# A look at quantum systems
## via p-adic 1-Lipschitz maps

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# Outline of the talk

1. Functions which are causal (=1-Lipschitz $p$-adic=automatic) with respect to ‘discrete’ time and which simultaneously are real continuous functions of real (=‘continuous’) time.
2. ‘Automatic wavefunctions’ over discrete time (that is, based on 1-Lipschitz $p$-adic functions $\mathbb{Z}_p \to \mathbb{Z}_p$) which approximate wavefunctions over continuous time.

# Part I

# Motivators and rationale

- **Volovich postulates** (see his numerous papers, books, talks on the $p$-adic mathematical physics)
  1. Only rational numbers can be observed; irrational numbers cannot.
  2. Distances smaller than Planck length cannot be measured.
  3. Fundamental physical laws should be invariant with respect to change of number field.

# Motivators and rationale

- **G. ‘t Hooft causality postulate** (see his numerous papers, talks, and his book “The Cellular Automaton Interpretation of Quantum Mechanics”)

  *It may well be that, at its most basic level, there is no randomness in Nature, no fundamentally statistical aspect to the laws of evolution. Everything, up to the most minute detail, is controlled by invariable laws. Every significant event in our universe takes place for a reason, it was caused by the action of physical law, not just by chance.*

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To be consistent with that postulate, a physical system must be *causal*; that is, the “effect”, which is the reaction of the system to a “cause”, i.e., to an impact the system has been exposed to, must be a function of the “cause” and of the “state” of the system. But the very notion of causality is based on the notion of “time” which must be a totally ordered set since the “effect” cannot happen earlier than the “cause” whose function the “effect” is.

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As it is impossible experimentally distinguish rational numbers from real numbers (cf. Volovich first postulate); then it is reasonable to assume that "time" is a totally ordered *countable* set. It is well known that any totally ordered countable set $T$ is order-isomorphic to a subset of $\mathbb{Q}$ with respect to the natural order $\leq$ on $\mathbb{Q}$. Time $T$ is called "continuous' if the ordering of elements in $T$ is dense; i.e., given $t_i, t_2 \in T$ there exists $t_3 \in T$ such that $t_1 < t_3 < t_2$. Time $T$ is called "discrete" if given any $t_i, t_2 \in T$ there are not more than finitely many $t_3 \in T$ such that $t_1 < t_3 < t_2$.

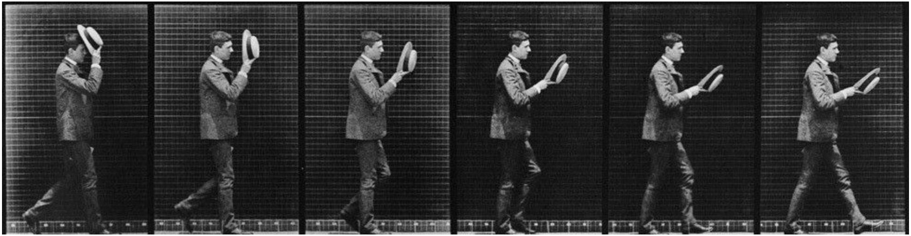
# Motivators and rationale

"Continuous" physical models are based on the assumption that any temporal/spatial interval can be divided into smaller intervals ad infinitum. The "discrete" models assume that spacetime should somehow be "quantized" at the smallest of scales; i.e., that there exist the smallest spatial/temporal intervals which can not be divided into smaller ones. In the latter case it would be reasonable to try to construct a mathematical theory assuming that total amount of that "indivisible" values can be increased ad infinitum.

- In the both cases, as well as in respective physical theories, the "infinity" just stands for a value which is extremely small (or extremely large) compared to a given value so that calculations involving the notion of infinity result in values which agree with respective measured values up to a small real number, the error.
- Therefore if theories of either type adequately describe physical reality at respective "ends of scale" , the theories must "meet one another somewhere in the middle of the scale". Is this possible to somehow agree the 'discrete' and the 'continuous' theories?

# Motivators and rationale

The discreteness implies that both “cause” and “effect” are sequences of “elementary causes” and “elementary effects” which happen at discrete time instants $0, 1, 2, \ldots$. In simple words, both ‘cause’ and ‘effect’ are like movies, the films consisting of sequences of frames:



**The finiteness assumption:** Both “elementary causes” and “elementary events” constitute finite sets either containing at least two elements since it is reasonable to assume that infinite number of events cannot happen in a physical world at a single instant of time: The infinity is a mathematical rather than a physical concept. So the shortest temporal interval is the one during which an observer cannot determine whether two elementary events happen simultaneously or not.

# Motivators and rationale

**Definition (R. E. Kalman—P. L. Falb—M. A. Arbib, and many others)**

*Causal functions over discrete time* $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ are exactly the functions $f$ which satisfy the following conditions:

1. the domain (the "causes") and range (the "effects") of $f$ are, accordingly, all sequences $\bar{\chi} = (\chi_i)_{i\in\mathbb{N}_0}$ and $\bar{\xi} = (\xi_i)_{i\in\mathbb{N}_0}$ over respective sets $\mathcal{I}$, the "elementary causes", and $\mathcal{O}$, the "elementary effects',
2. if $f(\bar{\chi}) = (\xi_i)_{i\in\mathbb{N}_0}$ then $\xi_i$ does not depend on $\chi_{i+1}, \chi_{i+2}, \ldots$, for all $i \in \mathbb{N}_0$.

**Under the finiteness assumption, the class of all causal functions coincides with a class of all automata functions:** The function $f$ is causal iff there exists a sequence $(\varphi_i^f)_{i=0}^{\infty}$ of maps $\varphi_i^f\colon \mathcal{I}^{i+1} \to \mathcal{O}$, $(i \in \mathbb{N}_0)$, s.t. $f(\mathbf{a}) = (\varphi_i^f(\chi_0, \ldots, \chi_i))_{i\in\mathbb{N}_0}$; i.e., iff $f$ is an automaton function. In particular, iff $f$ is a $p$-adic 1-Lipschitz function provided $\#\mathcal{I} = \#\mathcal{O} = p$.

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- As only rational numbers could be observed, the domain and range of *functions* $f$ *representing physical laws which can be experimentally verified must be the field* $\mathbb{Q}$ *of rational numbers*;

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- In order to investigate the laws $f$ when values of variables are "very large" or "very small" w.r.t. a reasonable metric, one has to expand $f$ from the field $\mathbb{Q}$ to a bigger field which is complete w.r.t. that metric; therefore, that bigger field can only be the field of real numbers $\mathbb{R}$ and/or $p$-adic fields $\mathbb{Q}_p$; but in order to be invariant w.r.t. the change of the number field, a *restriction to* $\mathbb{Q}$ *of any such expansion of* $f$ *to a bigger field* $\mathbb{F} \supset \mathbb{Q}$ *must be the same irrespectively of what that bigger field* $\mathbb{F}$ *is,* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{Q}_p$.

# Motivators and rationale

- To be consistent also with the ‘t Hooft causality postulate, the functions $f$ should be causal; however, as it has been argued before, the ‘time’ with respect to that the functions are causal must be order-isomorphic to a subset of $\mathbb{Q}$.

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- But since according to Volovich postulates no temporal interval smaller than Planck’s time cannot be measured, the temporal intervals can only be multiples of Planck’s time; therefore the ‘time’ over which the functions $f$ are causal must be order-isomorphic to a subset of $\mathbb{Z}$. Thus to up to order isomorphism the time is either $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ or $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Assuming that the ‘beginning of time’ exists, the time must be $\mathbb{N}_0$.

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- However, causal functions over discrete time $\mathbb{N}_0$ can be treated as $p$-adic 1-Lipschitz functions whose domain and range are $p$-adic integers $\mathbb{Z}_p$. As a ‘common part’ of $\mathbb{Z}_p$ and $\mathbb{R}$ is the ring $\mathbb{Z}_p \cap \mathbb{Q}$ of *rational* $p$-adic integers, to be consistent with Volovich postulates the causal functions must take values in $\mathbb{Z}_p \cap \mathbb{Q}$; also the functions must be expandable to the field $\mathbb{R}$ since $\mathbb{Z}_p \cap \mathbb{Q}$ is a dense subset of $\mathbb{R}$ and of $\mathbb{Z}_p$.

