Entanglement entropy of finite regions in spherically symmetric black holes

T.Rusalev, Steklov Mathematical Institute, Russian Academy of Sciences

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Goals of work

The goals of the work are for asymptotically flat spherically symmetric eternal black holes:

1. *Explicitly calculate the entanglement entropy of infinite regions and check the basic properties of the entanglement entropy that must be hold for the pure state of the total system*
2. *Consider the time evolution of the entanglement entropy for different types of finite regions using the island formula. The finite region setup in asymptotically flat black holes is a "toy model" for considering, for instance, the island formula for a Schwarzschild-de Sitter black hole.*

Information paradox in terms of entanglement entropy

It is known [1] that due to semiclassical effects a black hole emits approximately thermal Hawking radiation. The information paradox (unitarity violation in the process of radiation from a black hole) can be formulated in terms of the time dependence of the entanglement entropy of Hawking radiation collected in an infinite region R. Let the factorization of Hilbert spaces be given

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{BH} \otimes \mathcal{H}_R, \text{ total pure state } \rho_{tot}, \text{ (BH means black hole).} \tag{1}$$

The entanglement entropy is defined as the von Neumann entropy for reduced density matrix

$$S(R) \equiv S(\rho_R) = -\operatorname{Tr} \rho_R \log \rho_R, \text{ where } \rho_R = \operatorname*{Tr}_{\mathcal{H}_{BH}} \rho_{tot}.$$

The violation of the upper bound

$$S(R) \le S^{thermod}(BH) \propto Area(horizon). \tag{2}$$

where $S^{thermod}(BH)$ is thermodynamic entropy of black hole, is interpreted as an information paradox [2–3]. It is believed that the correct behavior of the entanglement entropy is obtained by [4–5]

$$\textbf{Island formula: } S(R) \simeq \min_{\partial I} \left\{ \operatorname*{ext}_{\partial I} \left[\frac{\text{Area}(\partial I)}{4G_{\text{N}}} + S_{\text{matter}}(R \cup I) \right] \right\}.$$

We consider four-dimensional asymptotically flat eternal Schwarzschild and Reissner-Nordström black holes. The length element is given as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}, \tag{3}$$

where G is the Newton constant, M and Q are the mass and charge of the black hole ($Q = 0$ for Schwarzschild black hole). Maximally extended spacetime by transition to Kruskal coordinates

$$U = -\frac{1}{\kappa} e^{-\kappa(t - r_*(r))}, \; V = \frac{1}{\kappa} e^{\kappa(t + r_*(r))}, \; r_*(r) = \int \frac{dr'}{f(r')}, \; \kappa_{Sch} = \frac{1}{4GM}, \; \kappa_{RN} = \frac{\sqrt{G^2M^2 - GQ^2}}{(GM + \sqrt{G^2M^2 - GQ^2})^2}. \tag{4}$$

The length element in Kruskal coordinates is

$$ds^2 = -e^{2\rho(r)} dU dV + r^2 d\Omega^2. \tag{5}$$

Reduction of a high-dimensional problem to two-dimensional one

Reduction logic: spherically symmetric black hole background $\Rightarrow$ the massless field in the high-dimensional problem is expanded in terms of spherical harmonics Y_{lm} $\Rightarrow$ the lowest harmonic $l = 0$, the s-wave, corresponds to an effective massless field theory $\Rightarrow$ we *assume* that the entanglement entropy in such a theory approximates the entanglement entropy in the original problem (**s-wave approximation**) [4,6].

The entanglement entropy of free massless Dirac fermions for region $R = [x_1, y_1] \cup \ldots \cup [x_N, y_N]$ [7]

$$S(R) = \frac{c}{3} \left(\sum_{i,j=1}^{N} \log \frac{d(\mathbf{x}_i, \mathbf{y}_j)}{\varepsilon} - \sum_{i<j}^{N} \log \frac{d(\mathbf{x}_i, \mathbf{x}_j)}{\varepsilon} - \sum_{i<j}^{N} \log \frac{d(\mathbf{y}_i, \mathbf{y}_j)}{\varepsilon} \right), \tag{6}$$

where ε is the UV cut-off and

$$d^2(\mathbf{x}, \mathbf{y}) = [U(\mathbf{x}) - U(\mathbf{y})] [V(\mathbf{y}) - V(\mathbf{x})] e^{\rho(\mathbf{x})} e^{\rho(\mathbf{y})}. \tag{7}$$

Regularization of spacelike infinities and its removal

The following properties for entanglement entropy must be satisfied when the total system is in a pure state:

1. entanglement entropy of the total pure state must be equal to zero: $S(R_{total}) = 0$
2. entanglement entropies of the interval R and its complement $\overline{R}$ are equal: $S(R) = S(\overline{R})$

To check these properties for an eternal asymptotically flat spherically symmetric black hole using the entanglement entropy (6) of the effective CFT_2, it requires the regularization of spacelike infinities.

