p-adic Mathematical Physics and V.S.Vladimirov Works

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Steklov Mathematical Institute

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(Vladimirov-100)

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Plane

- **Principles of p-adic mathematical physics**
- **V.S.Vladimirov works**
  - **spectral theory in p-adic quantum mechanics**
  - **Vladimirov's operator of fractional differential**
  - **higher derivatives**
- **Quantum zeta-function and motives**

Principles of p-adic mathematical physics. Rational numbers

I.V. *p-adic string*, Class.Quant. Gravity, 4, 1987, L83

- **Only rational numbers can be observed.**
- **Irrational numbers (infinite decimals) are not observed.**
- **On the field of rational numbers $\mathbb{Q}$ there are two norms: ordinary real and p-adic (Ostrowsky theorem)**
- **One cannot measure distances less the Planck length,**

$$\Delta x > \ell_{Planck},$$

because of the black hole creation.

- **Non-Archimedean geometry appears**

Number theory as an ultimate physical theory

- **In ultimate physical theory on can put the principle**

  according to which fundamental physical laws

  should be invariant with respect to change of number field
  (I.V. 1987).

- **It is in some sense a further generalization of the Einstein principle of general relativity.**

V.S.Anashin -automata
V.V.Zharinov -talk on functional mechanics

p-adic numbers

$\mathbb{Q}$ **field of rational numbers**

$x = p^{\nu}\frac{m}{n} \in \mathbb{Q}$

p-adic norm: $|x|_p = \frac{1}{p^{\nu}}$

Non-Archimedean norm: $|x+y|_p \leq \mathbf{max}(|x|_p, |y|_p)$

$\mathbb{Q}_p$ **field of p-adic numbers is a completion of** $\mathbb{Q}$ **with respect to p-adic norm**

p-adic numbers $x = p^{\nu}\sum_{n=0}^{\infty} a_n p^n$

Vladimirov fractional differential operator

$$D^\alpha \psi(x) = \int_{\mathbb{Q}_p} |\xi|_p^\alpha \chi(-\xi x) \tilde{\psi}(\xi) d\xi, \qquad \alpha > 0$$

$$D^\alpha \psi(x) = \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{\psi(x) - \psi(y)}{|x-y|_p^{\alpha+1}} dy, \qquad \alpha > 0$$

$$D^\alpha \chi(\xi x) = |\xi|_p^\alpha \, \chi(\xi x)$$

- D^α **unbounded self-adjoint operator in** $L^2(\mathbb{Q}_p)$
- **Spectrum:** $p^{\alpha n}$, $n \in \mathbb{Z}$ and 0

Spectral theory of the Vladimirov operator

Theorem. *Every function* $f \in L^2(\mathbb{Q}_p)$ *is expanded in the Fourier-series in eigen-functions* $\left\{\varphi^l_{N,k,\epsilon_1}\right\}$ *of the operator* D^α :

$$f(x) = \sum_{N\in\mathbb{Z}} \sum_{1\le l<\infty} \sum_{1\le k\le p-1} \sum_{\varepsilon_l} f^l_{N,k,\varepsilon_l} \varphi^l_{N,k,\varepsilon_l}(x) \qquad (*)$$

where

$$f^l_{N,k,\varepsilon_l} = \int_{\mathbb{Q}_p} f(x)\bar{\varphi}^l_{N,k,\epsilon_l}(x)dx.$$

The series (*) converges in $L^2(\mathbb{Q}_p)$, and the Parseval-Steklov equality is valid

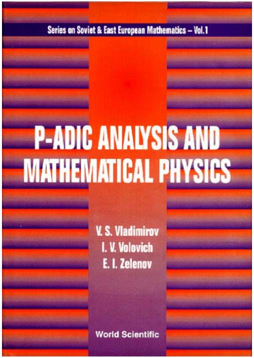
$$\|f\|^2 = \sum_{N\in Z} \sum_{1\le l<\infty} \sum_{1\le k\le p-1} \sum_{\varepsilon} \left|f^l_{N,k,\varepsilon_l}\right|^2 .$$

