The most beautiful formula in mathematics. Euler identity history

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$$ln(cos\varphi + isin\varphi) = i\varphi$$

$$cos\varphi + isin\varphi = e^{i\varphi}$$

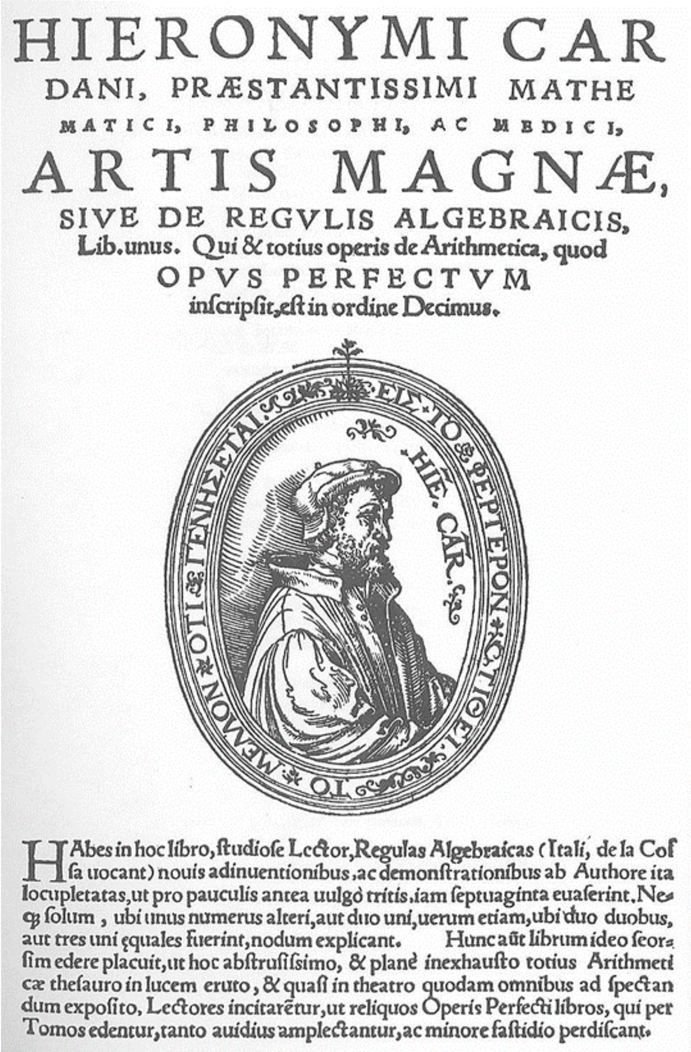
$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

1545

Girolamo Cardano

First appearance of complex numbers



HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inſcripſit, eſt in ordine Decimus.

HAbes in hoc libro, ſtudioſe Lector, Regulas Algebraicas (Itali, de la Coſa uocant) nouis adinuentionibus, ac demonſtrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam ſeptuaginta euaſerint. Neq; ſolum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo ſeorſim edere placuit, ut hoc abſtruſiſsimo, & planè inexhauſto totius Arithmeticæ theſauro in lucem eruto, & quaſi in theatro quodam omnibus ad ſpectandum expoſito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore faſtidio perdiſcant.