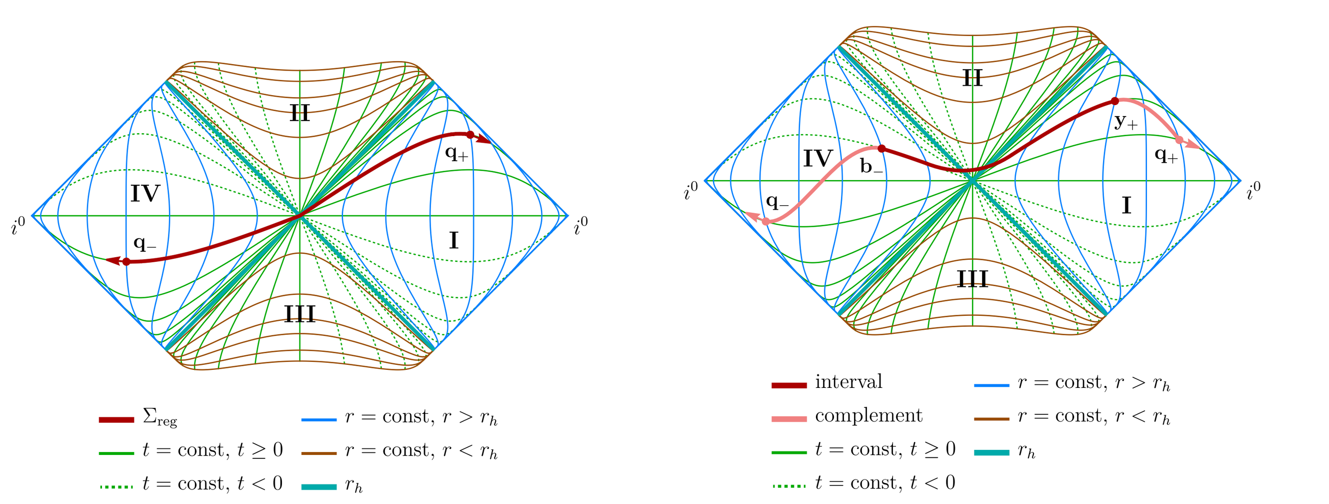

Figure 1.

To regularize Cauchy surface Σ, we take a finite spacelike interval $\Sigma_{\text{reg}} \subset \Sigma$ such that $\Sigma_{\text{reg}} = [\mathbf{q}_-, \mathbf{q}_+]$ with

$\mathbf{q}_+ = \{r_{q_+}, t_{q_+}\}$ for spacelike infinity in **left** wedge, $\quad \mathbf{q}_- = \{r_{q_-}, t_{q_-}\}$ for spacelike infinity in **right** wedge

Removing the regularization: for $S(\Sigma_{\text{reg}})$ we consider the limit $\mathbf{q}_\pm$ tend to spacelike infinities i^0 in appropriate wedges along an arbitrary spacelike geodesics $t_{q_\pm} = t_{q_\pm}(r_{q_\pm})$. But this limit exists only along the curves

$$\kappa \left[r_*(r_{q_+}) - r_*(r_{q_-}) \right] = c_1 + \alpha(r_{q_+}, r_{q_-}), \; \kappa \left[t_{q_+}(r_{q_+}) - t_{q_-}(r_{q_-}) \right] = c_2 + \beta(r_{q_+}, r_{q_-}), \tag{8}$$

where c_1, c_2 are arbitrary constants and $\lim\limits_{r_{q_\pm} \to \infty} \alpha(r_{q_+}, r_{q_-}) = \lim\limits_{r_{q_\pm} \to \infty} \beta(r_{q_+}, r_{q_-}) = 0$. Then along these curves

$$S(\Sigma) = \lim_{q_\pm \to \infty} S\left(\Sigma_{\text{reg}}\right) = \frac{c}{3} \log \frac{2}{\kappa_h \varepsilon} + \frac{c}{6} \log \left(\frac{\cosh c_1 + \cosh c_2}{2} \right). \tag{9}$$

The best we can get is to let $c_1 = c_2 = 0$, which leads to $r_{q_+} - r_{q_-} = o(1)$, $t_{q_+} - t_{q_-} = o(1)$ at $r_{q_\pm} \to \infty$

$$S(\Sigma)\Big|_{c_1 = c_2 = 0} = \frac{c}{3} \log \left(\frac{2}{\kappa\varepsilon} \right). \tag{10}$$

The same calculations can be made for the part $R \subset \Sigma$ of the Cauchy surface and its complement $\overline{R} = \Sigma / R$.

