lambda \int d^4x \, L(\tau)$$

$$L = \frac{1}{4(\operatorname{Im}\tau)^2}\Big[\mathcal{D}^2\tau\mathcal{D}^2\bar\tau - 2(R_{mn} - \tfrac{1}{3}Rg_{mn})\nabla^m\tau\nabla^n\bar\tau\Big]$$

$$+ \frac{1}{48(\operatorname{Im}\tau)^4}\Big(\nabla^m\tau\nabla_m\tau\nabla^n\bar\tau\nabla_n\bar\tau + 2\nabla^m\tau\nabla_m\bar\tau\nabla^n\tau\nabla_n\bar\tau\Big)$$

$$\mathcal{D}^2\tau \equiv \nabla^2\tau + \frac{i}{\operatorname{Im}\tau}\nabla^m\tau\nabla_m\tau, \quad \mathcal{D}^2\bar\tau \equiv \nabla^2\bar\tau - \frac{i}{\operatorname{Im}\tau}\nabla^m\bar\tau\nabla_m\bar\tau$$

$SL(2,R)$ and Weyl invariant

for $\tau = ie^{-\phi} \quad \to \quad L = (\partial^2\phi)^2 + (\partial\phi)^4$

One-loop effective action and beta-function

start with Euclidean $\partial^2\phi\partial^2\phi + g(\partial_m\phi\, \partial^m\phi)^2$ theory ($g > 0$)

expand near background $\phi = \varphi + \tilde{\phi}$

$$L = \partial^2\tilde{\phi}\partial^2\tilde{\phi} - V^{mn}(\varphi)\partial_m\tilde{\phi}\partial_n\tilde{\phi} + O(\tilde{\phi}^3)$$
$$V_{mn}(\varphi) = -2g(\delta_{mn}\partial^k\varphi\partial_k\varphi + 2\partial_m\varphi\partial_n\varphi)$$

one-loop effective action

$$\Gamma_1 = \tfrac{1}{2}\log\det\big[\partial^4 + \partial_m\big(V_{mn}(\varphi)\partial_n\big)\big]$$

log UV divergence from $\nabla^4 + ...$ analog of 't Hooft algorithm for $-\nabla^2 + V(\varphi)$ [Fradkin, AT (1981)]

$$(\Gamma_1)_\infty = -\tfrac{1}{(4\pi)^2}\log\tfrac{\Lambda}{\mu}\int d^4x\, b_4\ , \qquad b_4 = \tfrac{1}{24}V_{mn}V_{mn} + \tfrac{1}{48}(V^m_m)^2$$

single real scalar: $b_4 = 5g^2(\partial_m\varphi\partial_m\varphi)^2 \rightarrow$ renormalization of g

RG equation

$$\frac{dg}{dt} = \tfrac{1}{(4\pi)^2} 5g^2 \ , \qquad\qquad t = \log \mu$$

$g \to 0$ in IR: not asymptotically free (like standard ϕ^4)
not defined at short scales beyond Landau pole

• generalization to N scalars ϕ^a: two dim 0 couplings $g_1, \ g_2$

$$L = \partial^2\phi^a\partial^2\phi^a + g_1(\partial_n\phi^a\partial_n\phi^a)^2 + g_2(\partial_n\phi^a\partial_m\phi^a)(\partial_n\phi^b\partial_m\phi^b)$$

for $N = 1$ reduces to previous action with $g = g_1 + g_2$
for $\phi_a = \varphi_a + \tilde{\phi}_a$:

$$\begin{aligned} L &= \partial^2\tilde{\phi}_a\partial^2\tilde{\phi}_a - V^{ab}_{mn}(\varphi)\partial_m\tilde{\phi}_a\partial_n\tilde{\phi}_b + O(\tilde{\phi}^3) \\ V^{ab}_{mn} &= -2g_1(\delta_{ab}\delta_{mn}\partial_k\varphi_c\partial_k\varphi_c + 2\partial_m\varphi_{(a}\partial_n\varphi_{b)}) \\ &\quad - 2g_2(\delta_{ab}\partial_m\varphi_c\partial_n\varphi_c + \delta_{mn}\partial_k\varphi_a\partial_k\varphi_b + \partial_m\varphi_{(b}\partial_n\varphi_{a)}) \end{aligned}$$