1572

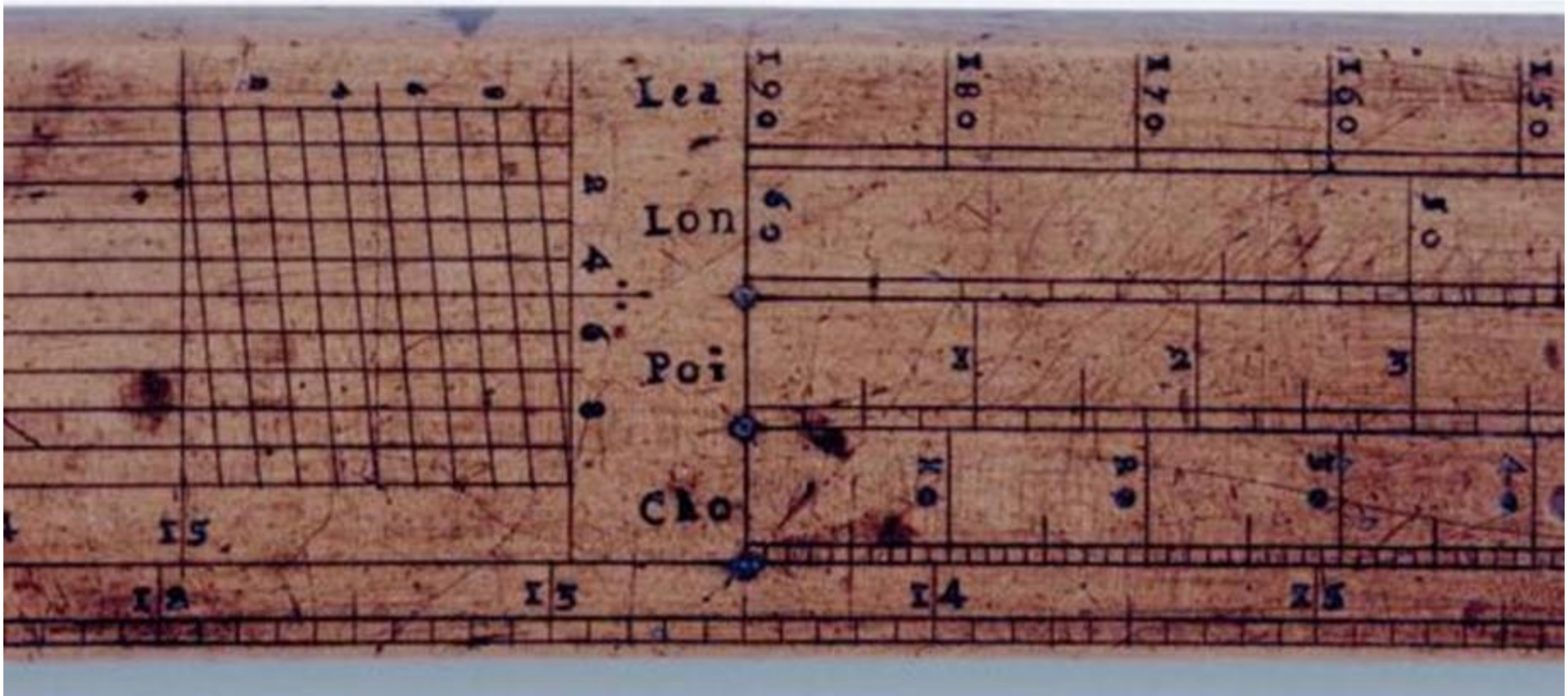
Rafael Bombelli

Addition and multiplication of complex numbers

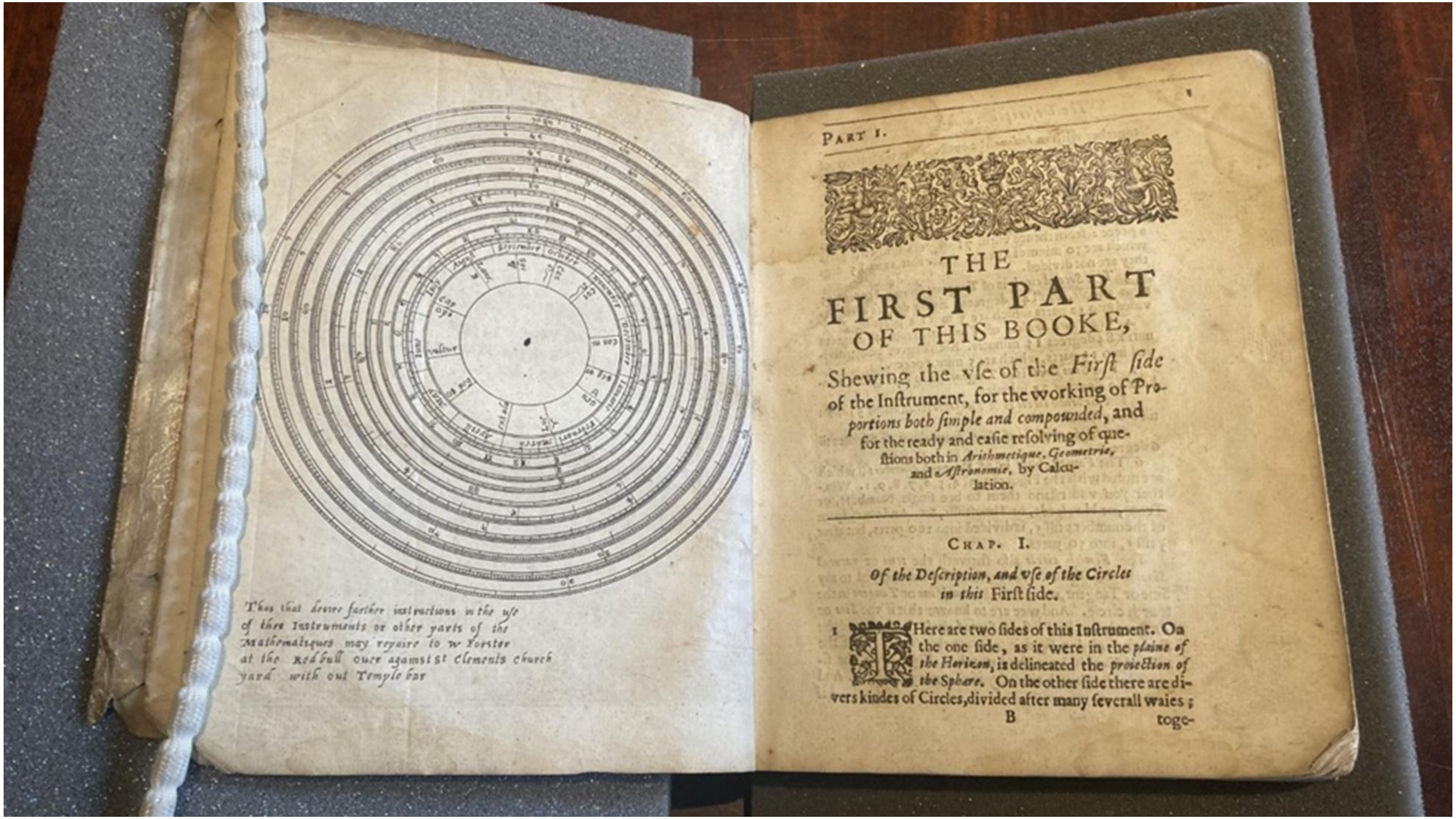

L'ALGEBRA
OPERA
Di RAFAEL BOMBELLI da Bologna
Diuisa in tre Libri.
Con la quale ciascuno da se potrà venire in perfetta cognitione della teorica dell' Arimetica.
Con vna Tauola copiosa delle materie, che in essa si contengono.
Posta hora in luce à beneficio delli Studiosi di detta professione.
IN BOLOGNA,
Per Giouanni Rossi. MDLXXIX.
Con licenza de' Superiori

1614, John Napier, logarithms

1620-1622,
First slide rules



William Oughtred (1574-1660)
Clavis Mathematicae



1637, **Rene Descartes**
on the status of complex numbers

LA

GEOMETRIE.

LIVRE PREMIER.

Des problesmes qu'on peut construire sans y employer que des cercles & des lignes droites.

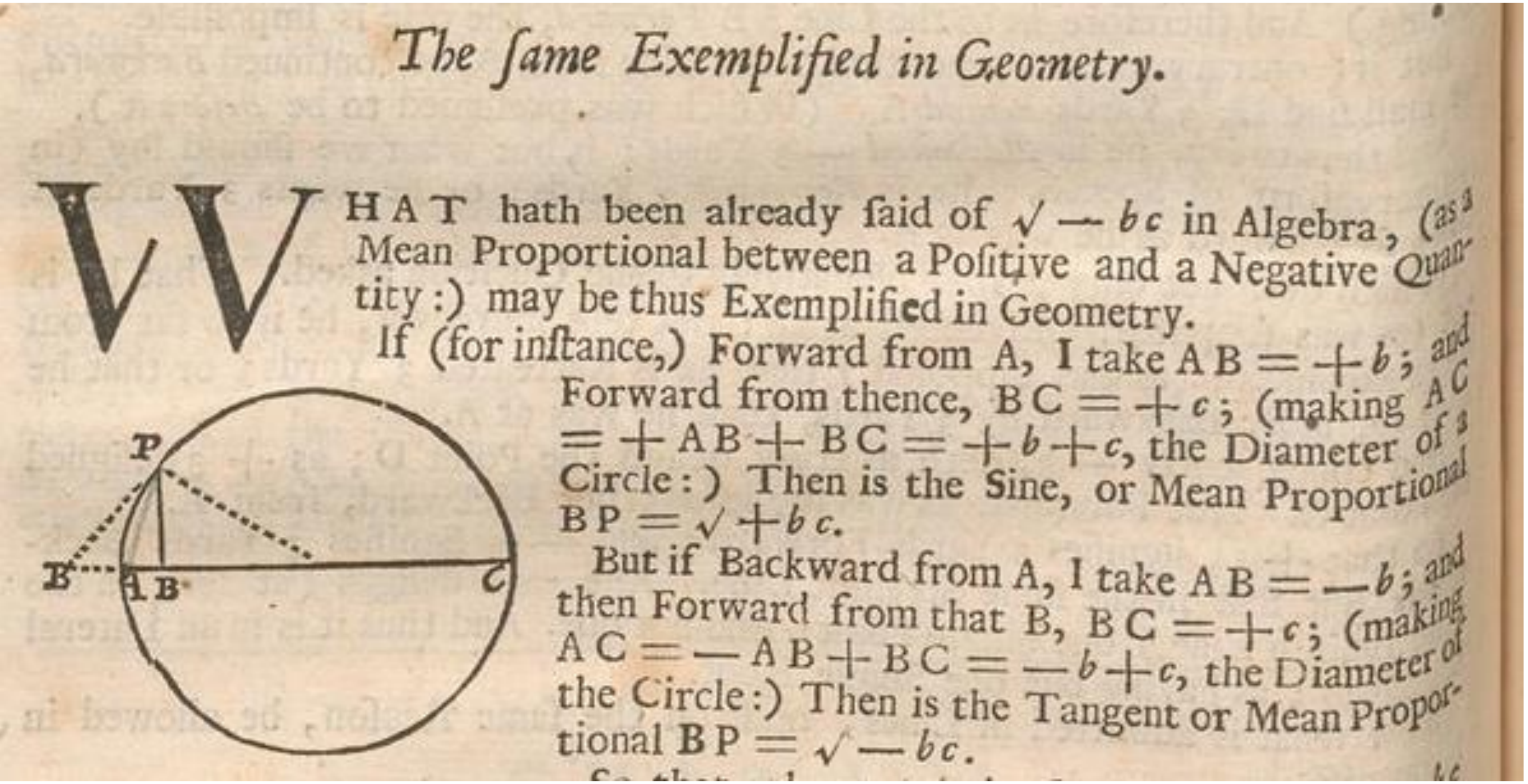
Tous les Problesmes de Geometrie se peuuent facilement reduire a tels termes, qu'il n'est besoin par aprés que de connoistre la longeur de quelques lignes droites, pour les construire.

Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, qui sont l'Addition, la Soustraction, la Multiplication, la Diuision, & l'Extraction des racines, qu'on peut prendre pour vne espece de Diuision : Ainsi n'at'on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connuës, que leur en adiouster d'autres, ou en oster, Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit à l'vne de ces deux, comme l'autre est a l'vnité, ce qui est le mesme que la Multiplication ; oubien en trouuer vne quatriesme, qui soit a l'vne de ces deux, comme l'vnité

Commēt le calcul d'Arithmetique se rapporte aux operations de Geometrie.

1685, **John Wallis,**

an attempt at interpreting complex numbers



The ſame Exemplified in Geometry.

WHAT hath been already ſaid of $\sqrt{-bc}$ in Algebra, (as a Mean Proportional between a Poſitive and a Negative Quantity:) may be thus Exemplified in Geometry.

If (for inſtance,) Forward from A, I take $AB = +b$; and Forward from thence, $BC = +c$; (making $AC = +AB + BC = +b + c$, the Diameter of a Circle:) Then is the Sine, or Mean Proportional $BP = \sqrt{+bc}$.

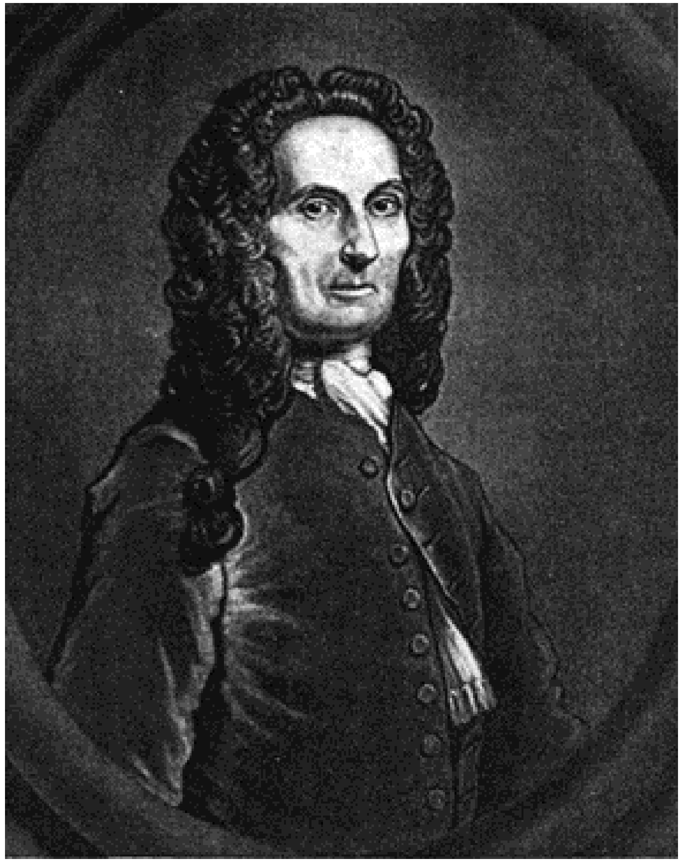
But if Backward from A, I take $AB = -b$; and then Forward from that B, $BC = +c$; (making $AC = -AB + BC = -b + c$, the Diameter of the Circle:) Then is the Tangent or Mean Proportional $BP = \sqrt{-bc}$.

