Lower-dimensional Regge-Teitelboim gravity

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Introduction

We study modified gravity theory known as Regge-Teitelboim approach [3], in which the gravity is represented by dynamics of a surface isometrically embedded in a flat bulk. We obtain some particular solutions of Regge-Teitelboim equations corresponding to a central symmetric vacuum 2+1-dimensional spacetime. In contrast with GR, this vacuum spacetime is not flat, so it is possible for the gravitational field to exist even without matter or cosmological constant.

Result and discussions

The 2+1-dimensional metric with $SO(2)\times\mathbf{R}$ symmetry can be written as:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\phi^2. \tag{1}$$

When embedding pseudo-riemannian manifold with this metric into 5-dimensional flat space, we can receive 6 types of surfaces [1], 4 of which can be described using one expression [2]:

$$\begin{cases} y^0 = kt + \frac{1}{\alpha}h(r) \\ y^1 = \frac{1}{\alpha}f(r)\sqrt{\varepsilon}\sin(\sqrt{\varepsilon}(\alpha t + \omega(r)) \\ y^2 = \frac{1}{\alpha}f(r)\cos(\sqrt{\varepsilon}(\alpha t + \omega(r)) \\ y^3 = r\sin\phi \\ y^4 = r\cos\phi \end{cases}$$

(2)

The functions $f(r), \omega(r), h(r)$ can be found using the induced metric conditions

$$g_{\mu\nu} = \partial_\mu y^a \partial_\nu y^b \eta_{ab}, \tag{3}$$

where $\eta_{ab} = \{\lambda, \mu\varepsilon, \mu, -1, -1\}$ and $\lambda, \mu, \varepsilon = \pm 1$:

$$\omega(r) = -\int \frac{k\lambda h_r}{\mu\varepsilon f^2}dr, \tag{4}$$

$$f(r) = \sqrt{\frac{A - k^2\lambda}{\mu\varepsilon}}, \tag{5}$$

$$h(r) = \int\sqrt{\frac{\alpha^2}{\lambda A}(1-B)(A-\lambda k^2) - \frac{A'^2}{4\lambda\varepsilon A}}dr. \tag{6}$$

The RT equations can be obtained from the Einstein-Hilbert action if we substitute the induced metric condition (3) in it:

$$\partial_\mu(\sqrt{g}G^{\mu\nu}\partial_\nu y^a) = 0. \tag{7}$$

The components of Einstein tensor in the case of metric (1) take the form

$$\begin{cases} G^{tt} = \frac{AB_r}{2rB^2} \\ G^{rr} = \frac{A_r}{2rA}. \end{cases} \tag{8}$$

Therefore, only one contribution of RT equations for $a = 0$ stays:

$$\partial_r(\sqrt{g}G^{rr}\partial_r y^0) = 0. \tag{9}$$

Integrating, we obtain the first RT equation. The second one may be chosen as a linear combination of equations for $a = 1$ and $a = 2$.

$$\begin{cases} \sqrt{g}G^{rr}\partial_r y^0 = C \\ \partial_t y^1(\sqrt{g}G^{tt}\partial_t\partial_t y^2 + \partial_r(\sqrt{g}G^{rr}\partial_r y^2)) - \\ \qquad \partial_t y^2(\sqrt{g}G^{tt}\partial_t\partial_t y^1 + \partial_r(\sqrt{g}G^{rr}\partial_r y^1) = 0, \end{cases} \tag{10}$$

where C is an integration constant. From the first first equation we can find $h(r) = \int\frac{2\alpha CAB^2}{A_r\sqrt{AB}}dr$. Substituting $h(r), \omega(r), f(r)$ from (4), we can rewrite the equations:

$$\begin{cases} AA_r(k^2\lambda - A)(BA_rA_{rr} + B_r(\varepsilon A^2B(k^2\lambda - A) - \frac{3A_r^2}{4}))+ \\ \qquad +BA_r^4(k^2\lambda - \frac{A}{2}) + 4\varepsilon\alpha^2k^2C^2A^2B^4 = 0 \\ \frac{A_r^4}{4} - \varepsilon\alpha^2(B-1)(k^2\lambda - A)A_r^2 + 4\lambda\varepsilon\alpha^2A^2B^3C^2 = 0 \end{cases} \tag{11}$$

Particular solutions

1. **C = 0**

If the integration constant equals to zero, then we shall conclude that $h_r = 0$ otherwise $G^{rr} = G^{tt} = 0$ and the spacetime is flat. Now we can find

$$A_r = \pm\frac{\alpha\sqrt{\varepsilon}A^{\frac{1}{6}}(A-\lambda k^2)^{\frac{5}{6}}}{\sqrt{A^{\frac{1}{3}}(A-\lambda k^2)^{\frac{2}{3}} - \beta\varepsilon(\lambda k^2 - A)}}. \tag{12}$$

Since there is no explicit solution for this type of equation, we can impose $A \sim r^n$ when $r \to \infty$. Equation (12) can be fulfilled by setting $n = 2$, so

$$\begin{cases} A(r) \to \left(\frac{r}{R}\right)^2 \\ B(r) \to const \end{cases} \quad r \to \infty. \tag{13}$$

2. **B=const**

In this case, the equation on A takes the form

$$A_r rA - A_r^2 = (1-\beta)\lambda\varepsilon\alpha^2k^2 \tag{14}$$

and the general solution is

$$A(r) = a\left(1 + \frac{r}{R}\right)^2 + \lambda\varepsilon\alpha^2k^2(1-\beta)r^2/2, \tag{15}$$

therefore $\beta = 1$ or $k = 0$. The case $b = \pm 1$ is physically undesirable, since the quantity $1 - b^2$ corresponds to the angular deficit in BTZ geometry [4] and is related to a mass of a point source. In order to keep a nontrivial angular deficit and thus a nonzero mass of a source, one must conclude that $k = 0$. Therefore, let us discuss this case in more detail.

3. **k=0**

The first RT equation from (11) admits the first integral:

$$\frac{A_r^2}{2\varepsilon\alpha^2AB^{\frac{3}{2}}} + \frac{2}{\sqrt{B}} = K, \tag{16}$$

and the solution for A_r is

$$A_r^2 = 2\varepsilon\alpha^2AB^{\frac{3}{2}}(K - \frac{2}{\sqrt{B}}). \tag{17}$$

After the substitution into the second equation of (11) one can obtain

$$B = const, \qquad A(r) = \left(1 + \frac{r}{R}\right)^2. \tag{18}$$

Summary and conclusions

We ensured that RT equations are satisfied by surfaces with the following metric:

$$ds^2 = \left(1 + \frac{r}{R}\right)^2 dt^2 - b^2dr^2 - r^2d\phi^2. \tag{19}$$

And the appropriate embedding function with signature $(++---)$ takes form:

$$\begin{cases} y^0 = Cb^3Rr \\ y^1 = \frac{1}{\alpha}\left(1 + \frac{r}{R}\right)\sinh\alpha t \\ y^2 = \frac{1}{\alpha}\left(1 + \frac{r}{R}\right)\cosh\alpha t \\ y^3 = r\cos\phi \\ y^4 = r\sin\phi. \end{cases} \tag{20}$$

Such metric gives rise to a nonzero Einstein tensor, so the corresponding spacetime is non-flat and bears a gravitational field, whereas in 2+1-dimensional GR vacuum spacetimes must be flat.

References

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