# Motivators and rationale

We finally summarize **the consistency conditions**: *A continuous real function $\breve{f}\colon\mathbb{R}\to\mathbb{R}$ which is consistent with both Volovich postulates and the 't Hooft causality postulate must share the properties listed below.*

1. *For $p$ a prime, the restriction $\breve{f}|_{\mathbb{N}_0}$ must be a causal function over discrete time $\mathbb{N}_0$; i.e., the restriction $\breve{f}|_{\mathbb{N}_0}$ must satisfy a $p$-adic Lipschitz condition with a constant 1. That is, for all $m,n\in\mathbb{N}_0$ there must hold the inequality $\|\breve{f}|_{\mathbb{N}_0}(m)-\breve{f}|_{\mathbb{N}_0}(n)\|_p\le\|m-n\|_p$.*
2. *Since $\mathbb{N}_0$ is a dense subset in $\mathbb{Z}_p$, by p. 1 there exists a unique extension of $\breve{f}|_{\mathbb{N}_0}$ to the function $f_p\colon\mathbb{Z}_p\to\mathbb{Z}_p$ which 1-Lipschitz w.r.t. the $p$-adic metric. Therefore, to be invariant w.r.t. the change of the field, the function $\breve{f}\colon\mathbb{R}\to\mathbb{R}$ must act on the set $\mathbb{Z}_p\cap\mathbb{Q}$* (the 'common part' of $\mathbb{R}$ and $\mathbb{Z}_p$) *of all rational $p$-adic integers exactly as the function $f_p$ does; that is, the restriction $\breve{f}|_{\mathbb{Z}_p\cap\mathbb{Q}}$ to rational $p$-adic integers $\mathbb{Z}_p\cap\mathbb{Q}$ must coincide with the restriction $f_p|_{\mathbb{Z}_p\cap\mathbb{Q}}$ to $\mathbb{Z}_p\cap\mathbb{Q}$: $\breve{f}|_{\mathbb{Z}_p\cap\mathbb{Q}}(r)=f_p|_{\mathbb{Z}_p\cap\mathbb{Q}}(r)$, for all $r\in\mathbb{Z}_p\cap\mathbb{Q}$.*

- Note that the conditions imply that $\breve{f}(\mathbb{Z}_p\cap\mathbb{Q})\subset\mathbb{Z}_p\cap\mathbb{Q}$.

# Reminder: Systems, automata, and word maps

## Notion: *system*

A (discrete) system (or, a system with a discrete time $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$) is a 5-tuple $\mathfrak{A} = \langle \mathcal{I}, \mathcal{S}, \mathcal{O}, S, O \rangle$ where

- $\mathcal{I}$ is a non-empty finite set, the input alphabet;
- $\mathcal{O}$ is a non-empty finite set, the output alphabet;
- $\mathcal{S}$ is a non-empty (possibly, infinite) set of (epistemic) states;
- $S \colon \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ is a state transition function;
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The system is autonomous if neither $S$ nor $O$ depend on input (that is $S \colon \mathcal{S} \to \mathcal{S}$, $O \colon \mathcal{S} \to \mathcal{O}$); otherwise the system is non-autonomous.

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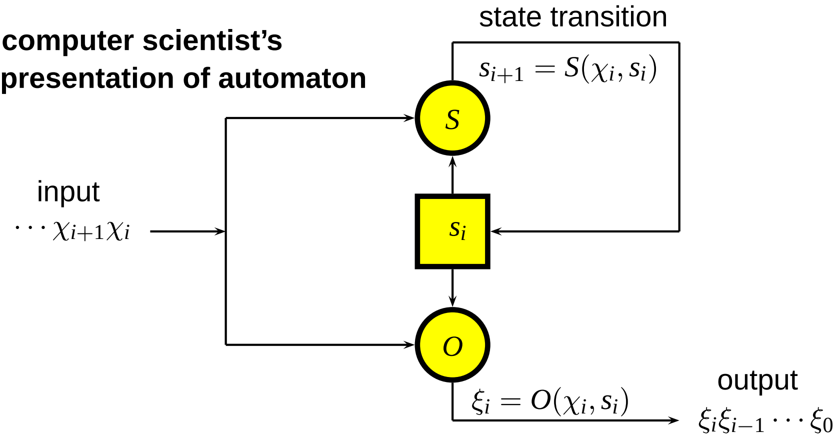
**A computer scientist's representation of automaton**



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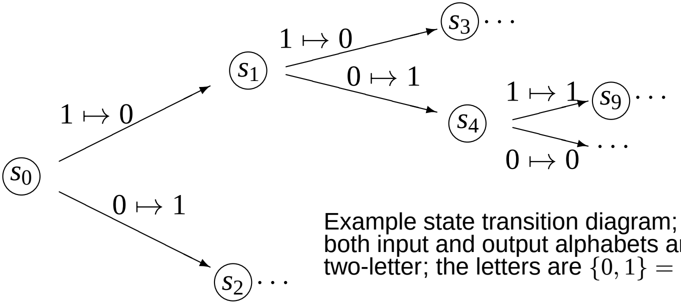
**State transition diagram of automaton=Moore diagram**

It is a directed graph whose vertices are states, whose arrows are state transitions; labels are $\text{input letter} \mapsto \text{output letter}$.




Example state transition diagram; both input and output alphabets are two-letter; the letters are $\{0, 1\} = \mathbb{F}_2$

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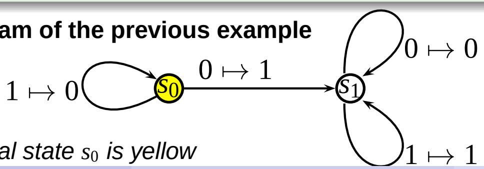
## Equivalent states; equivalent Moore diagrams

Two states $s_i, s_j \in \mathcal{S}$ of the automaton $\mathfrak{A}$ are *equivalent* if whenever taken as initial states, the words mappings performed by either automaton are equal one to another; i.e., if input words are equal one to another, then corresponding output words are also equal one to another. The tree-like state transition diagram can be reduced by that equivalence thus resulting in a *reduced diagram*.

**Reduced diagram of the previous example**




*Initial state $s_0$ is yellow*

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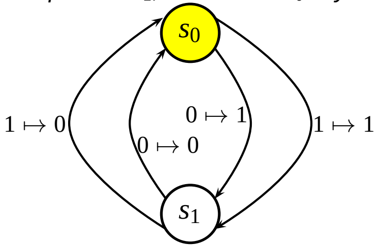
**Reduced state transition diagram of an autonomous automaton**
*(two-letter alphabet $\mathbb{F}_2$, initial state $s_0$ is yellow)*



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- Given two automata which perform the same word mapping, the reduced state transition diagram of the both coincide.
- An automaton is called *finite* (=finite-state) whenever there are only finite number of vertices in its reduced state transition diagram.