Transition from circle $x^2 + y^2 = 1$

to the hyperbola $x^2 - y^2 = 1$

by replacing y with $y\sqrt{-1}$

1702, **I. Newton, A. Moivre, I. Bernoulli**
researches on the development of methods for integrating rational functions



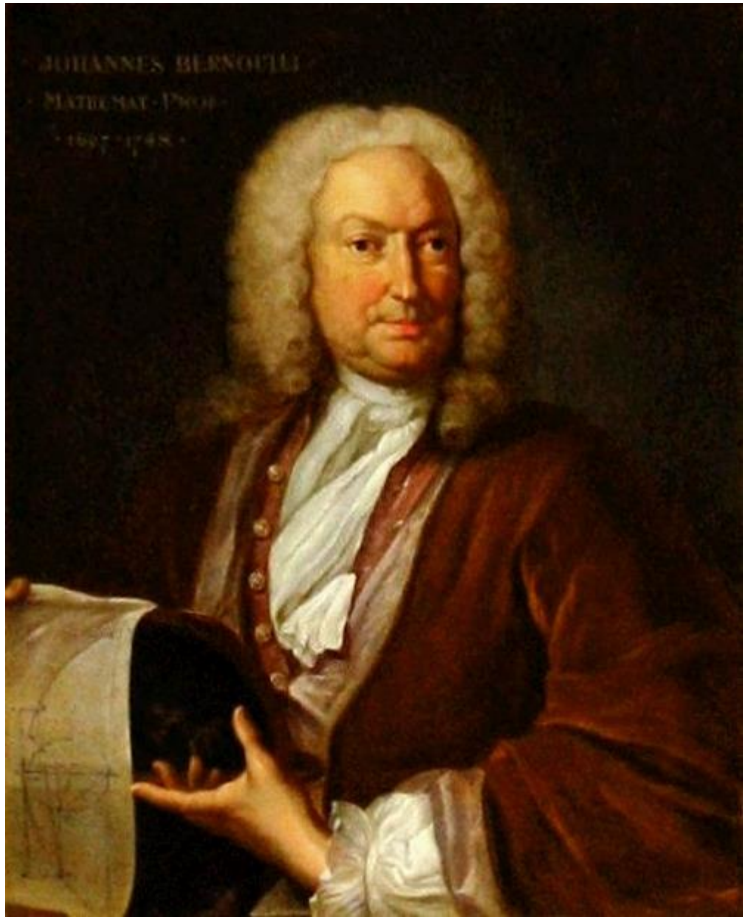
Bernoulli established in 1702 that the differential of the real circular sector $\frac{dz}{1+z^2}$ by substituting $z = \frac{t-1}{t+1}\sqrt{-1}$ is transformed into the differential of the imaginary logarithm.

On the other hand, as Bernoulli noted,

$$\frac{dz}{1+z^2} = \frac{1}{2} \cdot \frac{dz}{1+z\sqrt{-1}} + \frac{1}{2} \cdot \frac{dz}{1-z\sqrt{-1}},$$

it is equal to the sum of differentials of imaginary logarithms.

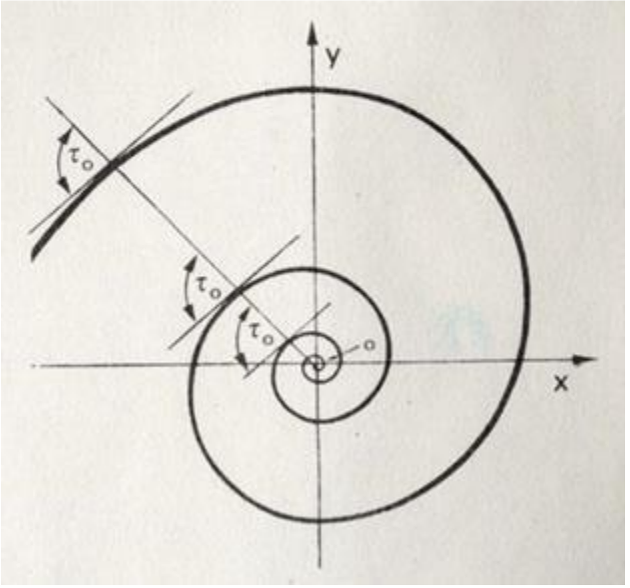
1712,
I. Bernoulli, G. Leibniz, J.-L. d'Alembert
on the meaning of the logarithm of a negative number



By 1712, Bernoulli and Leibniz were arguing in their correspondence about what the logarithm of a negative number is [50]. Leibniz substituted $x = -2$ into the formula for expanding the logarithm into a series and concluded that the logarithm of -1 cannot be zero. For a positive number a, we have $ln\sqrt{a} = \frac{1}{2} lna$. Continuing the reasoning, we can conclude that $\ln i = \ln\sqrt{-1} = \frac{1}{2}\ln(-1)$. But what is $\ln(-1)$ equal to?

Leibniz believed that it should be complex (imaginary), but he did not have a clear definition of this term either. Bernoulli [50, p. 294], and then d'Alembert [27, p. 210-230], believed that the logarithm of a negative number should be real. Bernoulli's arguments were based on integrating the logarithm of (-*x*) as an integral of $dz: z$ taken between the limits of 1 and (-*x*), but, as noted by I.Yu. Timchenko, he performed integration through the pole *z*=0 and obtained $\log(-x) = logx$ [4, p. 212-213]. Euler later proved that the logarithm of a negative number would be complex, adding that the logarithm had many values.

History of the logarithmic spiral

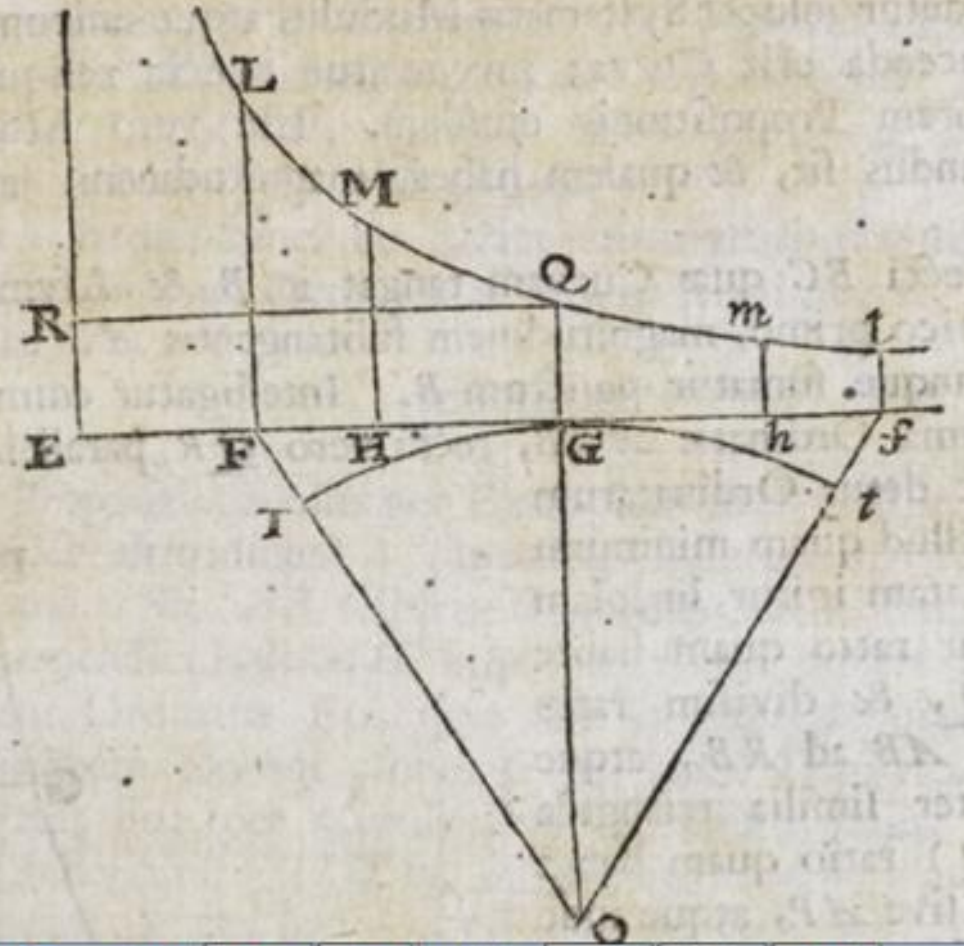


- The logarithmic spiral, known to us from the equation in polar coordinates $\rho = ae^{b\varphi}$, was first described by A. Durer (1525) as a spiral, at each point making a constant angle with the tangent (isogonal spiral). It was studied by R. Descartes (1638), E. Torricelli (1644), I. Newton (1687), Jacob I Bernoulli (1691, 1692), P. Varignon (1704).