Spectral theory of the Vladimirov operator

I kind $\left(l = 2, 3, \ldots, k = 1, 2, \ldots, p-1, \varepsilon_l = \varepsilon_0 + \varepsilon_1 p + \ldots + \varepsilon_{l-2} p^{l-2}\right)$

$$\varphi^l_{N,k,\varepsilon_l}(x) = p^{\frac{N+1-l}{2}} \delta\left(|x|_p - p^{l-N}\right) \cdot \delta\left(x_0 - k\right) \chi_p\left(\varepsilon_l p^{l-2N} x^2\right)$$

Various Formulations of p-adic Quantum Mechanics



Vladimirov, I.V. *"p-adic quantum mechanics*, Comm. Math. Phys., 123 (1989), 659-676,
300 citations

Vladimirov, I. V., Zelenov, *p-adic analysis and mathematical physics*, 1994 - World Scientific
1500 citations

- 4 numbers fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$
- p-adic quantum oscillator $(L_2(\mathbb{Q}_p), U_t, W(z))$
- Unitary representations, Vladimirov operator, Schrodinger type equation ...
- Yu.I.Manin - adelic space-time
- B.Dragovich - adelic quantum mechanics

Spectral theory in p-adic quantum mechanics and representation theory

- $(\mathcal{H}, U_t), \quad t \in \mathbb{R}, \quad (L^2(X, \mu), V_t), \quad V_t\phi(x) = e^{it\omega_x}\phi(x)$, $\phi \in L^2$, $\omega_x \in \mathbb{R}$

- $(\mathcal{H}, U_t), \quad t \in \mathbb{Q}_p, \quad (L^2(X, \mu), V_t), \quad V_t\phi(x) = \chi(\omega_x t)\phi(x)$, $\phi \in L^2$, $\omega_x \in \mathbb{Q}_p$

- p-adic classical oscillator

$$\partial_t \chi(\omega t) = i|\omega|_p \chi(\omega t)$$

$$\partial_t^\alpha = e^{i\pi\alpha/2} D_t^\alpha$$

- Generalization of the Jacobi theorem V.Zh. Sakbaev talk

Integrability of real and p-adic quantum dynamics

I.V. *"Remarks on the complete integrability of quantum and classical dynamical systems,"* P-Adic Numbers Ultrametric Anal. Appl., 11:4 (2019), 328-334, arXiv:1911.01335 .

- **Any real or p-adic quantum dynamical system is completely integrable.**
- **Moreover, it is unitary equivalent to a set of classical noninteracting harmonic oscillators**
- **Any classical dynamical system with an invariant measure is also completely integrable**
- **Explicit integrability wide classes of classical and quantum multidimensional PDE by using wave operators.**

Quantum dynamical systems

- **Quantum dynamical system:** $(\mathcal{H},\ H)$ **or** $(\mathcal{H},\ U_t)$**,**

$$U_t = e^{i\,t\,H}$$

- **Schrödinger equation**

$$i\frac{\partial\psi}{\partial t} = H\psi$$

- $(\mathcal{H},\ U_t)$**,** $t \in \mathbb{Q}_p$**, or locally compact commutative group** G**.**
- **Systems of harmonic oscillators**

$$(L^2(X,\mu), V_t), \qquad V_t\phi(x) = e^{-it\omega_x}\phi(x)$$

Integrability. Theorem

- **Theorem. Let $\mathcal{H}$ be a separable Hilbert space and H a self-adjoint operator with a dense domain $D(H) \subset \mathcal{H}$. Then the Schrödinger equation**

$$i \frac{\partial}{\partial t} \psi(t) = H\psi(t), \ \ \psi(t) \in D(H), \ t \in \mathbb{R}$$

completely integrable in the sense that this equation is unitary equivalent to the complexified system of equations for a family of classical non-interacting harmonic oscillators.

Equivalently, quantum dynamical systems $(\mathcal{H}, \ U_t)$ is unitary equivalent to a system of harmonics oscillators $(L^2(X, \mu), V_t)$.

There exists a set of non-trivial integrals of motion for the arbitrary Schrödinger equation.

