Asymptotic symmetries of general relativity in the BV-BRST formalism

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Abstract

- An important role in modern field theory is played by theories defined on manifolds with boundaries, in particular asymptotic ones. In this context, gauge field theories are of particular interest.
- The theory of general relativity on the asymptotic boundary has been widely studied for many years, for example by Penrose [1], but in last years interest in this topic has greatly increased. Recently, in [2], a new approach was proposed in terms of the so-called tractor calculus, which had previously been successfully applied to problems of conformal geometry.
- We will show how the theory of gravity can be compactly reformulated and put on the boundary in the Gauge PDE formalism, some version of the BV-BRST formalism. Moreover, our constructions allow us to build a Gauge PDE version of the tractor calculus. A closely related approach to boundary values of AdS_{d+1} gauge fields was previously shown in [3].

Q-manifolds

- A Q-manifold is a $\mathbb{Z}$-graded supermanifold equipped with a degree one vector field Q, which satisfies the nilpotency condition $Q^2 = 0$.
- The simplest example of a Q-manifold is $T[1]X$, the shifted tangent bundle to a smooth manifold X. If θ^α are coordinates on the fibres of $T[1]X$ in the basis $\frac{\partial}{\partial x^\alpha}$, then $d_X = \theta^\alpha \frac{\partial}{\partial x^\alpha}$ is the de Rham operator.
- A Q-bundle is a fibered bundle, that is, a locally trivial $\mathbb{Z}$-graded bundle $\pi : (E, Q_E) \to (M, Q_M)$ over a Q-manifold M, such that $\pi^* \circ Q_M = Q_E \circ \pi^*$. A Q-section is $\sigma : M \to E$, such that $\pi \circ \sigma = Id$ and $\sigma^* \circ Q_E = Q_M \circ \sigma^*$.

Gauge PDE formalism

- Gauge PDE [4] is a $\mathbb{Z}$-graded Q-bundle (E, Q) over $(T[1]X, d_X)$, where $(T[1]X, d_X)$ is considered as a graded Q-manifold with the form degree and the de Rham differential d_X.
- The concept of Gauge PDE allows us to encrypt information about any gauge theory (at the level of PDE). Basic example is a BV-BRST formulation of any gauge theory on jets. Then $E = J^\infty(F_X)$, $Q = s + dx^a D_a$. Here F_X is the space of fields, ghosts, antifields, etc., s is the standart BV-BRST differential of the theory and D_a are vectors from the Cartan distribution on jets such that $\pi_* D_a = \frac{\partial}{\partial x^a}$. In this language, σ satisfies the equations of motion $\iff$ σ is a Q section.
- But the Gauge PDE formalism is much more flexible than just reformulating BV-BRST on jets. This is due to the fact that E may not be a jet bundle. Moreover, there is a well-defined notion of Gauge PDE equivalence. Locally, it boils down to the fact that we can exclude from consideration the fiber variables forming contractible pairs, i.e. w^α and v^α such that $Qw^\alpha = v^\alpha$.
- The boundary theory is naturally defined as the pullback bundle i^*E, where $i : T[1]\mathcal{J} \to T[1]X$ is the embedding of the boundary, induced from the ordinary $i' : \mathcal{J} \to X$. Additional relations on the theory fields, boundary conditions, can also be imposed on the boundary.

Gauge PDE example: General relativity

- Consider a specific Gauge PDE corresponding to the general relativity. In the fiber we have coordinates $\{D_{(a)}\tilde{g}_{mn}, D_{(a)}\xi^n\}$, where (a) is multiindex, grading gh: gh($\tilde{g}_{mn}$)=0, gh(ξ^n)=1 and $QD_a = D_a Q$,
$$Q\tilde{g}_{mn} = \xi^a D_a \tilde{g}_{mn} + \tilde{g}_{an} D_m \xi^a + \tilde{g}_{ma} D_n \xi^a, \quad Q\xi^m = \xi^a D_a \xi^m.$$
In addition, we have algebraic relations on jets of the metric: $D_{(a)}(R_{mn}[\tilde{g}] - \Lambda \tilde{g}_{mn})$=0, i.e. Einstein equations and its jet prolongations. Now we can search for contractible pairs in this system, throw them out and get equivalent formulations of GR. But we will follow a slightly different way.
- One can notice that if we add to the fiber coordinates $\{D_{(a)}\Omega, D_{(a)}\lambda\}$ such that
$$Q\Omega = \xi^a D_a \Omega + \lambda\Omega, \qquad Q\lambda = \xi^a D_a \lambda$$
and $\Omega > 0$, nothing will change since these coordinates form contractible pairs. We can use this and make an inversible change of coordinates $g_{mn} = \Omega^2 \tilde{g}_{mn}$. Then
$$Qg_{mn} = \xi^a D_a g_{mn} + g_{an} D_m \xi^a + g_{ma} D_n \xi^a + 2\lambda g_{mn}.$$

