Backlund transformation for the nonlinear Schrodinger equation

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Nonlinear Schrodinger equation

The equation

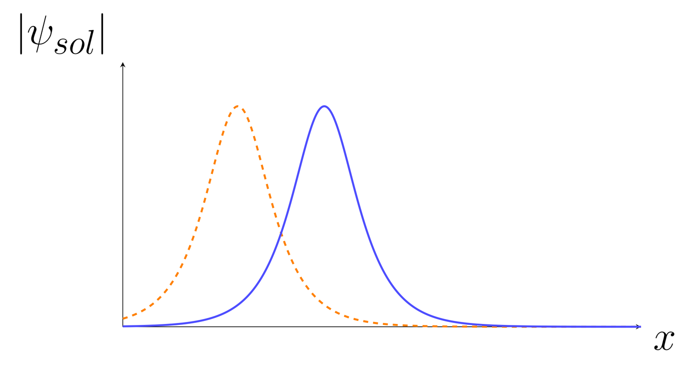
$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + 2c\bar{\psi}\psi^2 \tag{1}$$

on a complex-valued function $\psi(x,t)$ appears in a description of many physical phenomena and is called *nonlinear Schrodinger equation*.

The transformation $\mathcal{B}_v$ defined by relations

$$\begin{aligned} \bar{\varphi} &= v\bar{\psi} - i\partial_x\bar{\psi} - c\bar{\psi}^2\varphi, \\ \psi &= v\varphi + i\partial_x\varphi - c\bar{\psi}\varphi^2 \end{aligned} \tag{2}$$

represents a map $(\psi,\bar{\psi}) \mapsto (\varphi,\bar{\varphi})$ between solutions of the equation (1) with a parameter $v \in \mathbb{C}$. It is called *Backlund transformation* and together with its inverse $\mathcal{B}_v^{-1}$ can be used to generate new nontrivial solutions[1]. For example, from $\psi = 0$ this transformation generates a famous soliton solution.



$$\psi_{sol}(x,t) = \frac{a}{2\sqrt{-c}} \, \frac{\exp\left[\dfrac{ib}{2}x + \dfrac{i(a^2-b^2)}{4}t\right]}{\cosh\left(\dfrac{a}{2}[x-bt]\right)}$$

Figure 1. Soliton

Rebirth of interest in such transformations was caused by discovered connections with quantum integrable systems and separation of variables phenomenon[2].

Integrability and quantization

The nonlinear Schrodinger equation (1) is integrable and can be investigated by Inverse scattering method[3]. For this it is written as a zero-curvature condition

$$\partial_t L - \partial_x A + [L, A] = 0$$

where A and L are time and coordinate components of some "potential" (A, L). Both components are 2×2 matrices that depend on $\psi(x,t)$, the coordinate component being

$$L(u) = i\begin{pmatrix} -u/2 & c\,\bar{\psi} \\ -\psi & u/2 \end{pmatrix}, \qquad u \in \mathbb{C}.$$

In particular, assuming $x \in [-\ell,\ell]$ and periodic boundary conditions $\psi(-\ell,t) = \psi(\ell,t)$, this formulation implies that for any solution $\psi(x,t)$ the so-called *transfer-matrix*

$$t(u) = \operatorname{tr}\, \mathrm{Pexp}\left[\int_{-\ell}^{\ell} dx\, L(u)\right]$$

is conserved and generates infinite tower of local integrals of motion

$$\ln\left(\frac{e^{iu\ell}}{ic}\,t(u)\right) = u^{-1}H_1 + u^{-2}H_2 + u^{-3}H_3 + \dots$$