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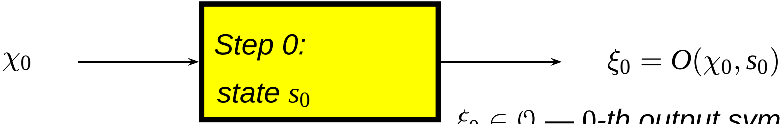
*Automaton* $\mathfrak{A} = \langle \mathcal{I}, \mathcal{S}, \mathcal{O}, S, O, s_0 \rangle$: $\mathcal{I}$ – *finite input alphabet;* $\mathcal{O}$ – *finite output alphabet;* $\mathcal{S}$ – *state set (not necessarily finite);* $S\colon \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ *state transition function;* $O\colon \mathcal{I} \times \mathcal{S} \to \mathcal{O}$ – *output function;* $s_0 \in \mathcal{S}$ – *initial state*

$\chi_0 \in \mathcal{I}$ — 0-*th input symbol*




$\xi_0 \in \mathcal{O}$ — 0-*th output symbol*

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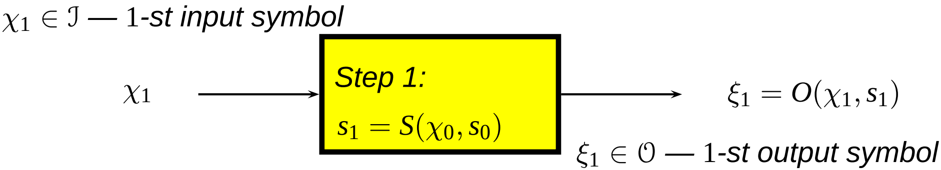

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$\chi_2 \in \mathcal{I}$ — 2*-nd input symbol*

$\chi_2$ → *Step 2:* $s_2 = S(\chi_1, s_1)$ → $\xi_2 = O(\chi_2, s_2)$

$\xi_2 \in \mathcal{O}$ — 2*-nd output symbol*

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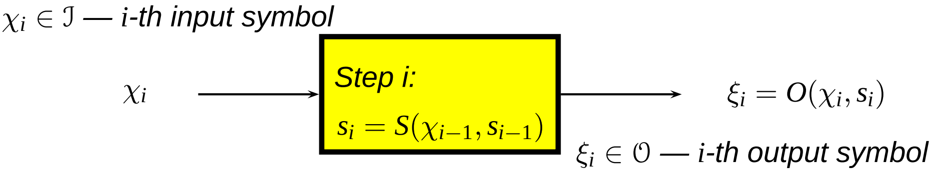

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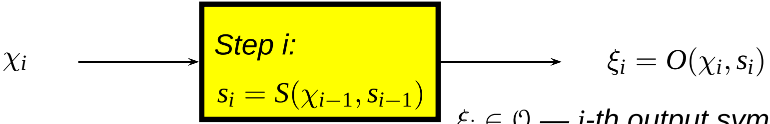
$\chi_i \in \mathcal{I}$ — *i-th input symbol*




$\xi_i \in \mathcal{O}$ — *i-th output symbol*

The automaton $\mathfrak{A}$ determines the automaton function $f_{\mathfrak{A}}$ that maps words over the alphabet $\mathcal{I}$ to words over the alphabet $\mathcal{O}$: $f_{\mathfrak{A}} \colon \ldots \chi_2\chi_1\chi_0 \mapsto \ldots \xi_2\xi_1\xi_0$

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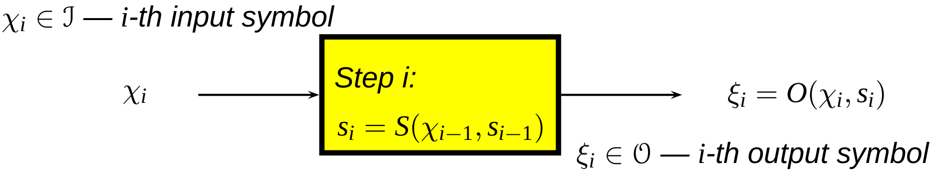


Every output symbol $\xi_i$ *depends only on symbols* $\chi_0, \ldots, \chi_i \in \mathcal{I}$ *which the automaton has been already fed*: $\xi_i = \varphi_i(\chi_0, \ldots, \chi_i) \in \mathcal{O}$

Therefore we may treat an automaton as a mathematical formalism of the *causality law:* (input symbols=elementary causes; output symbols=elementary effects)

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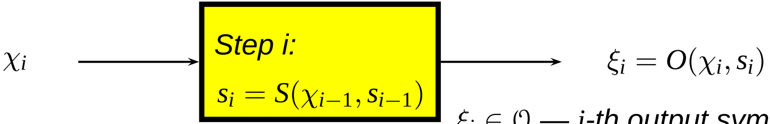


$\xi_i \in \mathcal{O}$ — *i-th output symbol*

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The automaton function $f_{\mathfrak{A}}\colon \ldots \chi_2\chi_1\chi_0 \mapsto \ldots \xi_2\xi_1\xi_0$ is completely determined by the sequence of maps $\varphi_i\colon \mathcal{I}^{i+1} \to \mathcal{O}$, $i \in \mathbb{N}_0$; and vice versa, every such sequence of maps determines an automaton function.

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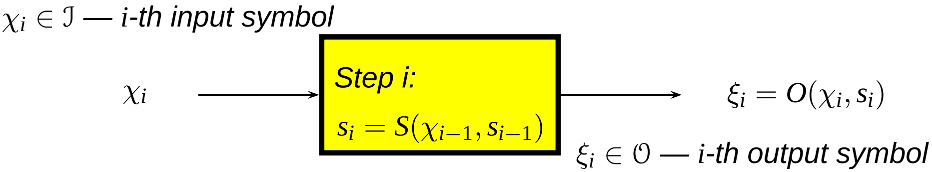


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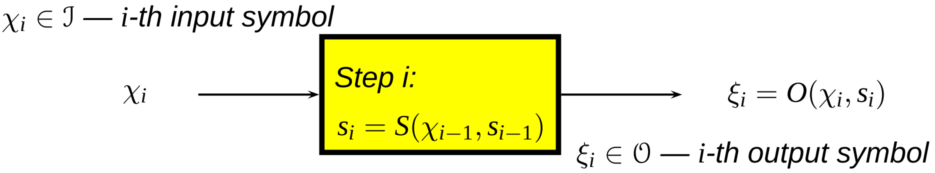
The automaton function maps infinite words (=left-infinite sequences) over $\mathcal{I}$ to infinite words over $\mathcal{O}$.

# Reminder: Systems, automata, and word maps

*Automaton* $\mathfrak{A} = \langle \mathcal{I}, \mathcal{S}, \mathcal{O}, S, O, s_0 \rangle$: $\mathcal{I}$ – *finite input alphabet;* $\mathcal{O}$ – *finite output alphabet;* $\mathcal{S}$ – *state set (not necessarily finite);* $S\colon \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ *state transition function;* $O\colon \mathcal{I} \times \mathcal{S} \to \mathcal{O}$ – *output function;* $s_0 \in \mathcal{S}$ – *initial state*




Every output symbol $\xi_i$ *depends only on symbols* $\chi_0, \ldots, \chi_i \in \mathcal{I}$ *which the automaton has been already fed*: $\xi_i = \varphi_i(\chi_0, \ldots, \chi_i) \in \mathcal{O}$

If $\mathcal{I} = \mathcal{O} = \{0, 1, \ldots, p-1\}$ the infinite words can be viewed as canonical representations of $p$-adic integers $\ldots\gamma_2\gamma_1\gamma_0 \longleftrightarrow \sum_{i=0}^{\infty} \gamma_i p^i$; then the automaton function is a map $\mathbb{Z}_p \to \mathbb{Z}_p$. The map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is an automaton (=causal) function *iff* it is 1-Lipschitz w.r.t. $p$-adic metric.