coefficient of log divergence

$$b_4 = \frac{1}{24} V^{ab}_{mn} V^{ab}_{mn} + \frac{1}{48} (V^{ab})^m_m (V^{ab})^n_n$$

gives beta-functions for g_1 and g_2

$$\frac{dg_1}{dt} = \frac{1}{(4\pi)^2} \left[(2N + \frac{7}{3}) g_1^2 + (N + \frac{17}{3}) g_1 g_2 + \frac{1}{12} (N + 15) g_2^2 \right]$$
$$\frac{dg_2}{dt} = \frac{1}{(4\pi)^2} \left[\frac{2}{3} g_1^2 + \frac{10}{3} g_1 g_2 + \frac{1}{6} (N + 21) g_2^2 \right]$$

in agreement with [Bowick et al (2001)]

for $N = 1$: previous RG eq. for $g = g_1 + g_2$

• if g_1 and $g_2 > 0$ beta-functions have no zeroes

in general: no common zeroes, i.e. Landau pole in UV

• supersymmetric generalization of this model?

possibility of non-trivial fixed-points?

Minkowski signature theory

$\phi\partial^4\phi + g(\partial\phi)^4$: expected acausality and non-unitarity

• compare to non-renormalizable but unitary model

$$L_2 = \phi\,\partial^2\phi + \bar{g}(\partial^m\phi\,\partial_m\phi)^2\ , \qquad\qquad \bar{g} = M^{-4}g$$

here ϕ and M have dim 1

may be treated as an effective field theory [Weinberg (1995)]

unitary in low-energy perturbation theory ($s = E^2 < M^2$)

• positivity of $\bar{g}$ from condition of causal (subluminal)

propagation of perturbations in $\varphi = u_m x^m,\ u_m$=const

$$L_2 = -K^{mn}\partial_m\tilde{\phi}\,\partial_n\tilde{\phi} + \ldots\ , \qquad\qquad \phi = \varphi + \tilde{\phi}$$

$$K_{mn} = (1 - 2\bar{g}u^2)\eta_{mn} - 4\bar{g}u_m u_n$$

dispersion relation ($\partial_m \to ip_m$)

$$(1 - 2\bar{g}u^2)p^2 - 4\bar{g}(u^m p_n)^2 = 0$$

in perturbation theory $1 - 2\bar{g}u^2 > 0$ so for subluminal propagation $p_0^2 = v^2\vec{p}^2$ with $v^2 < 1$, i.e. $p^2 \equiv -p_0^2 + \vec{p}^2 \geq 0$ should have $\bar{g} > 0$ [Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi (2006)]

- same argument does not apply to $\phi\,\partial^4\phi + g(\partial\phi)^2$ model

$$K_{mn} = \eta_{mn}(\Box - 2gu^2) - 4gu_m u_n$$
$$p^4 + 2gu^2p^2 + 4g(u^m p_n)^2 = 0$$

for $g > 0$ (required for positivity of Euclidean action) subluminal $p^2 > 0$ solution exists only if u_m is time-like ($u^2 < 0$); suggests breakdown of causality (and related analyticity of S-matrix)

- background “resolves" dipole ghost: effectively get a massive ghost
- same in curved space: if start with Weyl-inv analog of $\Box^2$

$\Delta_4 = \nabla^4 + 2\nabla^m[(R_{mn} - \frac{1}{3}Rg_{mn})\nabla^n]$

in Einstein space $R_{mn} = \frac{1}{4}Rg_{mn}$ get $\Delta_4 = (-\nabla^2 + \frac{1}{6}R)(-\nabla^2)$ can be “diagonalized" (but with R^{-1} factors singular in flat limit)

Definition of S-matrix?

• asymptotic states? scattering amplitudes?
how to see expected non-unitarity in S-matrix?