The third integral in this series

$$H_3 = \int_{-\ell}^{\ell} dx\,\left(\partial_x\bar{\psi}\,\partial_x\psi + c\,\bar{\psi}^2\psi^2\right)$$

can be regarded as a Hamiltonian of the nonlinear Schrodinger equation, since the latter can be written in the form

$$\partial_t\psi = \{H_3, \psi\}$$

assuming standard Poisson brackets between the functions $\psi, \bar{\psi}$

$$\{\psi(x,t), \bar{\psi}(y,t)\} = i\delta(x-y), \qquad \{\psi,\psi\} = \{\bar{\psi},\bar{\psi}\} = 0.$$

The Hamiltonian formulation allows to quantize this equation by turning functions into operators with the corresponding commutation relations

$$\psi, \bar{\psi} \longmapsto \hat{\psi}, \hat{\psi}^\dagger, \qquad [\hat{\psi}(x,t), \hat{\psi}^\dagger(y,t)] = \delta(x-y).$$

These operators act on a Fock space generated from the vector $|0\rangle$, such that $\hat{\psi}|0\rangle = 0$. Any other vector from the space can be written in the following way

$$f(\hat{\psi}^\dagger)|0\rangle = f_0|0\rangle + \sum_{n=1}^{\infty}\frac{1}{\sqrt{n!}}\int d^n\boldsymbol{x}\, f_n(\boldsymbol{x})\,\hat{\psi}^\dagger(x_1)\cdots\hat{\psi}^\dagger(x_n)|0\rangle. \tag{3}$$

In the n-particle sector the quantum model coincides with the famous Lieb-Liniger model defined by the Hamiltonian

$$H_{LL} = -\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + c\sum_{i\neq j}\delta(x_i - x_j).$$

Results*

We give a new derivation of the Backlund transformation $\mathcal{B}_v$.

It allows us to construct its quantum counterpart, the Baxter Q-operator, and to prove its key properties.

* Based on the works:
N. Belousov, S. Derkachov, *The Q-Operator for the Quantum NLS Model*, Journal of Mathematical Sciences, **242** (2019), 608–627.
N. Belousov, *Backlund transformation for the nonlinear Schrodinger equation*, Journal of Mathematical Sciences, **264** (2022), 203–214.

Backlund transformation from gauge invariance

Any transformation $(\psi,\bar{\psi}) \mapsto (\varphi,\bar{\varphi})$ that preserves transfer-matrix

$$t(\varphi,\bar{\varphi}) = t(\psi,\bar{\psi})$$

preserves all integrals of motion and, in particular, the Hamiltonian H_3. Moreover, if it is canonical, then it also preserves Poisson brackets and, as a consequence, the nonlinear Schrodinger equation.

The transfer-matrix is preserved under the gauge transformations of the matrix L

$$L(\varphi,\bar{\varphi}) = ML(\psi,\bar{\psi})M^{-1} + \big(\partial_x M\big)M^{-1}. \tag{4}$$

As it was demonstrated for other integrable models[2], in order to have a canonical mapping, matrix M should be searched among the matrices that satisfy the same Yang-Baxter equation as the matrix L.

We found that for the matrix M corresponding to the so-called *DST-chain* integrable model

$$M = \begin{pmatrix} v - u - c\,\bar{\psi}\varphi & c\,\bar{\psi} \\ -\varphi & 1 \end{pmatrix}$$

the matrix equation above (4) is equivalent to the Backlund transformation (2). This transformation is canonical since is can be written in the form

$$\bar{\varphi} = \frac{\delta F_v}{\delta\varphi}, \qquad \psi = \frac{\delta F_v}{\delta\bar{\psi}}$$

with the generating function

$$F_v(\bar{\psi},\varphi) = \int dx\,\left(\bar{\psi}\,(v + i\partial_x)\,\varphi - \frac{c}{2}\bar{\psi}^2\varphi^2\right). \tag{5}$$

As a byproduct, from the degeneracy of the matrix M at $v = u$ we obtained an explicit formula that connects the transfer-matrix and Backlund transformation

$$t(u) = 2\cosh\left(-iu\ell + ic\int dx\,\bar{\psi}\varphi\right). \tag{6}$$

Quantum Backlund transformation

Analogously to canonical transformations, similarity transformations $\hat{A} \mapsto \hat{Q}\hat{A}\hat{Q}^{-1}$ preserve quantum commutation relations. In the case of integral operator $\hat{Q}$, its kernel should reproduce a generating function of the corresponding classical canonical transformation in the limit $\hbar \to 0$[4]. For some transformations (e. g. Fourier transform) this connection is exact.