Violation of Entanglement Entropy Properties

The main properties of the entanglement entropy for the pure total state **are violated** in the background of an eternal asymptotically flat spherically symmetric black hole:

1. $S(\Sigma) = \dfrac{c}{3} \log \left(\dfrac{2}{\kappa\varepsilon} \right) \neq 0$, where Σ is Cauchy surface
2. $S(\overline{R}) - S(R) = \dfrac{c}{3} \log \left(\dfrac{2}{\kappa\varepsilon} \right) \neq 0$, where $R \subset \Sigma$ and $\overline{R} = \Sigma / R$

The constant that violates the properties of the entanglement entropy has the following features:

- diverges ultraviolet at $\varepsilon \to 0$
- for a Reissner-Nordström black hole can be made arbitrarily large by approaching the extreme case $Q \to \sqrt{G}M$ due to the black hole temperature $\kappa \propto T \propto \sqrt{M^2 - G^{-1}Q^2} \to 0$

Finite region dynamics in Schwarzschild space-time

Let us now consider the dynamics of the entanglement entropy for finite region

$$R_{\text{finite}} = [\mathbf{q}_-, \mathbf{b}_-] \cup [\mathbf{b}_+, \mathbf{q}_+] \subset R_\infty = \left(i_L^0, \mathbf{b}_-\right] \cup [\mathbf{b}_+, i_R^0\big) \subset \Sigma,$$
$$r_b \equiv r_{b_+} = r_{b_-}; \; r_q \equiv r_{q_+} = r_{q_-}; \; t_{b_+} = -t_{b_-}, \; t_b \equiv |t_{q_+}| = |t_{q_-}| = |t_{b_+}|. \tag{11}$$

The dynamics is obtained by island formula – the "competition" of a phases with and without island

$$I = \overbrace{[\mathbf{a}_-, \mathbf{a}_+]}^{Island} \Rightarrow S_{gen}(R_{\text{finite}}, I) = \frac{\pi(r_{a_-}^2 + r_{a_+}^2)}{G} + S_{\text{matter}}([\mathbf{q}_-, \mathbf{b}_-] \cup [\mathbf{a}_-, \mathbf{a}_+] \cup [\mathbf{b}_+, \mathbf{q}_+])$$

extremization procedure: $\quad \partial_{r_{a_-}, r_{a_+}} S_{gen}(R_{\text{finite}}, I) = 0, \; \partial_{t_{a_-}, t_{a_+}} S_{gen}(R_{\text{finite}}, I) = 0$

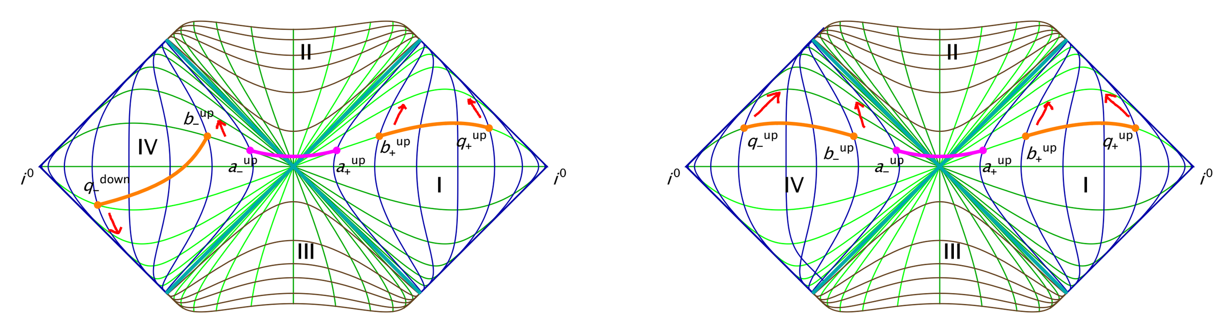

Figure 2. Asymmetric (**left**) and mirror-symmetric (**right**) regions (orange) with island (magenta).

Unitarity is preserved if the following inequality holds

$$S(R_{\text{finite}}) \le \frac{2\pi r_h^2}{G} + S(R_\infty / R_{\text{finite}}). \tag{12}$$

1. Asymmetric (AS) region: $t_{q_+} = t_{q_-}$

   Cauchy surface breaking for AS region: there is an *upper bound* on time t_b – during time evolution, the point $\mathbf{q}_-$ moves in time along the flow of the Killing vector ∂_t^-, while the point $\mathbf{b}_-$ moves in the opposite direction. The interval between them eventually becomes timelike at time $t_{\text{break}} = \dfrac{r_*(q) - r_*(b)}{2}$. Hence for $t_b > t_{\text{break}}$, the problem becomes ill-defined, since there is no longer a spacelike Cauchy surface.