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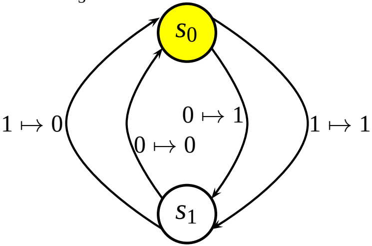
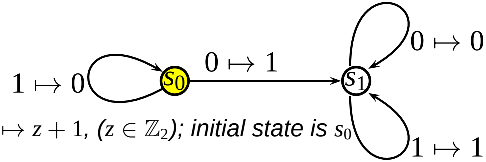
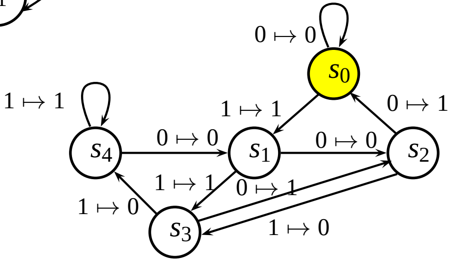
$z \mapsto -\frac{1}{3}$; $(z \in \mathbb{Z}_2)$, *initial state is* $s_0$




$z \mapsto z+1$, *$(z \in \mathbb{Z}_2)$; initial state is* $s_0$

$z \mapsto 5z$, $(z \in \mathbb{Z}_2)$
$s_0$ *is initial state*



# Reminder: Systems, automata, and word maps

- The automata are letter-to-letter transducers, the sequential Mealy machines (with maybe infinite memory). The class of mappings $\mathbb{Z}_p \to \mathbb{Z}_p$ they evaluate are all $p$-adic 1-Lipschitz functions; while $p$-adic continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ constitute the class of maps computed by letter-to-word transducers. General mappings $\mathbb{Z}_p \to \mathbb{Z}_p$ performed by Turing machines with a one-side infinite tape are not everywhere defined, not speaking of continuity. (Note that Turing machines with a one-side infinite tape are equivalent to Turing machines with both-side infinite tape).

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- To put it in simple words, the letter-to-letter transducers are the 'weakest computers' compared to cellular automata: The latter are equivalent to Turing machines and therefore are the 'most powerful' computers. The functions $\mathbb{N}_0^m \to \mathbb{N}_0^n$ the transducers can evaluate are necessarily primitive (rather than general) recursive functions, whereas all general recursive functions can be evaluated by Turing machines.

# Reminder: Systems, automata, and word maps

- The automata are letter-to-letter transducers, the sequential Mealy machines (with maybe infinite memory). The class of mappings $\mathbb{Z}_p \to \mathbb{Z}_p$ they evaluate are all $p$-adic 1-Lipschitz functions; while $p$-adic continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ constitute the class of maps computed by letter-to-word transducers. General mappings $\mathbb{Z}_p \to \mathbb{Z}_p$ performed by Turing machines with a one-side infinite tape are not everywhere defined, not speaking of continuity. (Note that Turing machines with a one-side infinite tape are equivalent to Turing machines with both-side infinite tape).
- To put it in simple words, the letter-to-letter transducers are the 'weakest computers' compared to cellular automata: The latter are equivalent to Turing machines and therefore are the 'most powerful' computers.
- From physical point of view, automata should be treated as models of *open systems* whereas cellular automata as models of *isolated systems*: contrasting to automata, a cellular automaton updates its states according only to a fixed local rule which does not depend on input.

# Properties of $\mathcal{C}_{\text{all primes}}(\mathbb{R})$-, $\mathcal{C}_p(\mathbb{R})$-, and $\mathcal{C}_P(\mathbb{R})$-functions

**Definition (complete consistency, $p$-consistency, $P$-consistency)**

We call a continuous real function $\breve{f}\colon \mathbb{R} \to \mathbb{R}$ completely consistent (both with Volovich and 't Hooft postulates) if

- $\breve{f}(\mathbb{Z}_p \cap \mathbb{Q}) \subset \mathbb{Z}_p \cap \mathbb{Q}$, for all prime $p$, and
- $\breve{f}|_{\mathbb{Z}_p\cap\mathbb{Q}}$ is 1-Lipschitz w.r.t. the $p$-adic metric.

We call $\breve{f}$ $p$-consistent (resp., $P$-consistent) if the conditions hold for a prime $p$ (resp., for a set $P$ of primes). Denote the classes of all respective functions via $\mathcal{C}_{\text{all primes}}(\mathbb{R})$, $\mathcal{C}_p(\mathbb{R})$, $\mathcal{C}_P(\mathbb{R})$.

- The definition immediately implies that a completely consistent function $\breve{f}$ maps integers to integers: $\breve{f}(\mathbb{Z}) \subset \mathbb{Z}$.

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### Theorem (Completely consistent functions; V. A.)

*A continuous function $\breve{f}:\mathbb{R}\to\mathbb{R}$ is completely consistent iff*

$$\breve{f}(x)=c_0+\sum_{i=1}^{k}c_i\cdot\operatorname{lcm}\{1,2,\ldots,i\}\cdot\binom{x}{i}=c_0+\sum_{i=1}^{k}c_i\cdot e^{\psi(i)}\cdot\binom{x}{i},$$

*where $k\in\mathbb{N}$, $c_0,c_i\in\mathbb{Z}$, $\operatorname{lcm}\{1,2,\ldots,i\}$ is the least common multiple of the numbers $1,2,\ldots,i$ and $\psi(i)=\sum_{q\le i,\ q\ \text{prime}}\lfloor\log_q i\rfloor\ln q$ is the second Chebyshev function, $i=1,2,\ldots$* (recall that $\psi(i)=i+o(i)$).

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Note that though all completely consistent functions are *integer-valued polynomials* (the polynomials which map $\mathbb{Z}$ into $\mathbb{Z}$) they are *not* necessarily polynomials over $\mathbb{Z}$; actually they are polynomials over $\mathbb{Q}$ (and thus map rational numbers to rational numbers). For instance, the polynomial $\frac{1}{2}x(x-1)(x-2)(x-3)$ is a completely consistent function .

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- The completely consistent functions are causal functions whose sets of 'elementary causes' and 'elementary effects' consist of $N > 1$ elements, where $N$ is arbitrary number, and not necessarily a prime; that is, whose domains and ranges are $N$-adic integers, where $N > 1$ may be taken arbitrarily. Thus, the functions are all polynomials from the class of pseudo-polynomials studied by R. R. Hall (1971).

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- If there are no integers in a real segment $[a,b] \subset \mathbb{R}$ then any continuous function $g\colon [a,b] \to \mathbb{R}$ can be uniformly approximated on $[a,b]$ w.r.t. real metric by completely consistent functions.

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- Any causal function $g\colon \mathbb{Z}_p \to \mathbb{Z}_p$ (i.e., a $p$-adic 1-Lipschitz function) can be uniformly approximated on $\mathbb{Z}_p$ w.r.t. $p$-adic metric by completely consistent functions.

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These facts may be treated as some evidence in favour of a disputable thesis that if Nature is causal, then an observer who performs measurements up to a non-zero error w.r.t. real metric can never determine whether time and Nature at the smallest of scales are discrete or continuous, and if they are discrete, what is that the smallest of scales; and if time is discrete at some scale, whether Nature is causal.

# Properties of $\mathcal{C}_{\text{all primes}}(\mathbb{R})$-, $\mathcal{C}_p(\mathbb{R})$-, and $\mathcal{C}_P(\mathbb{R})$-functions

Recall that given a prime $p$, the class $\mathcal{C}_p(\mathbb{R})$ consists of all continuous functions $\breve{f}\colon \mathbb{R} \to \mathbb{R}$ such that:

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The class $\mathcal{C}_p(\mathbb{R})$ is the main class we are focused at; further in the talk we may not differ $\breve{f}$ from $f$ when it is clear from the context what domain is considered.

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## Example

Given polynomials $u, v \in \mathbb{Z}[x]$ s.t. $v(z) \not\equiv 0 \pmod p$ for all $z \in \mathbb{Z}_p$ and $v(x) \neq 0$ for all $x \in \mathbb{R}$, the rational function $f(x) = \frac{u(x)}{v(x)}$ is in $\mathcal{C}_p(\mathbb{R})$.

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### Proposition

*There exist functions in $\mathcal{C}_p(\mathbb{R})$ which are not rational functions.*

# Properties of $\mathcal{C}_{\text{all primes}}(\mathbb{R})$-, $\mathcal{C}_p(\mathbb{R})$-, and $\mathcal{C}_P(\mathbb{R})$-functions

**Theorem (Finite automata $C^1$-functions = affine functions; V. A.)**

*Let a finite automaton function $f \in \mathcal{C}_p(\mathbb{R})$ be differentiable both over $\mathbb{R}$ and over $\mathbb{Z}_p$; let the derivatives of $f$ w.r.t. both real and $p$-adic metric exist, coincide on $\mathbb{Z}_p \cap \mathbb{Q}$, and let them be continuous w.r.t. respective metrics. Then $f$ is an affine function over $\mathbb{Z}_p \cap \mathbb{Q}$; i.e., $f(x) = ax + b$ for suitable $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$. Vice versa, all these affine functions are finite automata functions from $\mathcal{C}_p(\mathbb{R})$.*

**Note**

The theorem is true in multidimensional case as well.