For the quantum nonlinear Schrodinger the scalar product between the vectors (3) can be written using the Gaussian functional integral

$$\begin{aligned} \langle 0|\bar{g}(\hat{\psi})\, f(\hat{\psi}^\dagger)|0\rangle &= \bar{g}_0 f_0 + \sum_{n=1}^{\infty}\int d^n\boldsymbol{x}\,\bar{g}_n(\boldsymbol{x})f_n(\boldsymbol{x}) \\ &= \int \mathrm{D}\alpha^\dagger \mathrm{D}\alpha\, \exp\left[-\int dx\,\alpha^\dagger\alpha\right]\,\bar{g}(\alpha)f(\alpha^\dagger). \end{aligned}$$

We observed that the operator $\hat{Q}$ defined by its action on an arbitrary vector f

$$\left[\hat{Q}(v)f\right](\hat{\psi}^\dagger)|0\rangle = \int \mathrm{D}\alpha^\dagger\mathrm{D}\alpha\,\exp\left[-\int dx\,\alpha^\dagger\alpha\right]\,\mathcal{Q}_v(\hat{\psi}^\dagger,\alpha)f(\alpha^\dagger)|0\rangle$$

with the kernel reproducing exactly the generating function of the Backlund transformation (5)

$$\mathcal{Q}_v(\hat{\psi}^\dagger,\alpha) = \exp\left[F_v(\hat{\psi}^\dagger,\alpha)\right]$$

satisfies a number of remarkable properties:

- It can be written in a closed form
$$\hat{Q}(v) = :\exp\left[\int dx\,\left(\hat{\psi}^\dagger(v - 1 + i\partial_x)\hat{\psi} - \frac{c}{2}(\hat{\psi}^\dagger)^2\hat{\psi}^2\right)\right]:$$
using normal ordering $:\ :$ symbol.
- It represents a commuting family of operators
$$[\hat{Q}(v), \hat{Q}(u)] = 0.$$
This is a counterpart of the commutation property of the Backlund transformation $\mathcal{B}_v \circ \mathcal{B}_u = \mathcal{B}_u \circ \mathcal{B}_v$.
- It also commutes with the quantum transfer-matrix
$$[\hat{Q}(v), \hat{t}(u)] = 0.$$
Since quantum transfer-matrix generates quantum conserved quantities, this property agrees with the fact that $\mathcal{B}_v$ preserves integrals of motion.
- It satisfies Baxter difference equation
$$\hat{Q}(u)\hat{t}(u) = e^{-iu\ell}\hat{Q}(u + ic) + e^{iu\ell}\hat{Q}(u - ic).$$
This is an analog of the classical formula (6).
- The wave functions $|u_1\dots u_n\rangle$ of the quantum model are known exactly (by Bethe ansatz). We proved that the eigenvalues of the Q-operator are polynomials with zeroes at the quantum numbers u_j
$$\hat{Q}(v)|u_1\dots u_n\rangle = \prod_{j=1}^{n}(v - u_j)|u_1\dots u_n\rangle.$$

The last two properties together give famous *Bethe equations* for the Lieb-Liniger model.

Finally, we remark that the quantum nonlinear Schrodinger admits a discretization that preserves integrability. It is given by $sl(2)$-invariant spin chain. We observed that the same Q-operator can be obtained in the continuum limit of the corresponding object from this spin chain.

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[4] V. Fock, *On the canonical transformation in classical and quantum mechanics*, Acta Physica, **27** (1969), 219–224.