   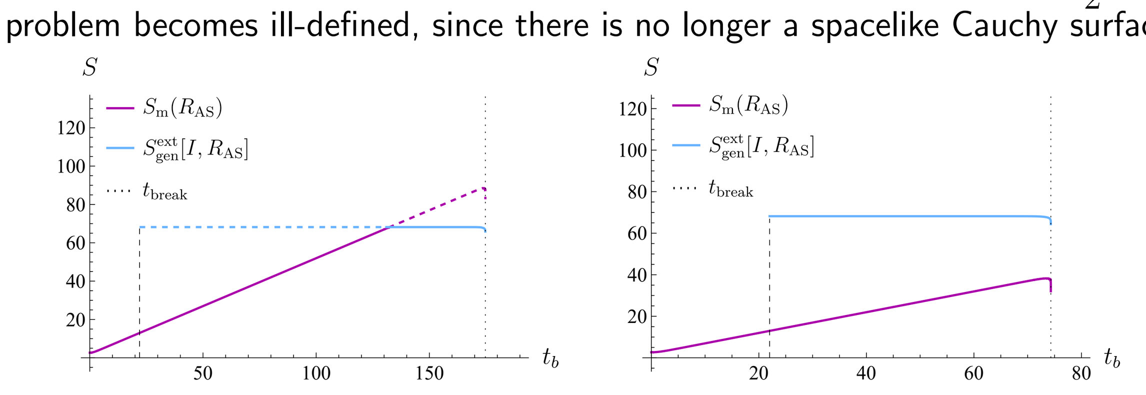

   Figure 3. For sufficiently large sizes of AS region **(left)**, the island dominates in the final stage removing monotonous growth. For sufficiently small sizes **(right)**, only the phase without the island dominates.

   Summing up: **a)** dynamics for the AS regions corresponds to the behavior of infinite regions [6] (correct limit to infinity); **b)** but there is *inevitable* a Cauchy surface breaking; **c)** for a large AS regions the island retains a unitary evolution; **d)** for a small AS regions the island is not dominated and is not required for unitarity.

2. Mirror-symmetric (MS) region: $t_{q_+} = -t_{q_-}$

   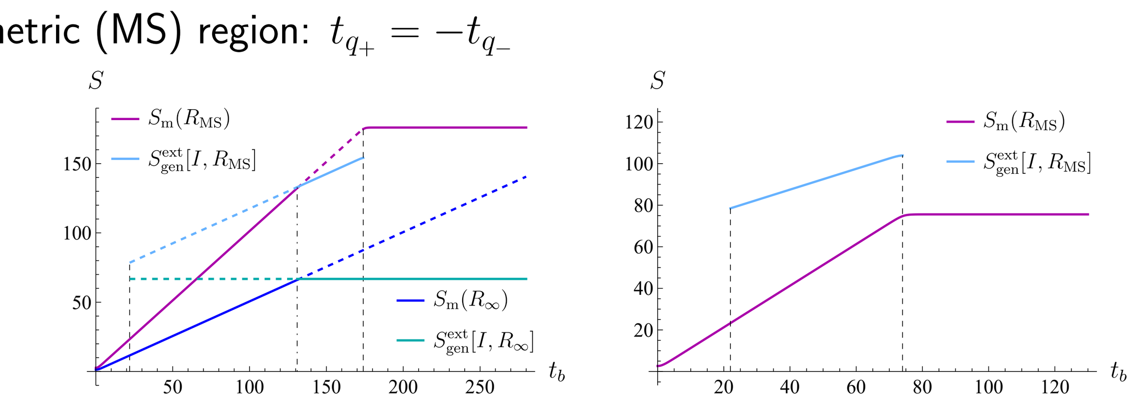

   Figure 4. For sufficiently large MS regions **(left)**, the island dominates for a finite time, after which there is a *discontinuous jump* to "plateau". Comparison with the dynamics of infinite regions R_∞ shows inconsistency with MS regions. For sufficiently small regions **(right)**, only the non-island phase dominates.

   Summing up: **a)** dynamics for the AS regions does not correspond to the behavior of infinite regions [6](incorrect limit to infinity); **b)** there is *no* Cauchy surface breaking; **c)** but for a large MS regions the island dominates for a finite time which leads to a discontinuous jump – it leads to non-unitary evolution over a finite time; **d)** for a small MS regions the island is not dominated and is not required for unitarity.

Conclusions

1. The basic properties of the entanglement entropy of radiation for asymptotically flat spherically symmetric eternal black holes, generally speaking, are not satisfied at least in the s-wave approximation
2. For sufficiently large finite regions, the island begins to dominate in the later stages; for small regions, only the non-island phase dominates
3. The dynamics of the AS region, corresponding to the correct limit to an infinite one, leads to "Cauchy surface breaking", after which the problem becomes ill-defined
4. The dynamics of the MS region does not correspond to the dynamics of infinite one. For large MS region, there is a discontinuous jump and violation of unitarity

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