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- This may serve as a sort of reasoning why mathematical formalism of QM is the theory of linear operators: If all 'real-world' systems may have only a finite number of states, then when duration of temporal interval measured in Planck time units becomes comparable to the number of the states, the finiteness reveals itself as the linearity.

# Properties of $\mathcal{C}_{\text{all primes}}(\mathbb{R})$-, $\mathcal{C}_p(\mathbb{R})$-, and $\mathcal{C}_P(\mathbb{R})$-functions

The following assertions hold true:

- Any continuous function $g\colon [a,b] \to \mathbb{R}$ can be uniformly approximated on $[a,b] \subset \mathbb{R}$ by $\mathcal{C}_p(\mathbb{R})$-functions.

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The *(time-)reversibility of an automaton means that the automaton function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is bijective*; but a 1-Lipschitz map $\mathbb{Z}_p \to \mathbb{Z}_p$ is bijective if and only if it is an isometry w.r.t. $p$-adic metric, or, equivalently, if and only if $f$ is measure-preserving w.r.t. normalized Haar measure on $\mathbb{Z}_p$. The latter measure is a natural probability measure on $\mathbb{Z}_p$.

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# Part II

1. Causal functions

2. Automata wavefunctions

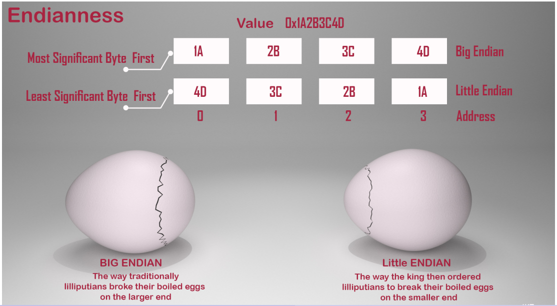
# Observations, measurement, data processing

Recalling *Gulliver's Travels* by Jonathan Swift, imagine two observers, the Little-endian and the Big-endian, whose capabilities of observation are different.

# Observations, measurement, data processing

Recalling *Gulliver's Travels* by Jonathan Swift, imagine two observers, the Little-endian and the Big-endian.

- Any $r \in \mathbb{Z}_p \cap \mathbb{Q} \cap (0,1)$ can be represented as $r = \frac{\tilde{r}}{p^n - 1}$ for some $n \in \mathbb{N}$ and $\tilde{r} \in \{1, 2, \ldots, p^n - 2\}$. When evaluating the number w.r.t. some $p$-adic error, the Little-endian observer sees some low order digits of the base-$p$ representation of $\tilde{r}$, while the Big-endian observer when evaluating $r$ w.r.t. some real error sees the higher order digits of the base-$p$ expansion of $\tilde{r}$ since $r = 0.(\zeta_{n-1}\zeta_{n-2}\ldots\zeta_0)^\infty \in \mathbb{R}$ and $r = 1 + \sum_{j=0}^{\infty}\sum_{i=0}^{n-1}(p - 1 - \zeta_i)p^{i+nj} \in \mathbb{Z}_p$ where $\tilde{r} = \zeta_0 + \zeta_1 p + \cdots + \zeta_{n-1}p^{n-1}$.



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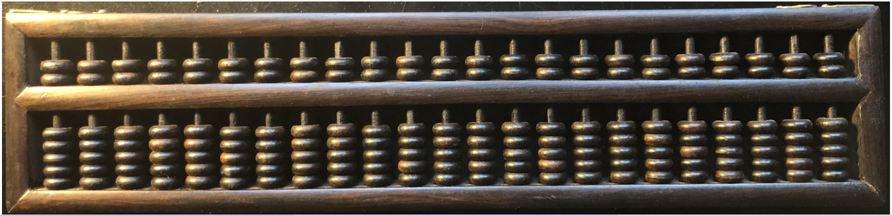
- The Little-endian uses abacus (the ‘non-Archimedean device’) to process measurement data by counting indivisible quantities represented by beads on rods. The ‘infinitely long abacus’ having a (right-)infinite number of rods can represent and process $p$-adic integers.



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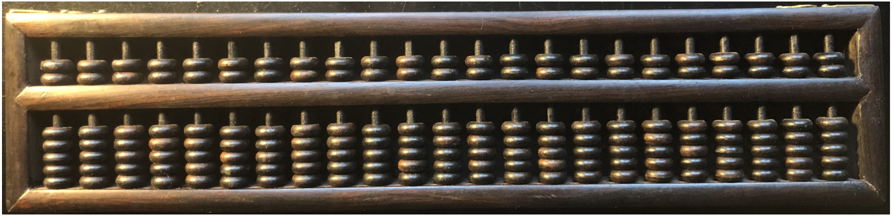
- The Big-endian uses inch tape (an ‘Archimedean device’) obtained by (potentially infinite) division of ‘standard rods’ into smaller pieces which are represented by the distances between respective markings. The inch tape having ‘infinite accuracy’ (rather than the accuracy 1/32 inch like the one depicted below) can represent real numbers.

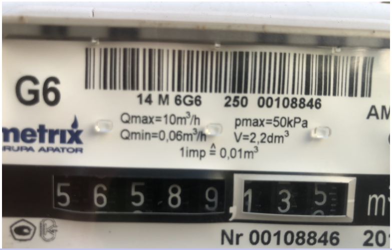
# Observations, measurement, data processing




The two ‘devices’ are absolutely different by their very nature and origin: The first one is aimed to count objects (thus, reflects arithmetical properties of the world at ‘human’s scale’) while the second one is basically a ruler designed to measure lengths of objects (thus, reflects geometry of the world at ‘human’s scale’).

The Little-endian uses abacus to count ‘Plank-size’ drops pouring into a beaker one by one while the Big-endian measures volume of liquid with the grades on the beaker.

The Little-endian only sees a few figures in the right end of the meter whereas the Big-endian only sees a few figures on the left end:



# Observations, measurement, data processing

**Both the Little- and the Big- endians are ‘thinking logarithmically’!!!**

That is, they represent/operate all the data (the numbers) in the base-$p$ expansion, where $p > 1$. Due to limited accuracy, the both deal with a finite non-overlapping blocks of values of consecutive digits of the number; the Little-endian can observe/process only the block of the *smallest* order digits whereas only a block of a *higher-order* digits is accessible to Big-endian.

- Therefore, the number of steps the automaton performs to reach some state from the initial state is *not* the time measured in Planck units, it is proportional to *logarithm* of that value.

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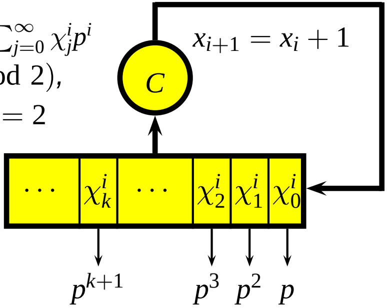
*Readout:* $x_i = \sum_{j=0}^{\infty} \chi_j^i p^i$

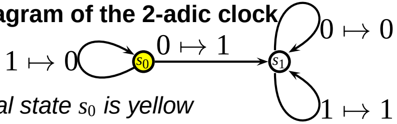
$$\chi_j^{i+1} \equiv \chi_j^i + \chi_{j-1}^i \cdots \chi_0^i \pmod 2,$$
$$\chi_0^{i+1} \equiv \chi_0^i + 1 \pmod 2 \text{ if } p = 2$$




*Periods:*

The Little-endian sees the the rightmost digits, the Big-endian sees the leftmost ones. If $x_0 = 0$ then $x_i = i = \sum_{j=0}^{\infty} \chi_j^i p^i$ is the number of *iterations* of $C$, i.e., time elapsed which is measured in Planck units.