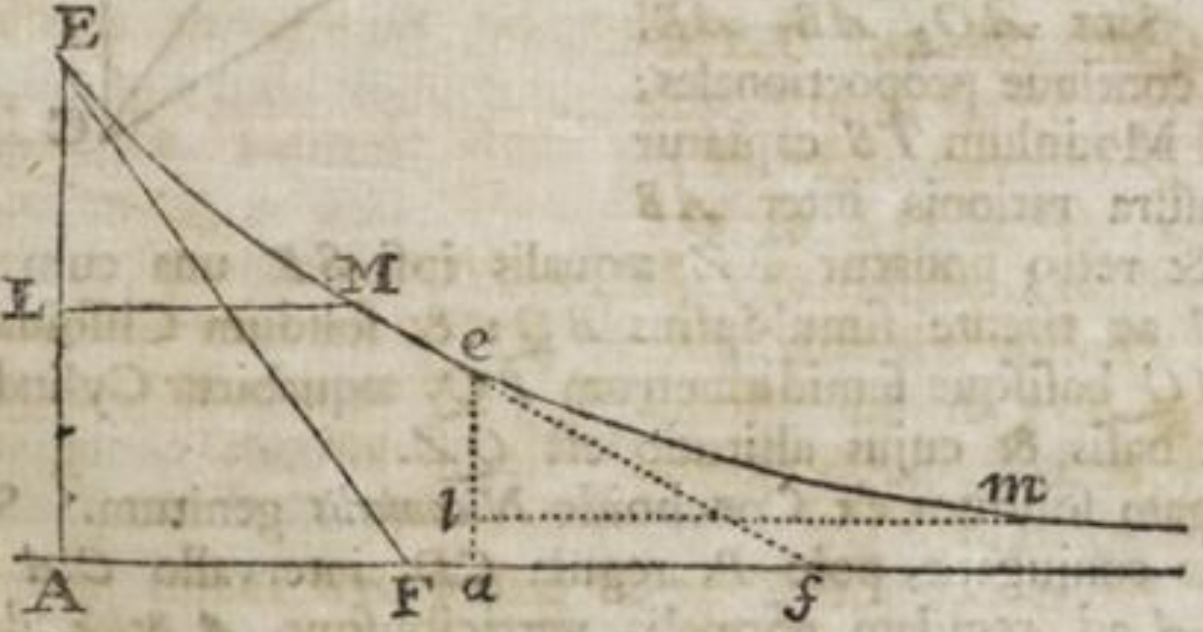
- Descartes found that in a logarithmic (isogonal) spiral, when the angle changes in an arithmetic progression, the radius vector changes exponentially and showed the equivalence to the fact that the polar angles for the points of the curve are proportional to the logarithms of the radius vectors.

- In 1714, the English astronomer and mathematician, student, editor and publisher of Newton, Roger Cotes, based on the study of its properties, developed his own computational method for solving problems of analysis.

ducatur ipsi *ER* parallela *hm* Hyperbolæ occurrens in *m*. De-



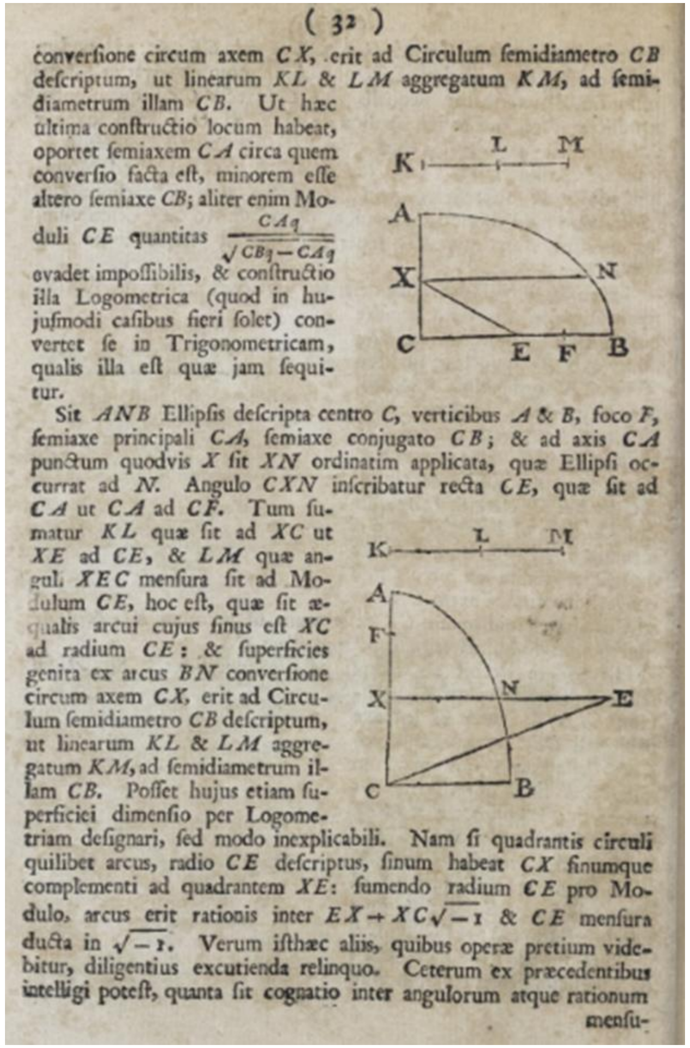
tionem; quam & hoc loco apponam in eorum gratiam qui hujus-



modi contemplationibus delectantur. Oblata ſit igitur Logiſtica *EMem*, cujus Aſymptotos *AFaf*: & quæratur longitudo cujuſvis arcus *Ee*. Demiſſis in Aſymptoton perpendiculis *ELA*, *ela*,

&

1714, **Roger Cotes,** The logarithm of a complex number. Relationship between arc and logarithmic functions



(32)

converſione circum axem CX, erit ad Circulum ſemidiametro CB deſcriptum, ut linearum KL & LM aggregatum KM, ad ſemidiametrum illam CB. Ut hæc ultima conſtructio locum habeat, oportet ſemiaxem CA circa quem converſio facta eſt, minorem eſſe altero ſemiaxe CB; aliter enim Moduli CE quantitas $\frac{CAq}{\sqrt{CBq - CAq}}$ evadet impoſſibilis, & conſtructio illa Logometrica (quod in hujuſmodi caſibus fieri ſolet) convertet ſe in Trigonometricam, qualis illa eſt quæ jam ſequitur.

Sit ANB Ellipſis deſcripta centro C, verticibus A & B, foco F, ſemiaxe principali CA, ſemiaxe conjugato CB; & ad axis CA punctum quodvis X ſit XN ordinatim applicata, quæ Ellipſi occurrat ad N. Angulo CXN inſcribatur recta CE, quæ ſit ad CA ut CA ad CF. Tum ſumatur KL quæ ſit ad XC ut XE ad CE, & LM quæ anguli XEC menſura ſit ad Modulum CE, hoc eſt, quæ ſit æqualis arcui cujus ſinus eſt XC ad radium CE: & ſuperficies genita ex arcus BN converſione circum axem CX, erit ad Circulum ſemidiametro CB deſcriptum, ut linearum KL & LM aggregatum KM, ad ſemidiametrum illam CB. Poſſet hujus etiam ſuperficiei dimenſio per Logometriam deſignari, ſed modo inexplicabili. Nam ſi quadrantis circuli quilibet arcus, radio CE deſcriptus, ſinum habeat CX ſinumque complementi ad quadrantem XE: ſumendo radium CE pro Modulo, arcus erit rationis inter $EX + XC\sqrt{-1}$ & CE menſura ducta in $\sqrt{-1}$. Verum iſthæc aliis, quibus operæ pretium videbitur, diligentius excutienda relinquo. Ceterum ex præcedentibus intelligi poteſt, quanta ſit cognatio inter angulorum atque rationum menſu-

Cotes has the formula

$$\ln(cosx + isinx) = ix$$

expressed in these words:

“**For if some arc of a quadrant of a circle described with radius *CE* has sine *CX* and sine of the complement of the quadrant *XE*, taking radius *CE* as modulus, the arc will be the measure of the ratio between $EX + XC\sqrt{-1}$ and *CE*, the measure having been multiplied $\sqrt{-1}$**”.

Thanks to Cotes, an algebraic formula appeared for the first time in geometry and analysis, connecting logarithmic and circular functions.

Philosophical Transactions
of the Royal Society of London
Volume 29, Issue 338 Mar 1714

LOGOMETRIA

Auctore

ROGERO COTES,

Trin. Coll. Cantab. Soc.

Aſtr. & Ph. Exp. Profeſſore PLUMIANO, & R. S. S.

HARMONIA MENSURARUM,

SIVE

ANALYSIS & SYNTHESIS

Per RATIONUM & ANGULORUM MENSURAS

PROMOTÆ:

ACCEDUNT ALIA

OPUSCULA MATHEMATICA:

PER

ROGERUM COTESIUM.