• subtlety: “dipole ghost" theory is not a smooth limit
of “massive ghost" model $\phi(\Box^2 + \mu\Box)\phi$ which is
“diagonal" combination of massless mode and massive ghost
then defn of S-matrix straightforward, lack of unitarity explicit

• $\phi\Box^2\phi \to 2\psi\Box\phi - \psi^2$ is non-diagonalisable

• $\Box^2\phi = 0$: which $D(E, j_1, j_2)$ irrep of $SO(2,4)$?
indecomposable module: $D(1, \frac{1}{2}, \frac{1}{2})$ with $D(0,0,0)$ [Binegar:1983kb]
indefinite inner product; states with $\Box\phi = 0$ have 0 norm;
corresponding rep. of Poincare is not unitary and indecomposable

• $\Box^2\phi = 0$: “massless" oscillating solutions of $\Box\phi = 0$
$\phi(x) \sim \tilde{\phi}(p)e^{ip\cdot x}$ $(p^2 = 0)$ plus growing solutions $\phi(x) \sim N_n(p)\, x^n e^{ip\cdot x}$

• how to define S-matrix?

heuristic approach:

(i) specify “in" asymptotic conditions;

(ii) solve non-linear eqs: $\phi = \phi_{\rm in} + \mathcal{O}(g)$

(iii) evaluate (effective) action on the solution;

(iv) expand in powers of $\phi_{\rm in}$ to get scattering amplitudes

[cf. Arefeva, Slavnov, Faddeev (1974)]

• can scatter only oscillating modes [Adamo, Nakach, AT (2018)]:

analogs of amplitudes for growing modes not well defined –

IR divergent, no mom. conservation (cf. scattering in external field)

another route: introduce an auxiliary field to have

standard 2-derivative kinetic term

2-derivative formulation

$$L(\phi,\psi) = 2\psi\Box\phi - \psi^2 + g(\partial^m\phi\,\partial_m\phi)^2$$

equivalent model if ψ has no sources:

scattering amplitudes for ϕ only – same as in $\Box^2$ theory

• special case of $\Phi_a = (\psi,\phi)$

$$L = h_{ab}\Phi_a\Box\Phi_b + m_{ab}\Phi_a\Phi_b + V(\partial\Phi)$$

kinetic operator $h_{ab}\Box + m_{ab}$ not symmetric $\to$ not diagonalizable

• $\Box^2$ model not a smooth limit of $\Box^2 + \mu\Box$:

$$2\psi\Box\phi - \psi^2 + \mu\,\phi\Box\phi + V(\partial\phi)$$
$$= \mu\,\phi'\Box\phi' - \mu^{-1}\psi\Box\psi - \psi^2 + V(\partial\phi' - \mu^{-1}\partial\psi)\,, \qquad \phi' = \phi + \mu^{-1}\psi$$

$\mu \neq 0$: diagonal combination of massless mode and massive ghost

non-positive; considering massless S-matrix only is inconsistent

- $\mu \to 0$ limit is singular: $\Box^2$ case should be analysed separately
- field equations

$$\Box\phi - \psi = 0\,, \qquad \Box\psi - 2g\partial^m\big(\partial_m\phi(\partial\phi)^2\big) = 0$$

$g \to 0$: asymptotic solutions

$$\psi_{\rm in} = \int d^4p\,\delta(p^2)\,\tilde{\psi}_{\rm in}(p)\,e^{ip\cdot x}, \qquad \Box\psi_{\rm in} = 0$$
$$\phi_{\rm in} = \int d^4p\,\delta(p^2)\,N_n(p)\,x^n\,e^{ip\cdot x}\,, \quad \Box^2\phi_{\rm in} = 0\,, \quad \Box\phi_{\rm in} = \psi_{\rm in}$$
$$\partial_n\phi_{\rm in} = \int d^4p\,\delta(p^2)\,[N_n(p) + ip_nN_m(p)\,x^m]\,e^{ip\cdot x}$$

if $\quad \tilde{\psi}_{\rm in}(p) = 2iN^n(p)p_n \neq 0\quad$ growing $\partial_n\phi_{\rm in}$:
amplitude from $g\int d^4x(\partial\phi_{\rm in})^4$ divergent – not well defined