**Reduced diagram of the 2-adic clock**




*Initial state $s_0$ is yellow*

# Discreteness+causality imply wavefunction

- Let a causal function $f$ over a $p$-symbol alphabet satisfy the first of Volovich postulates, i.e., let $f$ be a 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ s t. $f(\mathbb{Z}_p \cap \mathbb{Q}) \subset \mathbb{Z}_p \cap \mathbb{Q}$.

For instance, $f$ satisfies this property if $f \in \mathcal{C}_p(\mathbb{R})$ or if $f$ is an automaton function of a finite automaton (i.e., of an automaton having only finite number of states).

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- Define a map on the unit circle $\mathbb{S}_1 \subset \mathbb{C}$ as follows. Put

$$\hat{f}\colon e^{2\pi i z} \mapsto e^{2\pi i f(z)} \quad (z \in \mathbb{Z}_p \cap \mathbb{Q}).$$

As $f$ is causal, it is completely defined by its values on $\mathbb{Z}_p \cap \mathbb{Q} \cap [0,1)$; so the map $\hat{f}$ is completely defined by its values on $\mathbb{S}_1$. We denote that map $\mathbb{S}_1 \to \mathbb{S}_1$ by the same symbol $\hat{f}$ and call $\hat{f}$ the dual to $f$.

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- Note that if $f \in \mathcal{C}_p(\mathbb{R})$ then $\hat{f}$ is a continuous map from $\mathbb{S}_1$ to $\mathbb{S}_1$.

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- The pairs $(e^{2\pi i z}; e^{2\pi i f(z)})$ can be identified with the points on the surface of the torus $\mathbb{T}^2 \subset \mathbb{R}^3$ or, accordingly, with the points of the real unit square $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$ where $\mathbb{I} = [0,1]$. Denote corresponding point set via $\hat{\mathcal{E}}^2(f)$. Denote $\hat{\mathcal{P}}^2(f)$ the closure of $\hat{\mathcal{E}}^2(f)$ in $\mathbb{R}^2$; call $\hat{\mathcal{P}}^2(f)$ a (2-dimensional) plot of the 1-Lipschitz map $f$.
- Call a limit plot of $f$ the set $P\hat{\mathcal{P}}^2(f)$of all limit points of the plot $\hat{\mathcal{P}}^2(f)$.

The limit plot shows behaviour of $f$ after infinitely long time elapsed.

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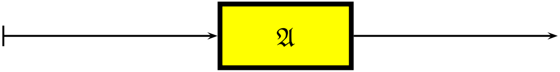
We give another look at the limit plot. Let $\alpha_2(f)$ be the Lebesgue measure of the closure of the following set $\mathcal{P}^2(f)$, in the unit square $\mathbb{I}^2 \subset \mathbb{R}^2$:

$$\mathcal{P}^2(f) = \left\{ e_k^f(x) = \left( \frac{x \bmod p^k}{p^k}; \frac{f(x) \bmod p^k}{p^k} \right), \text{ where } x \in \mathbb{Z}_p, k = 1, 2, \ldots \right\}$$

Both $x \bmod p^k$ and $f(x) \bmod p^k$ may be ascribed to $k$-letter input/output words of the automaton $\mathfrak{A}$ whose automaton function is $f$.




$$x \bmod p^k = \chi_{k-1} \cdots\cdots\cdots \chi_1\chi_0 \qquad \xi_{k-1} \cdots\cdots\cdots \xi_1\xi_0 = f(x) \bmod p^k$$

$$(0.\chi_{k-1} \ldots \chi_1\chi_0; 0.\xi_{k-1} \ldots \xi_1\xi_0)$$

# Discreteness+causality imply wavefunction

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- Note that the set of all limit points of $\mathcal{P}^2(f)$, the plot of $f$, is $P\hat{\mathcal{P}}^2(f)$, the limit plot.

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**Theorem** (*automata 0-1 law, V. A.*)

*Either $\alpha_2(f) = 0$, or $\alpha_2(f) = 1$ otherwise. Equivalently, either the plot $\mathcal{P}^2(f)$ is nowhere dense in $\mathbb{I}^2$ or $\mathcal{P}^2(f)$ is dense in $\mathbb{I}^2$, respectively.*

We will say for short that a 1-Lipschitz map (resp., an automaton) $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is of measure 0 iff $\alpha_2(f) = 0$, and of measure 1 otherwise.

# Discreteness+causality imply wavefunction

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**Theorem (V. A.)**

*If $f_{\mathfrak{A}}$ is an automaton function of a **finite** automaton $\mathfrak{A}$ then $\alpha_2(f_{\mathfrak{A}}) = 0$.*

The limit plot of a measure-0 automaton cannot contain ‘figures’ but it may contain ‘lines’.

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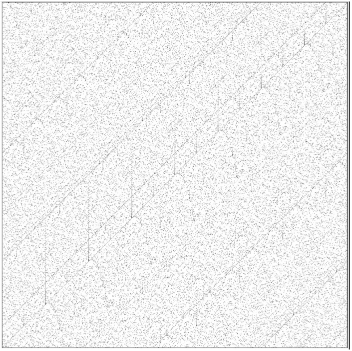
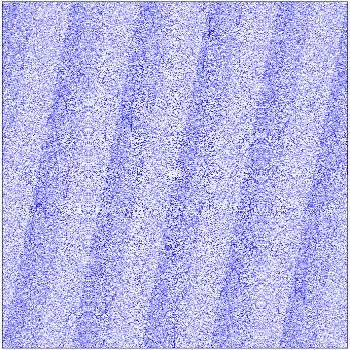




# Discreteness+causality imply wavefunction

It turns out that smooth lines from the limit plot of a finite automaton are necessarily windings of torus.

## Theorem (V. A.)

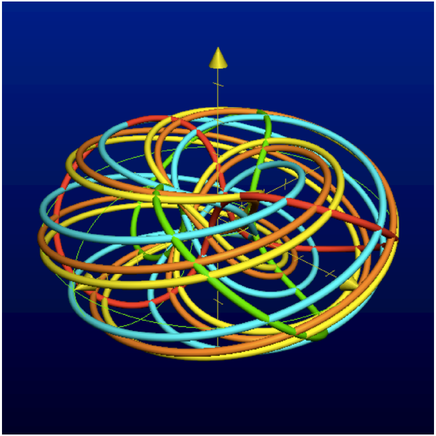
*Let $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ be an automaton function of a finite automaton; let $g$ be a $C^2$ real function with domain $[a,b] \subset [0,1)$ and range $[0,1)$. Let the graph $\mathbf{G}(g) = \{(x; g(x))\colon x \in [a,b]\}$ of $g$ lie completely in the limit plot $P\hat{\mathfrak{P}}^2(f)$. Then there exist $a, b \in \mathbb{Q} \cap \mathbb{Z}_p$ such that $g(x) = (ax+b) \bmod 1$ for all $x \in [a,b]$; moreover, there is a winding of the torus $\mathbb{T}^2$ which lies completely in the limit plot $P\hat{\mathfrak{P}}^2(f)$ and which contains the graph $\mathbf{G}(g)$ of the function $g$.*

*There are no more than a finite number of pairwise distinct windings of the unit torus $\mathbb{T}^2$ in $P\hat{\mathfrak{P}}^2(f)$; all of these are images of real affine functions $x \mapsto ax+b$ for $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ under the mapping $\mathrm{mod}1\colon \mathbb{R}^2 \to \mathbb{T}^2$.*

Hologram-likeness: If a plot contains the graph $\mathbf{G}(g)$ then the limit plot contains the whole link of torus knots to a winding of which the graph belongs.

# Discreteness+causality imply wavefunction

The smooth lines in the limit plot of a finite automaton constitute a set of overlapping torus links. Below is a limit plot of a 2-adic automaton.