——— *Si propius stes,*
Te capient magis. ———

EDIDIT ET AUXIT

ROBERTUS SMITH

Collegii S. *Trinitatis* apud *Cantabrigienses* Socius;
Astronomiæ & Experimentalis Philosophiæ
Post COTESIUM Professor.

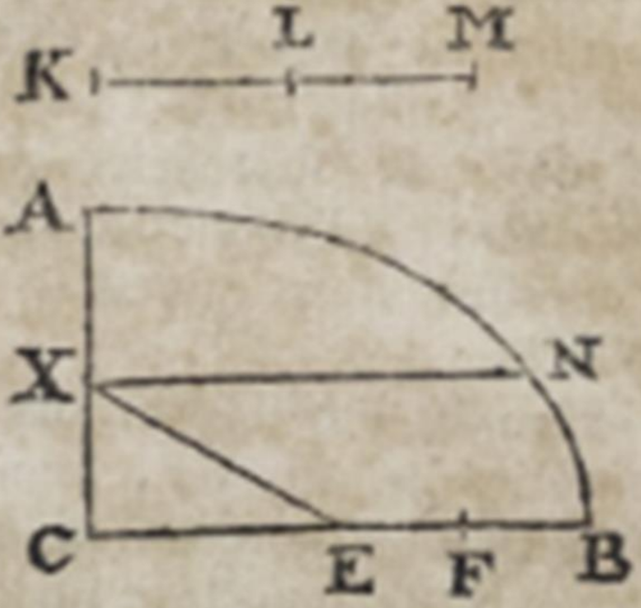
CANTABRIGIÆ, MDCCXXII.

Cotes for the first time in mathematics introduces an angle whose measure is always equal to the radius, i.e. providing such a ratio, whose measure is always equal to the modulus. This modular ratio is 2.71828..., the corresponding angle is 57.295 degrees, which is one radian. It was thanks to Coates that the radian measure of angles appeared, without which the appearance of the Euler identity would have been impossible.

merus Graduum, qui metitur Angulum Modularem modo definitum, hoc eſt, qui continetur in arcu Radio æquali. Eſt autem hic Numerus ad Gradus 180 ut Circuli Radius ad Semicircumferentiam, hoc eſt ut 1 ad 3. 14159265389 &c. Unde Modulus Canonis Trigonometrici prodibit 57. 2957795130 &c. Cujus Reciprocus eſt 0. 0174532925 &c. Hujus moduli ſubſidio (quem in chartula quadam Auctoris manu deſcriptum inveni) commodiſſime computabis menſuras angulares, quemadmodum oſtendam in Nota III.

Calculation of the surface of an oblate and prolate spheroid

converſione circum axem *CX*, erit ad Circulum ſemidiametro *CB* deſcriptum, ut linearum *KL* & *LM* aggregatum *KM*, ad ſemidiametrum illam *CB*. Ut hæc ultima conſtructio locum habeat, oportet ſemiaxem *CA* circa quem converſio facta eſt, minorem eſſe altero ſemiaxe *CB*; aliter enim Moduli *CE* quantitas $\frac{CAq}{\sqrt{CBq - CAq}}$ evadet impoſſibilis, & conſtructio illa Logometrica (quod in hujuſmodi caſibus fieri ſolet) convertet ſe in Trigonometricam, qualis illa eſt quæ jam ſequitur.

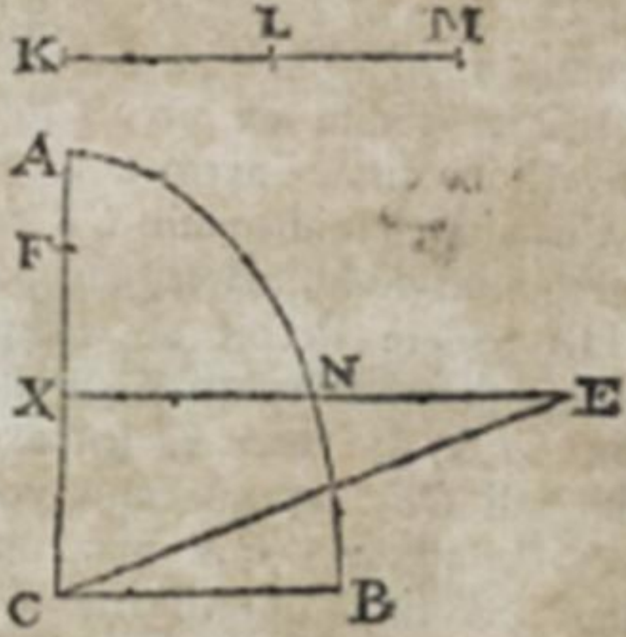


Cotes R. Logometria

Philosophical Transactions of the Royal Society of London. 1714, 29 (338). P. 5-45. – стр. 32.

“For if some arc of a quadrant of a circle described with radius *CE* has sine *CX* and sine of the complement of the quadrant *XE*, taking radius *CE* as modulus, the arc will be the measure of the ratio between $\boldsymbol{EX + XC\sqrt{-1}}$ and *CE*, the measure having been multiplied $\boldsymbol{\sqrt{-1}}$”.

Sit *ANB* Ellipſis deſcripta centro *C*, verticibus *A* & *B*, foco *F*, ſemiaxe principali *CA*, ſemiaxe conjugato *CB*; & ad axis *CA* punctum quodvis *X* ſit *XN* ordinatim applicata, quæ Ellipſi occurrat ad *N*. Angulo *CXN* inſcribatur recta *CE*, quæ ſit ad *CA* ut *CA* ad *CF*. Tum ſumatur *KL* quæ ſit ad *XC* ut *XE* ad *CE*, & *LM* quæ anguli *XEC* menſura ſit ad Modulum *CE*, hoc eſt, quæ ſit æqualis arcui cujus ſinus eſt *XC* ad radium *CE*: & ſuperficies genita ex arcus *BN* converſione circum axem *CX*, erit ad Circulum ſemidiametro *CB* deſcriptum, ut linearum *KL* & *LM* aggregatum *KM*, ad ſemidiametrum illam *CB*. Poſſet hujus etiam ſuperficiei dimenſio per Logometriam deſignari, ſed modo inexplicabili. Nam ſi quadrantis circuli quilibet arcus, radio *CE* deſcriptus, ſinum habeat *CX* ſinumque complementi ad quadrantem *XE*: ſumendo radium *CE* pro Modulo, arcus erit rationis inter $EX + XC\sqrt{-1}$ & *CE* menſura ducta in $\sqrt{-1}$. Verum iſthæc aliis, quibus operæ pretium videbitur, diligentius excutienda relinquo. Ceterum ex præcedentibus intelligi poteſt, quanta ſit cognatio inter angulorum atque rationum menſu-



The explanation for the last sentence is as follows.

CE is the radius of the circle,

угол XEC=θ, $\frac{CX}{CE} = sin\theta$; $\frac{EX}{CE} = cos\theta$;

$$\frac{EX}{CE} + \frac{CX}{CE}\sqrt{-1} = cos\theta + sin\theta\sqrt{-1}.$$

The logarithm is the measure of a given ratio,

i.e. $\ln(\frac{EX}{CE} + \frac{CX}{CE}\sqrt{-1}) = \ln(cos\theta + sin\theta\sqrt{-1})$.

Arc corresponding to angle XEC and radius CE,

measured in radians is equal to

$\theta\sqrt{-1}$.

So $\ln\left(cos\theta + sin\theta\sqrt{-1}\right) = \theta\sqrt{-1}$.

Cotes, having discovered, thanks to his method, the connection between circular and logarithmic canons (functions), called this connection the wonderful harmony of nature:

“Moreover, from the foregoing can be understood the extent of the relationship between the measures of angles and of ratios, further, by simple exchange among themselves, they are easily converted for different cases of the same problem”.

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L. Euler. Elements of complex analysis

Since 1730, Leonhard Euler developed the foundations of the theory of elementary functions of a complex variable. In 1734-1735. Euler obtained the condition

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

In 1743, Euler created a method for solving linear differential equations of higher orders, in which imaginary numbers arise when solving characteristic algebraic equations, and the general solution of the equation is real.

Euler used more convenient notation, namely, the letter "*e*" as the base of the natural logarithm from 1728, the letter π as the ratio of the circumference to the diameter from 1736, the letter *i* as the symbol for the imaginary unit from 1777, and gives it the modern definition

$$i^2 = -1, \frac{1}{i} = -i$$

1743, Euler formulas for sine (left) and cosine (right)

Hinc jam a priori assignare possum omnes radices seu factores hujus expressionis infinitæ

$$s - \frac{s^3}{1.2.3.} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.\ldots.7} + \frac{s^9}{1.2.3..9} - \&c.$$

Hæc enim expressio æquivalet isti $\frac{e^{s\sqrt{-1}} - e^{-s\sqrt{-1}}}{2\sqrt{-1}}$ denotante e numerum, cujus logarithmus est $=$ 1. & cum sit $e^z = \left(1 + \frac{z}{n}\right)^n$ existente n numero infinito, reducetur expressio infinita proposita ad hanc: $\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n - \left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2\sqrt{-1}}$: cujus primum factor simplex est s, quem quidem ipsa seriei inspectio monstrat. Ad reliquos factores eruendos compa-

§. XI.