• the only consistent choice: $\psi_{\rm in}=0,\ \phi_{\rm in}\neq 0$

$$\psi_{\rm in}=0\ , \qquad \Box\phi_{\rm in}=0\ , \qquad \phi_{\rm in}(x)=\int d^4p\,\delta(p^2)\,\tilde\phi_{\rm in}(p)\,e^{ipx}$$

$$\phi=\phi_{\rm in}-\Box^{-1}\psi, \qquad \psi=2g\Box^{-1}\partial^m(\partial_m\phi_{\rm in}(\partial\phi_{\rm in})^2)+\mathcal{O}((\partial\phi_{\rm in})^5)$$

$$L=g(\partial\phi_{\rm in})^4+\mathcal{O}((\partial\phi_{\rm in})^6)$$

• same as in $\Box^2$ theory with massless $\phi_{\rm in}$ as asymptotic states

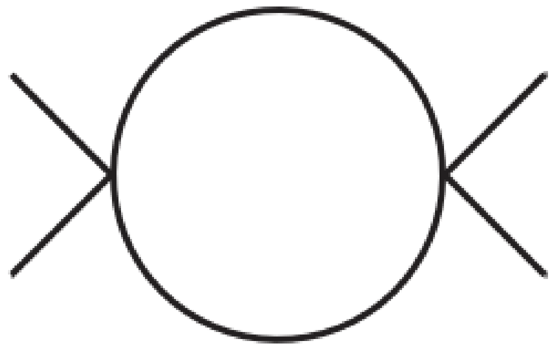
• massless scattering amplitude: same as in $\phi\Box\phi+\bar g(\partial\phi)^4$ theory

$$A_4^{\rm (tree)}\sim g(s^2+t^2+u^2)\ , \qquad s+t+u=0,\quad p_i^2=0$$

• how non-unitarity is reflected in massless S-matrix?

• Im part of 1-loop amplitude: generalized optical theorem holds in $\phi\Box\phi+\bar g(\partial\phi)^4$ but fails in $\phi\Box^2\phi+\bar g(\partial\phi)^4$

$\frac{1}{q^2}\to\frac{1}{q^4}$ internal propagators: Im $A_4^{(1-{\rm loop})}\sim A_4^{\rm (tree)}$ not $(A_4^{\rm (tree)})^2$



1-loop scattering amplitude

1-loop A_4 from 1-loop effective action on $\varphi = \varphi_{\rm in}$

$$\Gamma_1 \to \int \frac{d^4p_1}{(2\pi)^4}\dots\frac{d^4p_4}{(2\pi)^4} e^{ix\cdot\sum p_i}\, A_4(p_1,\dots,p_4)\,\varphi_{\rm in}(p_1)\dots\varphi_{\rm in}(p_4)$$

$$\mathrm{Tr}\Big[\Box^{-2}\partial_m(V^{mn}\partial_n)\Box^{-2}\partial_k(V^{kr}\partial_r)\Big] \to$$

$$A_4 \sim \mathcal{V}^{mn}(p_1,p_2)\,\mathcal{V}^{kr}(p_3,p_4)\int\frac{d^4q}{(2\pi)^4}\frac{(p_1{+}p_2{+}q)_m q_n (p_1{+}p_2{+}q)_k q_r}{(p_1{+}p_2{+}q)^4\ \ q^4}\ \delta^{(4)}(\textstyle\sum p_i)$$

$$\mathcal{V}_{mn}(p_1,p_2) = 2g(\delta_{mn}p_1\cdot p_2 + p_{1m}p_{2n} + p_{2n}p_{1m})\ , \qquad p_1^2 = p_2^2 = 0$$

• integral is IR finite but UV divergent

after renormalizing g in $A_4^{(\mathrm{tree})} + A_4^{(1-\mathrm{loop})}$ get

$$A_4 \sim g^2 \big[(s^2+t^2+u^2)\log(-\frac{s}{\mu^2}) + s^2 F(\frac{s}{t})\big]$$

classical theory is scale-invariant: same s^2 scaling as at tree level

$$A_4 \sim g^2 X_4 \,, \qquad X_4 = \int \frac{d^4 q}{(2\pi)^4} \frac{K(p_1,p_2,q)\; K(p_3,p_4,q)}{(p+q)^4\, q^4}$$

$$K(p_1,p_2,q) \equiv 2 p_1\cdot q\; p_2\cdot q + \tfrac{1}{2}p^2(p+q)^2 - \tfrac{1}{2}p^4, \qquad p = p_1 + p_2$$