Reduced model for GR

- Now we can use the well-known elimination of contractible pairs in conformal geometry from [5]. After that in the fiber we have coordinates $\{g_{mn}, \lambda, \nabla_a\lambda, \xi^n, \nabla_r\xi^n, \nabla_{(a)}\Omega, \nabla_{(a)}W^m{}_{nkp}\}$. Here $\nabla_{(a)}(\dots)$ can be obtained by changing coordinates from $D_{(a)}(\dots)$ and $W^a{}_{bcd}$ is the Weyl curvature. Einstein's equations in these coordinates will take an unusual form:
$$\Omega\rho + n^a n_a = \Lambda, \quad \nabla_m \nabla_n \Omega + g_{mn}\rho = 0, \quad \nabla_{a_1} \dots \nabla_{a_n}\Omega = 0, n \geq 3,$$
here $\rho \equiv -\frac{1}{d}\nabla_a \nabla^a \Omega$ and $n^a \equiv g^{ab}\nabla_b \Omega$. It can be shown that this system of equations is equivalent to Friedrich's conformal equations, which are known to be equivalent to Einstein's equations in the bulk, but are well defined at null-infinity. This system can also be used to build a Gauge PDE analog of the tractor calculus.

Boundary model for GR

- Our new system is equivalent to the system for Einstein's equations at $\Omega > 0$ and the crucial point is that it is well defined at $\Omega = 0$. Let's use this to study its boundary behavior. Suppose that the base manifold has a boundary and $i : T[1]\mathcal{J} \to T[1]X$ is the embedding of the boundary. Then the boundary theory is i^*E plus additional boundary conditions:
$$\Omega = 0, \quad Q\Omega = 0,$$
where the first goes back to Penrose [1] and is included in the definition of an asymptotically simple space-time, and the second is necessary for the self-consistency of the system.
- The equations of motion are induced from the bulk:
$$n_a n^a = \Lambda, \quad \nabla_m \nabla_n \Omega + g_{mn}\rho = 0, \quad \nabla_{a_1} \dots \nabla_{a_n}\Omega = 0, n \geq 3$$

Asymptotically AdS spacetimes

- Consider the case of $\Lambda < 0$. Reducing the contractible pairs and taking into account the equations of motion we obtain a system with the fiber coordinates $\{\xi^A, \nabla_A \xi^B, \lambda, \nabla_A \lambda, \nabla_{(a)} W^a{}_{bcd}\}$, where $A, B = 1, \dots, n \equiv dim X - 1$ and
$$Q\xi^A = \xi^B \nabla_B \xi^A - \xi^A \lambda, \quad Q\lambda = \xi^A \nabla_A \lambda,$$
$$Q\nabla_A \xi^B = \nabla_A \xi^C \nabla_C \xi^B + \nabla_A \lambda \xi^B - \nabla^B \lambda \xi_A + \frac{1}{2}\xi^C \xi^D W^B{}_{ACD},$$
$$Q\nabla_A \lambda = \nabla_A \xi^C \nabla_C \lambda - \lambda \lambda^A + \frac{1}{2(n-2)}\xi^B \xi^C \nabla_E W^{EA}{}_{BC}.$$
i.e. for zero curvatures, Q is the Chevalley-Eilenberg differential for n-dimensional conformal algebra, as might be expected.
- Additional restrictions on the curvatures can also follow from the equations of motion. For example, for $n = 4$, the Bach tensor on the boundary turns out to be zero (as expected, since the Bach tensor is the Fefferman-Graham obstruction tensor for $n = 4$).

Asymptotically flat spacetimes

- Consider the case of $\Lambda = 0$. After reducing the contractible pairs we obtain a system with the fiber coordinates $\{\xi^u, C^N, \xi^N, \nabla_M \xi^N, \lambda, \nabla^u \lambda, \nabla_M \lambda, \nabla_{(a)} W^a{}_{bcd}\}$, where $M, N = 1, \dots, n-1$ and for zero curvatures Q is the Chevalley-Eilenberg differential for the algebra of symmetries of the conformal n-dimensional Carroll geometry equipped with connection (induced from the bulk). This group is closely related to the well-known BMS (Bondi - Metzner - Sachs) group. Moreover, at the boundary connection has failed to be unique that is expressed in variable C^M.

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