Simili modo si consideremus hanc seriem:

$$1 - \frac{ss}{1.2} + \frac{s^4}{1.2.3.4} - \frac{s^6}{1.2.3.\ldots.6} + \frac{s^8}{1.2.3\ldots.8} - \&c.$$

ea reducetur ad hanc formam $\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n + \left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2}$ denotante n numerum infinitum. Divisores ergo binomii $\left(1+\frac{s\sqrt{-1}}{n}\right)^n + \left(1-\frac{s\sqrt{-1}}{n}\right)^n$ simul erunt divisores seriei propositæ, & quidem omnes. Comparata hac forma cum a^n

1749, Euler on the logarithms of negative and imaginary numbers

§ 29. Soit maintenant l'arc proposé φ de 180°, ou soit $\varphi = \pi$, et nous aurons $\sin\varphi = 0$ et $\cos\varphi = -1$. Cette supposition faite, l'équation générale trouvée se changera en cette forme

$$l(-1) = (\pi \pm 2n\pi)\sqrt{-1} \; (= 1 \pm 2n)\,\pi\sqrt{-1},$$

d'où nous tirons toute l'infinité des logarithmes du nombre négatif -1, car nous aurons

$$l(-1) = \pm\pi\sqrt{-1},\quad \pm 3\pi\sqrt{-1},\quad \pm 5\pi\sqrt{-1},\quad \pm 7\pi\sqrt{-1},\quad \text{etc.}$$

et de là nous voyons clairement, que tous les logarithmes de -1 sont imaginaires et tous différents des logarithmes de $+1$. Cela non obstant, les logarithmes de $(-1)^2$ qui seront

$$\pm 2\pi\sqrt{-1},\quad \pm 6\pi\sqrt{-1},\quad \pm 10\pi\sqrt{-1},\quad \text{etc.}$$

sont visiblement contenus dans les logarithmes de $+1$; ce qui suffit pour sauver les contradictions apparentes dont j'ai fait mention là-haut, quoiqu'il n'en suive pas réciproquement que les moitiés de tous les logarithmes de $+1$ soient logarithmes de -1: ce que la nature même des quantités ne permet pas, puisque -1 n'est pas la seule racine carrée de $+1$.

Euler (1747), after studying the work of Coates, found an easier way: he compared the antiderivatives of the two integrals $\int \frac{dx}{\sqrt{1-x^2}}$ and $\int \frac{dx}{\sqrt{1+x^2}}$ and noticed that one integral from the other one is obtained with the help of an imaginary substitution, after which the antiderivatives are compared.

Euler considers arcs on a unit circle, their sines and cosines, taking into account periodicity $\pm 2\pi n + \varphi$ for argument values $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ и т.д. It stands for $x = sin\varphi,\ y = cos\varphi.\ y = \sqrt{1 - x^2}$. The letter l denotes the natural logarithm. Because $d\varphi = \frac{dx}{y} = \frac{dx}{\sqrt{1-x^2}}$, Euler introduces the notation $x = z\sqrt{-1}$, so $d\varphi = \frac{dz\sqrt{-1}}{\sqrt{1+z^2}}$. Consequently, $\int \frac{dz\sqrt{-1}}{\sqrt{1+z^2}} = l(\sqrt{1 + z^2} + z) + C$. Respectively, $\bar{\varphi} = \sqrt{-1}\, l(\sqrt{1 - x^2} + \frac{x}{\sqrt{-1}}) + C$. Evidently, writes Euler, the constant $C = 0$.

Then $\bar{\varphi} = \frac{1}{\sqrt{-1}} l(\sqrt{1 - x^2} - x\sqrt{-1})$, so $\varphi = \frac{1}{\sqrt{-1}} l(\sqrt{1 - x^2} + x\sqrt{-1})$,

or $\varphi = \frac{1}{\sqrt{-1}} l(y + x\sqrt{-1})$

1750 г., Giulio Carlo de' Toschi di Fagnano

DIvidendo per $(1-t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}}$ il numeratore, e il denominatore di quella frazione, il logaritmo della quale fa il ſecondo membro dell' equazione (6), notata nella I. ſoluzione del II. problema, e dividendo per $(1+t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}}$ il numeratore, e il denominatore di quella frazione, il di cui logaritmo fa il ſecondo membro dell' equazione (7), ſi à ſempre

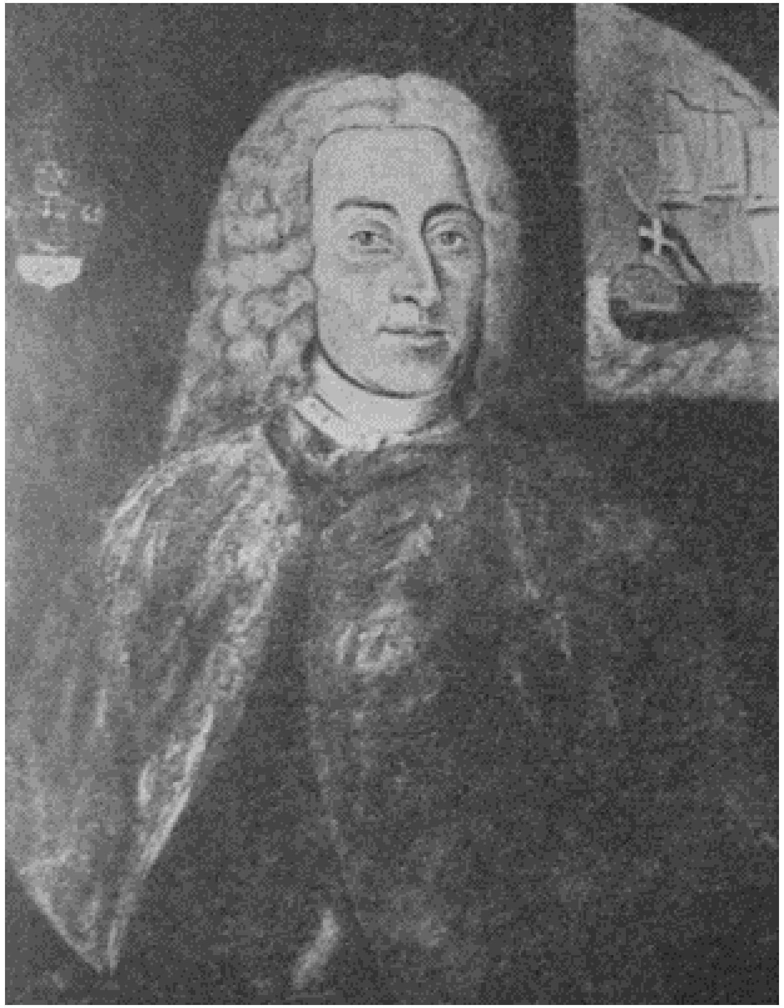
$$S.\frac{dt}{1+tt} = \log.(1-t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1+t\sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}$$

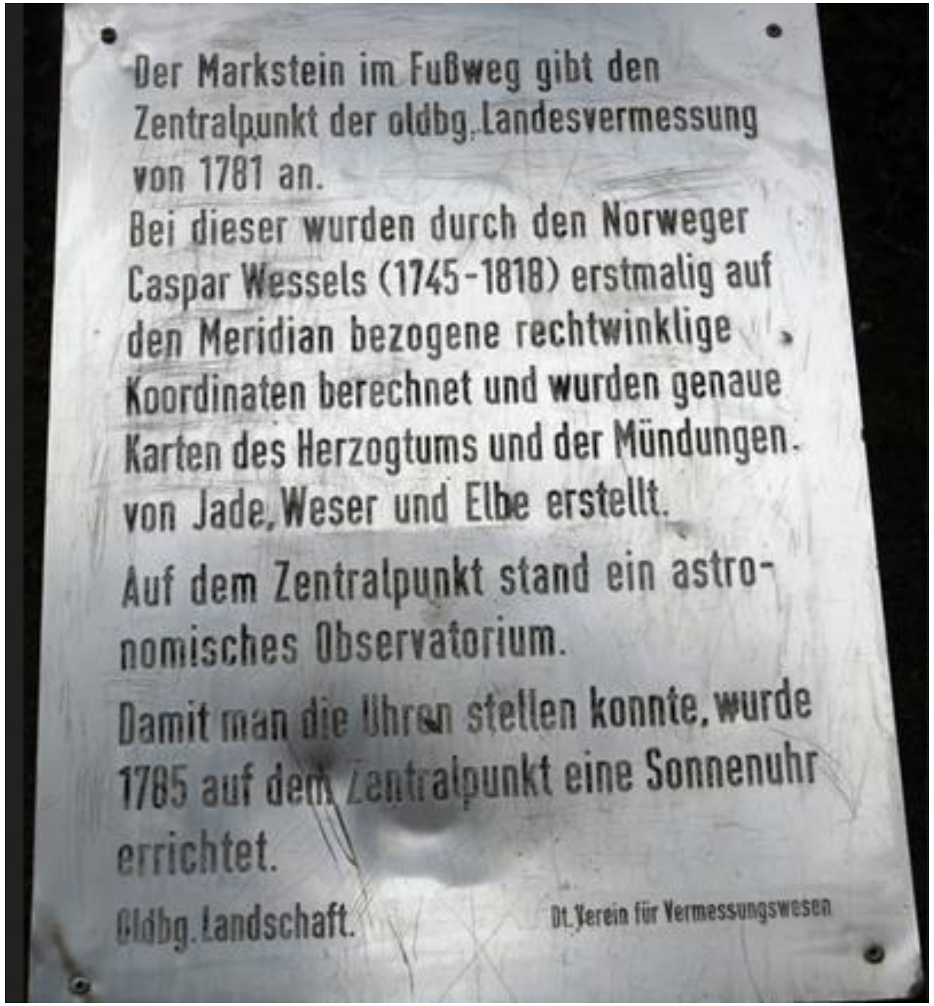
e in virtù del corollario precedente, ſi à ancora