Integrability of Quantum Dynamical Systems on $\mathbb{Q}_p$ and Locally Compact Commutative Groups

- $(\mathcal{H},\ U_t)$, $t \in \mathbb{Q}_p$ or G.

Quantization of the Riemann Zeta-Function and Cosmology

I.Aref'eva, I.V. Int. J. Geom. Meth. Mod. Phys.,4, 881-895 (2007), hep-th/0701284

- **Quantization of the Riemann zeta-function is proposed.**
- **We treated the Riemann zeta-function as a symbol of a pseudodifferential operator and studied the corresponding classical and quantum field theories.**
- **We show that the Lagrangian for the zeta-function field is equivalent to the sum of the Klein-Gordon Lagrangians with masses defined by the zeros of the Riemann zeta-function**

Quantization of the Riemann Zeta-Function. Main Definitions

The Riemann zeta-function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + i\tau, \ \sigma > 1$$

One introduces the Riemann ξ-function

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

The Hadamard representation for the ξ-function

$$\xi(s) = \frac{1}{2} e^{as} \prod_{\rho} (1 - \frac{s}{\rho}) e^{s/\rho}$$

Here ρ are nontrivial zeros of the zeta-function and

$$a = -\frac{1}{2}\gamma - 1 + \frac{1}{2} \log 4\pi \tag{1}$$

where γ is Euler's constant.

Quantization of the Riemann Zeta-Function. Pseudodifferential operator $F(\Box)$

If $F(\tau)$ is a function of a real variable τ then we define a pseudodifferential operator $F(\Box)$ by using the Fourier transform

$$F(\Box)\phi(x) = \int e^{ixk} F(k^2)\tilde{\phi}(k)dk\,.$$

Here $\Box$ is the d'Alembert operator $\Box = -\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_{d-1}^2}$,

$\phi(x)$ **is a function from** $x \in \mathbb{R}^d$, $\tilde{\phi}(k)$ **is the Fourier transform,**

$k^2 = k_0^2 - k_1^2 - ... - k_{d-1}^2$.

Quantization of the Riemann Zeta-Function. Lagrangian

We consider the following Lagrangian

$$\mathcal{L} = \phi\xi(\frac{1}{2} + i\Box)\phi\,. \qquad (*)$$

The integral in

$$\xi(\frac{1}{2} + i\Box)\phi(x) = \int e^{ixk}\xi(\frac{1}{2} + ik^2)\tilde{\phi}(k)dk$$

converges if $\phi(x)$ is a decreasing function since $\xi(\frac{1}{2} + i\tau)$ is bounded.

The quantized ξ-function can be expressed as

$$\xi(\frac{1}{2} + i\Box) = \frac{1}{2} - (\Box^2 + \frac{1}{4})\int_1^\infty x^{-\frac{3}{4}} \cos[\frac{\Box}{2}\log x]\omega(x)dx\,.$$

Quantization of the Riemann Zeta-Function. Spectrum

Proposition. *The Lagrangian (*) is equivalent to the following Lagrangian*

$$\mathcal{L}' = \sum_{\epsilon,n} \eta_{\epsilon n} \psi_{\epsilon n} (\Box + \epsilon m_n^2) \psi_{\epsilon n} \ , \tag{2}$$

where the notations are defined below.
Let

$$\rho_n = \frac{1}{2} + i m_n^2, \quad \bar{\rho}_n = \frac{1}{2} - i m_n^2, \quad m_n > 0, \quad n = 1, 2, \ldots \tag{3}$$

be the zeros at the critical line.
We shall show that the zeros m_n^2 of the Riemann zeta-function become the masses of elementary particles in the Klein-Gordon equation.