$f_1(z) = -2z + \frac{1}{3}$ (red and green windings);
$f_2(z) = \frac{3}{5}z + \frac{2}{7}$, (yellow, brown, and blue windings).

# Discreteness+causality imply wavefunction

Both limit plots of constants from $\mathbb{Z}_p \cap \mathbb{Q}$ and limit plots of affine automata functions $x \mapsto ax + b$ (where $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$) are links of torus knots; the knots are *windings of torus*, images of straight lines in $\mathbb{R}^2$ under the mapping $\bmod 1\colon (x; y) \mapsto (x \bmod 1; y \bmod 1) \in \mathbb{T}^2$.

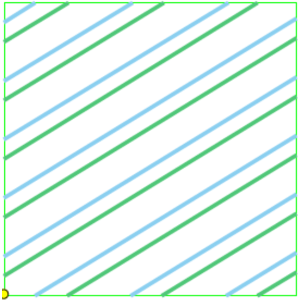


Figure: Limit plot of the function $f(z) = \frac{3}{5}z + \frac{1}{3}$, $z \in \mathbb{Z}_2$, in $\mathbb{R}^2$

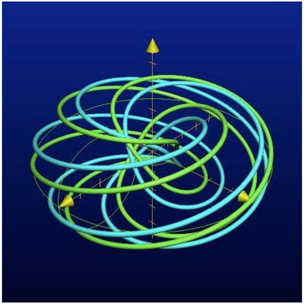


Figure: Limit plot of the same function on the torus $\mathbb{T}^2$

# Discreteness+causality imply wavefunction

In cylindrical coordinates the winding $\rho \mapsto \frac{\alpha}{\beta}\rho + \omega$ of a torus whose axis of rotation is $Z$ can be represented by parametric equations

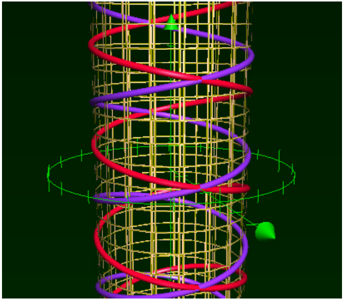
$$\begin{bmatrix} r_0 \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} R + r\cos\left(\frac{\alpha}{\beta}\rho + \omega\right) \\ \rho \\ r\sin\left(\frac{\alpha}{\beta}\rho + \omega\right) \end{bmatrix}, \ \rho \in \mathbb{R}.$$

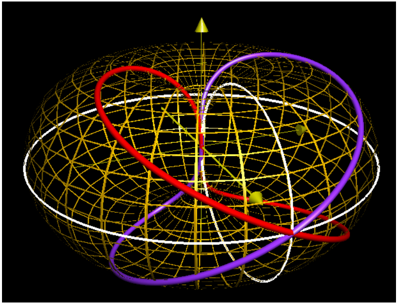
The winding winds $\beta$ times around $Z$-axis and $|\alpha|$ times around a circle (of radius $r$) in the interior of the torus; the sign of $\alpha$ determines whether the rotation is clockwise or counter-clockwise. Hence a 'physical meaning' of the coefficient $a = \frac{\alpha}{\beta}$ of the map $z \mapsto az+b$, ($z \in \mathbb{Z}_p$) is **wavenumber**. The sign $+$ or $-$ of $a$ may be treated as **polarization** since it shows 'clockwise'/'counter-clockwise' direction of rotation. To the map it corresponds the function $\mathbb{R} \times \mathbb{N}_0 \to \mathbb{C}$ (which also may be considered on $(\mathbb{Z}_p \cap \mathbb{Q}) \times \mathbb{Z}$):

$$\psi(\rho, k) = e^{i(\frac{\alpha}{\beta}\rho - 2\pi p^k b)} = e^{2\pi i(\frac{\alpha}{2\pi\beta}\rho - p^k b)}, \ (k \in \mathbb{N}_0; \text{ one may take } k \in \mathbb{Z} \text{ as well})$$

# Discreteness+causality imply wavefunction

The limit plot $\left\{\psi(\rho,k)=e^{i(\frac{\alpha}{\beta}\rho-2\pi p^k b)}\colon k\in\mathbb{N}_0\right\}$ of a finite affine automaton function $z\mapsto az+b$, ($z\in\mathbb{Z}_p$) may also be viewed as a solenoid.





The figures show the two representations of the limit plot of the 2-adic 1-Lipschitz map $f(z)=((z\ \mathrm{and}\ 1)-((\mathrm{not}(z))\ \mathrm{and}\ 1))\cdot z$, where and, not are respectively bitwise logical ‘and’ and bitwise logical ‘not’ operations on base-2 expansions of numbers (with no carries), ‘$\cdot$’ and ‘$-$’ are usual multiplication and subtraction of numbers (with carries).

# Discreteness+causality imply wavefunction

The solenoid limit plot of a finite automaton looks like a 'wave packet', a collection an overlapping sine waves.

- Note that all the mentioned links of torus knots are limits plots of automata functions of minimal automata; i.e., the automata whose *state transition diagrams are totally connected*: Given two states $s, t \in \mathcal{S}$, there is finite word $w$ in the alphabet $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ such that when the automaton in state $s$ accepts $w$, the automaton changes its state to $t$. If an automaton reaches a state which belongs to its sub-automaton, the automaton will never reach a state which does not belong to the sub-automaton.

# Discreteness+causality imply wavefunction

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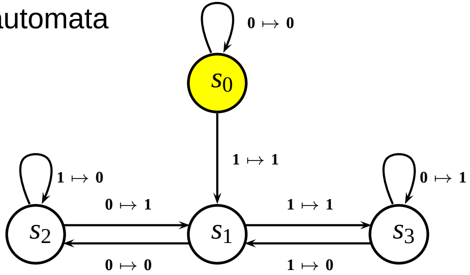



$s_0$ *is initial;* $f : z \mapsto -\frac{1}{3}z$, $(z \in \mathbb{Z}_2)$

$s_1$ *is initial;* $f : z \mapsto -\frac{1}{3}z - \frac{2}{3}$

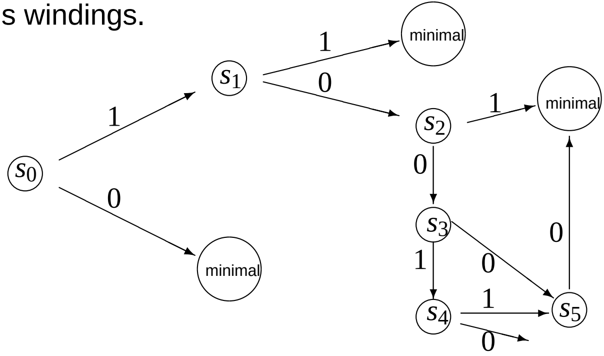
$s_2$ *is initial;* $f : z \mapsto -\frac{1}{3}z - \frac{1}{3}$

$s_3$ *is initial;* $f : z \mapsto -\frac{1}{3}z - 1$

*$f$ may depend on the choice of initial state, but the limit plot $P\hat{\mathfrak{P}}^2(f)$ may not*

*The automaton whose initial state is $s_0$ is not minimal. Initial state of a minimal sub-automaton is any of $s_1, s_2, s_3$.*
*Limit plots of all these automata coincide;*
*i.e. limit plot does not depend on what state of $s_0, s_1, s_2, s_3$ is taken as initial.*

State transition diagram of a finite automaton may be thought of as a graph each path in which ultimately reaches a minimal sub-automaton. There are no outgoing paths from minimal sub-automata. By feeding the automaton with random long words, to each minimal sub-automaton one assigns a probability of reaching states which belong to the minimal sub-automaton. If the sub-automaton is affine, one thus assigns a probability to its limit plot, i.e., to the map $\psi(\rho, k) = e^{i(a\rho - 2\pi p^k b)}$ which is a link of torus windings.