• compute integral in dimensional reg (in Euclid)

limit $\epsilon \equiv \frac{d-4}{2} \to 0$ gives for s-channel amplitude

$$X_4 = \tfrac{1}{96(2\pi)^2}\Big[-(13s^2+t^2+u^2)\big(\tfrac{1}{\epsilon} + \log(-s) + \gamma_E - \log(4\pi)\big) + \tfrac{1}{3}(26s^2 - t^2 - u^2)\Big]$$

$$A_4 = g^2 X_4^{(\mathrm{sym})}$$

$$= -\tfrac{g^2}{96(2\pi)^2}\Big[15(s^2+t^2+u^2)\big[\tfrac{1}{\epsilon}+\log(-s)+\gamma_E-\log(4\pi)-\tfrac{8}{15}\big]$$

$$+(13t^2+u^2+s^2)\log\tfrac{t}{s}+(13u^2+s^2+t^2)\log\tfrac{u}{s}\Big]$$

- to compare: in standard massless $\phi\Box\phi + g\phi^4$ theory

$$A_4 = -\tfrac{g^2}{16\pi^2}\big[\tfrac{1}{\epsilon}+\log(-s)+\gamma_E-\log(4\pi)-2\big]$$

- in unitary $\phi\Box\phi + \bar{g}(\partial\phi)^4$ theory:

$$X_4 = \int \tfrac{d^4q}{(2\pi)^4}\tfrac{K(p_1,p_2,q)\,K(p_3,p_4,q)}{(p+q)^2\;q^2}$$

$$A_4 = \bar{g}^2 X_4^{(\mathrm{sym})}$$

$$=\tfrac{1}{480(2\pi)^2}\Big(-27(s^2+t^2+u^2)^2\big[\tfrac{1}{\epsilon}+\log(-s)+\gamma_E-\log(4\pi)-\tfrac{1609}{30}\big]$$

$$+2\big[22t^4+5t^2(u^2+s^2)\big]\log\tfrac{t}{s}+2\big[22u^4+5u^2(s^2+t^2)\big]\log\tfrac{u}{s}\Big)$$

- UV div $\sim p^8$ requires counterterm $\sim (\partial\partial\phi)^4$
- imaginary part comes from

$$\log(-s) \to \log|s| + i\pi$$

same structure as square of tree-level amplitude $(s^2+t^2+u^2)^2$
agreement with generalized optical theorem as in ϕ^4 theory

- suggests optical theorem should fail in $\phi\Box^2\phi + g(\partial\phi)^4$ case:

same tree amplitude but Im of 1-loop $\sim s^2$ not s^4

Issue of unitarity

- unitarity of S-matrix $\to$ generalized optical theorem:

$\mathrm{S} = 1 + iT, \quad -i(T - T^\dagger) = T^\dagger T$

$\langle 1, 2|T|3, 4\rangle = A_4(p_1, ..., p_4)\, \delta^{(4)}(\sum p_i)$

$$- i\big[A_4(p_1, p_2 \to p_3, p_4) - A_4^*(p_3, p_4 \to p_1, p_2)\big] = \sum_n \prod_{i=1}^n \int \frac{d^4 q_i}{(2\pi)^4}\, \delta(q_i^2)$$
$$\times A_4(p_1, p_2 \to q_1, .., q_n)\, A_4^*(p_3, p_4 \to q_1, .., q_n)\, (2\pi)^4 \delta^{(4)}(\textstyle\sum_{i=1}^n q_i)$$

- for $n = 2$ internal propagators in $\phi\Box\phi + ...$ case

$\int d^4 q \frac{1}{q^2 (q+p)^2} = \int d^4 q_1 \int d^4 q_2 \frac{1}{q_1^2 q_2^2} \delta^{(4)}(p - q_1 - q_2)$

$$2\,\mathrm{Im}\, \frac{1}{q^2 + i\epsilon} = -\frac{2i\epsilon}{(q^2+\epsilon^2)^2} \to -2\pi i \delta(q^2)$$

- standard ϕ^4 theory: tree-level amplitude is const and get

$$2\,\mathrm{Im}\, A_4^{(1-\mathrm{loop})}(p_1, p_2, p_3, p_4) = (2\pi)^4 \int d\Pi(q_1) \int d\Pi(q_2)$$
$$\times A_4^{(\mathrm{tree})}(p_1, p_2, q_1, q_2)\, A_4^{(\mathrm{tree})}(q_1, q_2, p_3, p_4)\, \delta^{(4)}(p_1 + p_2 - q_1 - q_2)$$

$$\int \frac{d^4q}{(2\pi)^4} 2\pi\delta(q^2)\theta(q^0) \equiv \int d\Pi(q) = \int \frac{d^3\vec{q}}{(2\pi)^3\ 2|\vec{q}|}$$