Quantization of the Riemann Zeta-Function. Proof of Proposition

From the Hadamard representation we get

$$\xi(\frac{1}{2} + i\tau) = \frac{C}{2} \prod_{n=1}^{\infty} (1 - \frac{\tau^2}{m_n^4}) \tag{4}$$

Now if we define the fields $\psi_{\epsilon n}$ as

$$\psi_{\epsilon_0 n_0} \;=\; \frac{C}{2m_n^2} \prod_{\epsilon n \neq \epsilon_0 n_0} \left(1 + \frac{\Box}{\epsilon m_n^2}\right) \phi\,, \tag{5}$$

and the real constants $\eta_{\epsilon n}$ by

$$\eta_{\epsilon n} \;=\; \frac{1}{i\xi'(\frac{1}{2} - i\epsilon m_n^2)}\,, \tag{6}$$

then it is straightforward to see that the Lagrangians (*) and (2) are equivalent. The proposition is proved.

Quantization of the Riemann Zeta-Function. Summary

- **Quantization of the mathematics of Fermat-Wiles and the Langlands program is indicated.**
- **The Beilinson conjectures on the values of L-functions of motives are interpreted as dealing with the cosmological constant problem.**
- **Possible cosmological applications of the zeta-function field theory are discussed.**

Motives

I.V. *"D-branes, black holes and $SU(\infty)$ gauge theory,"* 2nd International Sakharov Conference on Physics, 20-23 May 1996, Proc. pp 618-621, hep-th/9608137.

- **Grothendieck's motives.**
  - **Motives are defined by algebraic correspondences modulo homological equivalence**
  - **Motivic cohomology is a kind of universal cohomology theory for algebraic varieties.**
  - **Realizations of a motive M over the field of rational numbers Q are linear spaces over Q and over the field of l-adic numbers Q_l.**
  - **If X is a smooth projective algebraic variety over Q then the realization of its motive is given by the Betti $H_B(X)$, De Rham $H_{DR}(X)$ and l-adic cohomology $H_l(X)$ of X.**
  - **Category of motives is isomorphic to category of finite dimensional representations of proalgebraic motivic Galois-Serre group G.**

Motives. L-functions

- **The group G plays the role of conformal group in the ordinary string theory.**
- **Motivic-theory deals with algebraic varieties over the field of rational numbers Q, so the theory is background independent and it is not based on a spacetime continuum.**
- **The partition function in motivic-theory is given by L-function of a motive.**
- **L-function of a motive M is defined by the Euler product:**

$$L(M,s)=\prod_p det(1-p^{-s}Frob_p|H_l(M)^{I_p})^{-1},$$

where $Frob_p$ is a Frobenius element in G_{Q_p} and I_p is the inertia group in G_{Q_p}.

Motives. Motivic-string

- **Motivic-string partition function can be expressed as inverse to the Mellin transform of L-function of a motive.**
- **Deligne has used L-function in the proof of the Ramanujan conjecture:**

$$|\tau(p)| \leq 2p^{11/2}.$$

The Ramanujan function $\tau(n)$ is defined by the relation

$$\Delta(q) = q\prod_n (1-q^n)^{24} = \sum_n \tau(n) q^n$$

We have: $\Delta(q)^{-1} = \mathbf{tr} q^H$, where $H = L_0 - \frac{1}{24}$ and L_0 is the Virasoro operator.

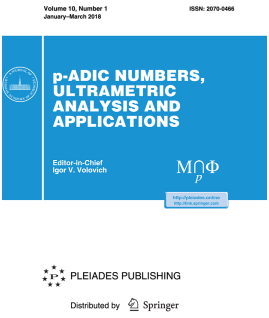
- **L-function of the motive M is the Dirichlet series:**

$$L(M,s) = \sum_n \tau(n) n^{-s} = \prod_p (1-\tau(p)p^{-s} - p^{11-2s})^{-1},$$

- **It is amusing that the motive M is eleven-dimensional.**
- **We suggest a relation between the cosmological constant problem and the Beilinson conjectures on values of L-functions.**

Conclusions

- **Principle of invariance of fundamental physical laws:**
  fundamental physical laws should be invariant
  with respect to change of number field
- **V.S.Vladimirov works on p-adic mathematical physics**
- **All quantum dynamical system $(\mathcal{H}, U_t)$ (real and p-adic) are completely integrable in the sense of the unitary equivalence to a system of classiacal harmonic oscillators**
- **Quantization of Riemann zeta-function**



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