$$(8)\quad S.\frac{dt}{1+tt} = \log.(AA-BB)$$

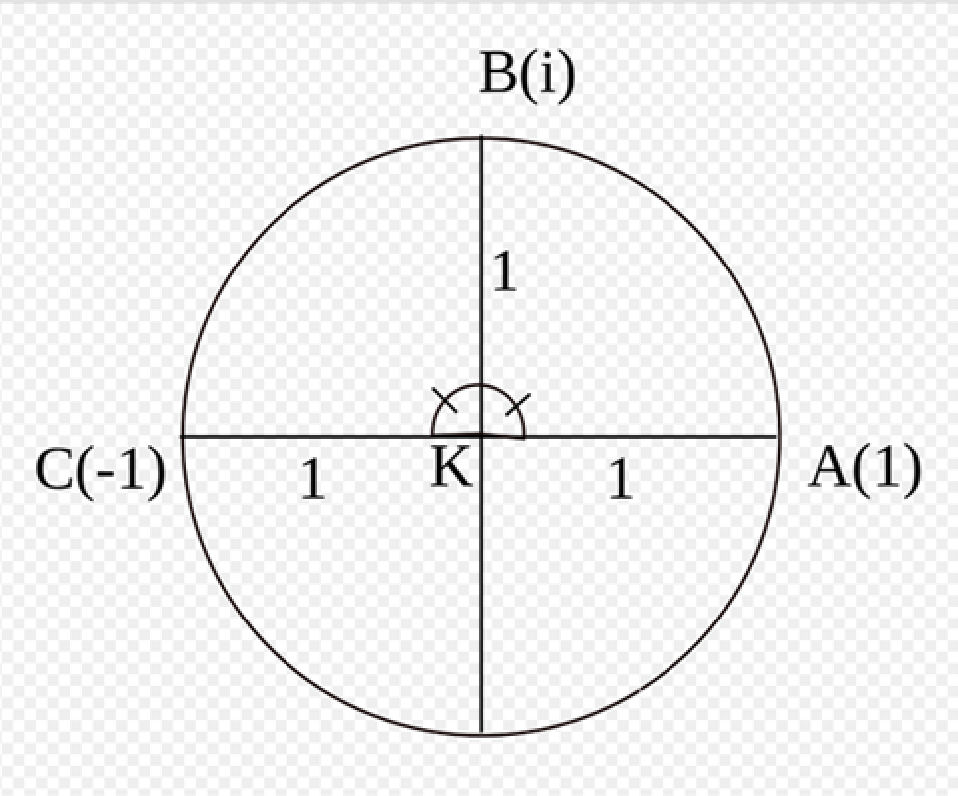
Queſte due ultime equazioni manifeſtano una nuova, e belliſſima proprietà del cerchio, ciaſcun arco del di cui quadrante à per ſuo elemento $\frac{dt}{1+tt}$, quando la t denota la tangente dell' arco medeſimo.

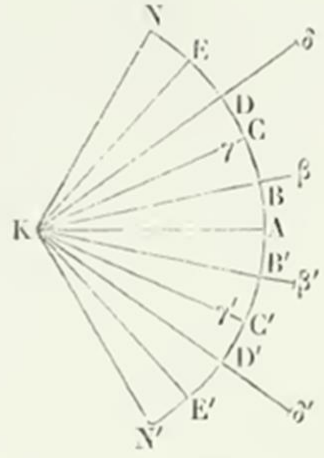
1797 (1799), Caspar Wessel
Complex number as a directed segment. Geometric interpretation of operations





1806, 1813/14, Argand.
Geometric diagrams





§ 1. Si AB, BC, . . . , EN (*fig.* 8) sont des arcs égaux au nombre de n, et qu'on fasse

$$\overline{KB} = u,$$

on aura

$$\overline{KC} = u^2,$$

$$\overline{KD} = u^3,$$

$$\ldots\ldots\ldots,$$

$$\overline{KN} = u^n.$$

1813, Jacques Frédéric Français

First appearance of the famous formula

Démonstration. La quantité $\pm a\sqrt{-1}$ est une moyenne proportionnelle, de grandeur et de position, entre $+a$ et $-a$, c'est-à-dire, entre a_0 et $a_{\pm\pi}$; donc, d'après le corollaire 1.er de la définition 3.e, la valeur de cette moyenne proportionnelle, de grandeur et de position, est $a_{\pm\frac{\pi}{2}}$; c'est-à-dire, qu'elle est perpendiculaire à l'axe des abscisses, et dirigée soit en dessus soit en dessous de cet axe; et l'on a $+a\sqrt{-1}=a_{+\frac{\pi}{2}}$, et $-a\sqrt{-1}=a_{-\frac{\pi}{2}}$. Réciproquement, toute perpendiculaire à l'axe des abscisses est représentée, d'après nos notations, par $a_{\pm\frac{\pi}{2}}$: elle est, par conséquent, d'après le corollaire 1.er de la définition 3, une moyenne proportionnelle entre a_0 et $a_{\pm\pi}$, ou entre $+a$ et $-a$: elle est donc une quantité imaginaire de la forme $\pm a\sqrt{-1}$.

Corollaire 1.er. Il suit de là que $\pm\sqrt{-1}$ est un signe de position qui est identique avec $1_{\pm\frac{\pi}{2}}$.

Corollaire 2. De plus, puisqu'on a $-1=1_{\pm\pi}=e^{\pm\pi\sqrt{-1}}$, on a aussi $\pm\sqrt{-1}=1_{\pm\frac{\pi}{2}}=e^{\pm\frac{\pi}{2}\sqrt{-1}}$.

Corollaire 3. Les quantités dites *imaginaires* sont donc tout aussi réelles que les quantités positives et les quantités négatives, et n'en diffèrent que par leur position qui est perpendiculaire à celle de ces dernières.

Remarque générale. Cette théorie des signes de position est une con-

$$\boldsymbol{+1 = e^{0\pi\sqrt{-1}}}$$

$$\boldsymbol{-1 = e^{\pm\pi\sqrt{-1}}}$$

Français, J. F. Nouveaux principes de géométrie de position, et interprétation géométrique des symboles imaginaires. *Annales de Mathématiques pures et appliquées*, tome 4 (1813-1814), p. 61-71
- p. 64

Corollaire 2. De plus, puisqu'on a $-1 = 1_{\pm\pi} = e^{\pm\pi\sqrt{-1}}$, on a aussi $\pm\sqrt{-1} = 1_{\pm\frac{\pi}{2}} = e^{\pm\frac{\pi}{2}\sqrt{-1}}$.

1821, Augustin Louis Cauchy

Analyse algébrique

$$(19)\qquad l((\alpha+\beta\sqrt{-1}))=l(\rho)+\zeta\sqrt{-1}+\pi\sqrt{-1}+l((1)).$$

Si dans cette dernière équation on fait en particulier $\alpha+\beta\sqrt{-1}=-1$, c'est-à-dire, $\alpha=-1$, $\beta=0$, et par suite $\rho=1$, $\zeta=0$, on obtiendra la suivante

$$(20)\qquad l((-1))=\pi\sqrt{-1}+l((1)).$$

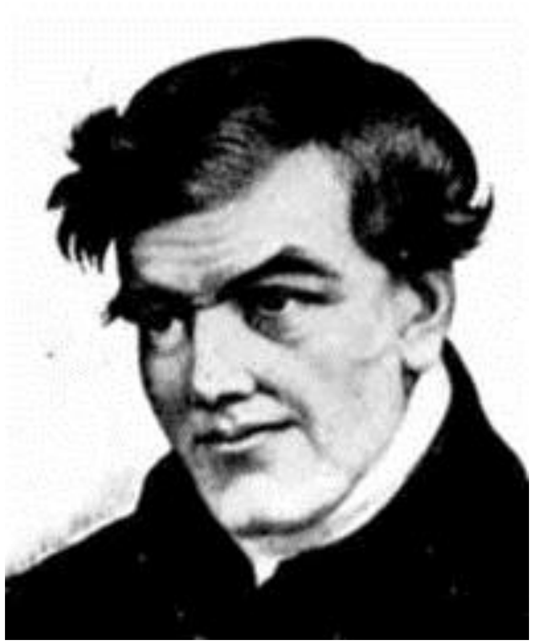
Il en résulte qu'on aura généralement, pour des valeurs négatives de α,