- in c.o.m. $p = p_1 + p_2 = (p^0, 0, 0, 0)$ match Im part of $A_4^{(1-\text{loop})}$

$$\int d^4q \frac{1}{(q^2+i\epsilon)((q+p)^2+i\epsilon)} \to \int d^4q\, \delta(q^2)\, \delta((q+p)^2)$$

$$\to \quad 4\pi \int dq^0 \ \int d|\vec{q}|\ |\vec{q}|^2\ \frac{\delta(q^0-|\vec{q}|)}{2|\vec{q}|}\ \delta((p^0)^2 - 2p^0q^0) \to \pi$$

- same argument applies to $\phi\Box\phi + \bar{g}(\partial\phi)^4$ theory:

$$\int d^4q\, \delta(q^2)\, \delta((q+p)^2)\ K(p_1, p_2, q)\ K(p_3, p_4, q)$$

$$\to \tfrac{1}{4} \int dq^0 \ \int d^3\vec{q}\, \delta(q^0 - |\vec{q}|)\ \delta(q^0 - \tfrac{1}{2}p^0)\ K(p_1, p_2, q)\ K(p_3, p_4, q)$$

reduces to Im of 1-loop amplitude

and = phase space integral of product of tree-level amplitudes

• $\phi\Box^2\phi + g(\partial\phi)^4$ theory: unitarity not be expected
external massless states but $1/q^4$ propagators imply more virtual “states"
• argument for optical theorem used $1 = \sum |...\rangle\langle...|$
may imply summing also over “growing" modes
but corresponding tree-level amplitudes are not well defined
• Minkowski-space analog of $\Box^{-2} \sim \log x^2$ or $1/q^4$?
in 2-derivative formulation $\Box \to \Box - i\epsilon$ or $q^2 \to q^2 + i\epsilon$ implies

$$\frac{1}{q^4} \quad \to \quad \frac{1}{(q^2+i\epsilon)^2}$$

consistent with prescription [d’Emilio, Mintchev (1982)]

$$\Box^{-2} \quad \to \quad \frac{1}{(q^2-a^2+i\epsilon)^2} + i\pi \log\frac{a^2}{\mu^2}\,\delta^{(4)}(q)\ , \qquad a \to 0$$

(in case of derivative interactions $\delta^{(4)}(q)$ terms drop out)

• get analog of standard cut relation

$$\frac{1}{(q^2+i\epsilon)^2}-\frac{1}{(q^2-i\epsilon)^2}=-\frac{4i\epsilon q^2}{[(q^2)^2+\epsilon^2]^2}\to 2\pi i\frac{\partial}{\partial q^2}\delta(q^2)$$

$$\delta(x)=\tfrac{1}{\pi}\tfrac{\epsilon}{x^2+\epsilon^2}\Big|_{\epsilon\to 0},\quad \delta'(x)=-\tfrac{1}{\pi}\tfrac{2\epsilon x}{(x^2+\epsilon^2)^2}\Big|_{\epsilon\to 0}$$

• then from 1-loop integral

$$\mathcal{V}_{ij}(p)\mathcal{V}_{kr}(-p)\int d^4q d^4q'\delta^{(4)}(q+p-q')\; q_j q_r \tfrac{\partial}{\partial q^2}\delta(q^2)\; q'_i q'_k \tfrac{\partial}{\partial q'^2}\delta(q'^2)$$

• standard proof of optical theorem does not apply:
$\frac{\partial}{\partial q^2}\delta(q^2)$ instead of $\delta(q^2)$ factors
modification of phase space measure?

Concluding remarks

• massless S-matrix not good observable in dim 0 scalar theory?

• unitarity issue may be resolved beyond perturbation theory?
relaxing causality condition?
(cf. attempts in massive ghost models $\Box^2 + \mu\Box$)

• extra interactions may modify $\frac{1}{q^4}$ propagator?
elementary fields "confined", do not appear as asymptotic states
but only via virtual loops?

• view $\Box^2$ theory as $\epsilon \to 0$ limit of ghost-free non-local theory
e.g. $\phi\Box\frac{e^{\epsilon\Box}-1}{\epsilon}\phi + g(\partial\phi)^4$
[cf. Koshelev and Tokareva (2021)]