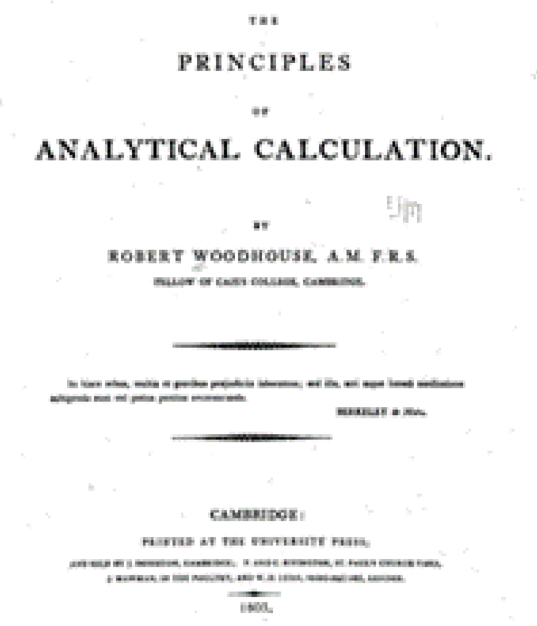
$$(21)\qquad l((\alpha+\beta\sqrt{-1}))=l(\rho)+\zeta\sqrt{-1}+l((-1)).$$

First third of the 19th century, Britain.
An attempt at logical interpretation

- From the beginning of the 19th century, mathematicians in England paid attention to the elimination of differences in English and continental mathematical terminology, dating back to the works of Newton and Leibniz, as well as to the justification of algebraic operations.
- In terms of mathematical preparation, England lagged noticeably behind European countries. Young British professors advocated reforming the outdated education system. At that time, the best textbooks and scientific studies were in French.
- Young mathematicians of Trinity College (Cambridge) J. Peacock, C. Babbage, J. Herschel and R. Woodhouse by 1812-1815 founded the Analytical Society for their own research aimed at the problems of substantiating the sciences taught, primarily algebra. At the same time, they made an attempt to give the analysis an algebraic form.

J. Peacock, C. Babbage, J. Herschel and R. Woodhouse



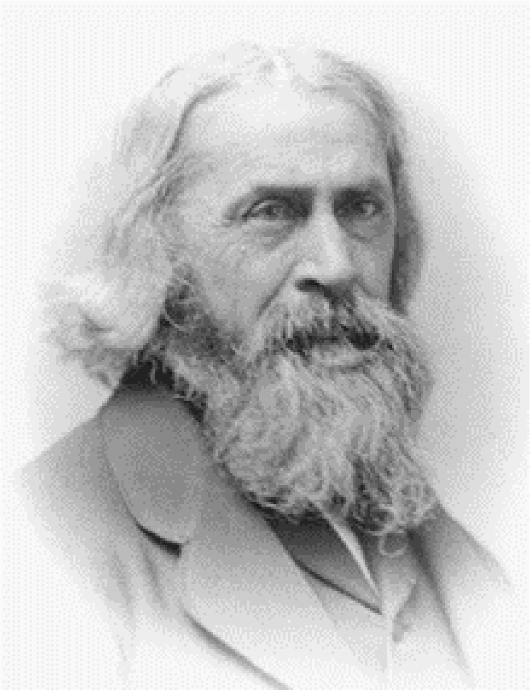


Now the question concerning the logarithms of negative quantities, in a precise form, and freed from its verbal ambiguities, is this; is the symbol which, substituted for x in the developement of e^x, makes y or $e^x = -1$, the sign of a real quantity or not?

In the expression e^x, x is the logarithm of e^x, and, by extension, $x\sqrt{-1}$ is to be called the logarithm of $e^{x\sqrt{-1}}$. Now $e^{x\sqrt{-1}}$ is the symbol for $1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3}$ &c. or $\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \text{\&c.}\right) + \sqrt{-1}\left(x - \frac{x^3}{1.2.3} + \text{\&c.}\right)$ or $e^{x\sqrt{-1}}$ may be said to be $= \cos. x + \sqrt{-1} \sin. x$. Hence, $x\sqrt{-1}$ is the logarithm of $\cos. x + \sqrt{-1} \sin. x$; when the arc x is equal 0, or 2π, or 4π, or 6π, or generally $2m\pi$, its $\cos. x = 1$.

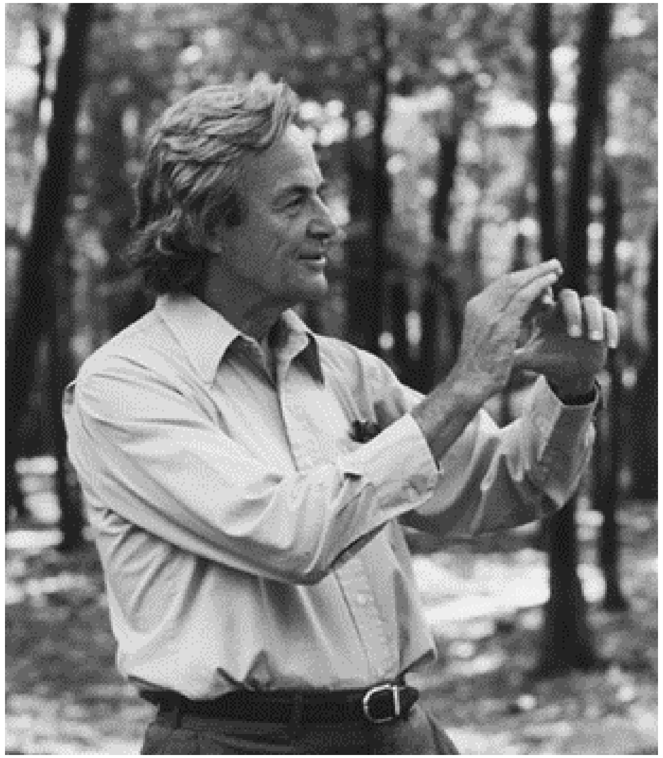
Mid 19th century, USA. B. Peirce

$$e^{\frac{\pi}{2}} = \sqrt[i]{i}$$



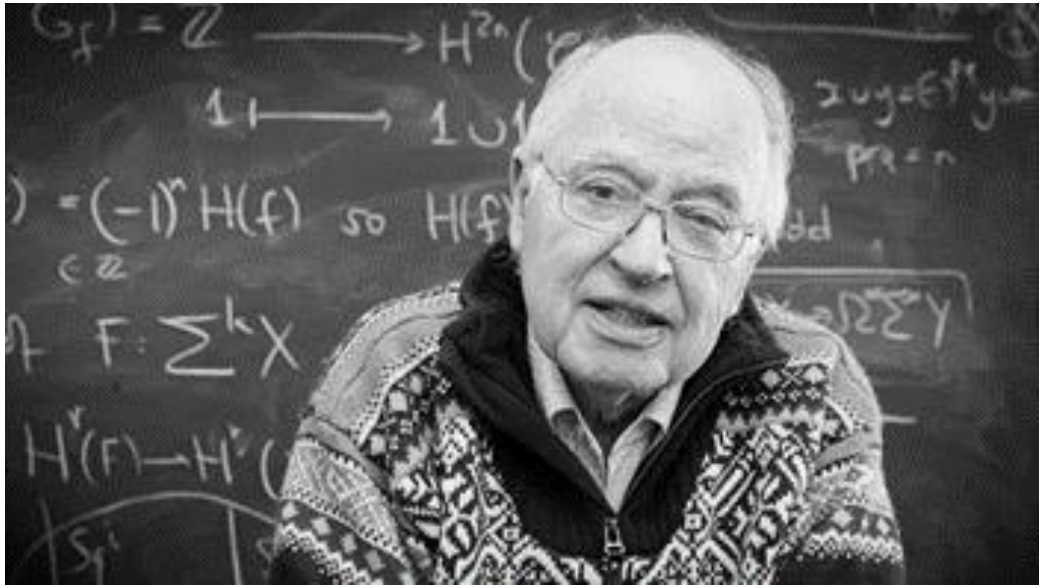
"Gentlemen, that is surely true, it is absolutely paradoxical, we can't understand it, and we don't know what it means, but we have provide it, and therefore we know it must be the truth."

1933, Richard Feynman



Feynman called Euler's equation "the most remarkable result in mathematics… To those who do not know mathematics it is difficult to get agross a real feeling as to the beauty, the deepest beauty of nature… If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speacs in”.

Michael Francis Atiyah



Michael Atiyah called this formula "...the mathematical analogue of Hamlet's phrase - "to be or not to be" - very short, very concise, and at the same time very deep."

The most beautiful math formula

In 1988, mathematician David Wells conducted a survey among readers of the mathematical magazine *The Mathematical Intelligencer*, asking them to choose the most beautiful theorem out of 24. In 1990, he summed up the results of the poll in the article Most Readers Chose Euler's Identity.

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Thank you for attention