# Discreteness+causality imply wavefunction

By feeding the automaton with random long words, to each sub-automaton one assigns a probability of reaching states which belong to the sub-automaton. If the sub-automaton is affine, one thus assigns a probability to its limit plot, i.e., to the map $\psi(\rho,k)=e^{i(a\rho-2\pi p^k b)}$

- This way one assigns to the automaton the sum

$$\sum_{a,b\in\mathbb{Z}_p\cap\mathbb{Q}} q_{a,b}e^{i(a\rho-2\pi p^k b)},$$

where $q_{a,b}$ is a probability assigned to the limit plot of $z\mapsto az+b$.

In what follows,

- we do not distinguish affine automata whose limit plots coincide, so actually the sum is taken over classes of affine sub-automata having coinciding limit plots;
- we use terms '$p$-adic integer', 'infinite word over $p$-symbol alphabet', and 'infinite path in a state transition diagram' as synonyms.

# Discreteness+causality imply wavefunction

By feeding the automaton with random long words, to each sub-automaton one assigns a probability of reaching states which belong to the sub-automaton. If the sub-automaton is affine, one thus assigns a probability to its limit plot, i.e., to the map $\psi(\rho, k) = e^{i(a\rho - 2\pi p^k b)}$

- This way one assigns to the automaton the sum

$$\tilde{\Psi}_{\mathfrak{A}}(\rho, k) = \sum_{a,b \in \mathbb{Z}_p \cap \mathbb{Q}} q_{a,b} e^{i(a\rho - 2\pi p^k b)},$$

where $q_{a,b}$ is a probability assigned to the limit plot of $z \mapsto az + b$.

- Let an automaton $\mathfrak{A}$ be such that all paths ultimately reach finite affine sub-automata. That is, every path begins with an initial finite word (=prefix) which reaches a state that belongs to a finite affine sub-automaton. Then $\sum q_{a,b} = 1$, the automaton function $f_{\mathfrak{A}} \colon \mathbb{Z}_p \to \mathbb{Z}_p$ is such that $f_{\mathfrak{A}}(\mathbb{Z}_p \cap \mathbb{Q}) \subset \mathbb{Z}_p \cap \mathbb{Q}$, the series $\tilde{\Psi}_{\mathfrak{A}}$ converges absolutely whence defines a complex-valued function $\tilde{\Psi}_{\mathfrak{A}} \colon \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$.

# Discreteness+causality imply wavefunction

- One assigns to the automaton $\mathfrak{A}$ the sum

$$\tilde{\Psi}_{\mathfrak{A}}(\rho, k) = \sum_{a,b \in \mathbb{Z}_p \cap \mathbb{Q}} q_{a,b} e^{i(a\rho - 2\pi p^k b)},$$

where $q_{a,b}$ is a probability assigned to the limit plot of $z \mapsto az + b$.
- Let a (not necessarily finite) automaton $\mathfrak{A}$ be such that almost all (w.r.t. Haar measure) paths ultimately reach finite affine sub-automata. That is, almost every infinite path begins with an initial finite word (=prefix) which reaches a state that belongs to a finite affine sub-automaton. Then $\sum q_{a,b} = 1$, the series $\tilde{\Psi}_{\mathfrak{A}}$ converges absolutely whence defines a complex-valued function $\tilde{\Psi}_{\mathfrak{A}} \colon \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$.
- The probability $q_{a,b}$ is Haar measure of the disjoint union $W_{a,b}$ of balls in $\mathbb{Z}_p$ having non-zero radii; i.e., $W_{a,b}$ is a set of all infinite paths which ultimately reach a sub-automaton whose limit plot is affine map $z \mapsto az + b$.

# Discreteness+causality imply wavefunction

- One assigns to the automaton $\mathfrak{A}$ the sum

$$\tilde{\Psi}_{\mathfrak{A}}(\rho,k)=\sum_{a,b\in\mathbb{Z}_p\cap\mathbb{Q}} q_{a,b}e^{i(a\rho-2\pi p^k b)},$$

where $q_{a,b}$ is a probability assigned to the limit plot of $z\mapsto az+b$.

- Let a automaton $\mathfrak{A}$ be such that almost all (w.r.t. Haar measure) paths ultimately reach finite affine sub-automata. That is, almost every infinite path begins with an initial finite word (=prefix) which reaches a state that belongs to a finite affine sub-automaton. The series $\tilde{\Psi}_{\mathfrak{A}}$ converges absolutely whence defines a complex-valued function $\tilde{\Psi}_{\mathfrak{A}}\colon\mathbb{R}\times\mathbb{Z}\to\mathbb{C}$.

**Proposition**

*Given convergent series $\sum_{j=0}^{\infty} q_j=1$ of non-negative real numbers $q_j\in\mathbb{R}_{\geq 0}$ and pairs $(a_j,b_j)$ of rational $p$-adic integers there exists an automaton $\mathfrak{A}$ such that $\tilde{\Psi}_{\mathfrak{A}}=\sum_{j=0}^{\infty} q_je^{i(a_j\rho-2\pi p^k b_j)}$.*

# Discreteness+causality imply wavefunction

- One assigns to the automaton $\mathfrak{A}$ the sum

$$\tilde{\Psi}_{\mathfrak{A}}(\rho, k) = \sum_{a,b \in \mathbb{Z}_p \cap \mathbb{Q}} q_{a,b} e^{i(a\rho - 2\pi p^k b)},$$

where $q_{a,b}$ is a probability assigned to the limit plot of $z \mapsto az + b$.

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**Proposition (All discrete random variables can be modelled on $\mathbb{Z}_p$)**

*Given convergent series $\sum_{j=0}^{\infty} q_j = 1$ of non-negative real numbers $q_j \in \mathbb{R}_{\geq 0}$ there exist pairwise disjoint open sets $W_j \supset \mathbb{Z}_p$ such that the (normalized) Haar measure of $W_j$ is $q_j$, $j = 0, 1, 2, \ldots$.*

# Discreteness+causality imply wavefunction

**Proposition**

*Given convergent series* $\sum_{j=0}^{\infty} q_j = 1$ *of non-negative real numbers* $q_j \in \mathbb{R}_{\geq 0}$ *and pairs* $(a_j, b_j)$ *of rational* $p$*-adic integers there exists an automaton* $\mathfrak{A}$ *such that* $\tilde{\Psi}_{\mathfrak{A}} = \sum_{j=0}^{\infty} q_j e^{i(a_j\rho - 2\pi p^k b_j)}$.

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Speaking loosely, this means that

- any discrete random variable can be modelled by an automaton;
- any (absolutely) continuous random variable can be approximated by the 'automatic' discrete random variables (by empirical cdfs).

# Discreteness+causality imply wavefunction

## Proposition

*Given convergent series* $\sum_{j=0}^{\infty} q_j = 1$ *of non-negative real numbers* $q_j \in \mathbb{R}_{\geq 0}$ *and pairs* $(a_j, b_j)$ *of rational* $p$*-adic integers there exists an automaton* $\mathfrak{A}$ *such that* $\tilde{\Psi}_{\mathfrak{A}} = \sum_{j=0}^{\infty} q_j e^{i(a_j\rho - 2\pi p^k b_j)}$.

Let the series $\sum_{j=0}^{\infty} q_j = 1$ be s.t. $\sum_{j=0}^{\infty} \sqrt{q_j}$ converges. As the set $\mathbb{Z}_p \cap \mathbb{Q}$ of rational $p$-adic integers is countable, the set of all (pairwise distinct) limit plots $\mathcal{P}^2(a, b)$ of automata functions $z \mapsto az + b$, where $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$, is countable also. Therefore the series

$$\hat{\Psi} = \sum_{(j)} \sqrt{q_j} e^{i(a_j\rho - 2\pi p^k b_j)}$$

(where $j = 0, 1, 2, \ldots$ is enumeration of limit plots) converges absolutely and thus defines a complex-valued function of $\rho \in \mathbb{R}$, $k \in \mathbb{Z}$.

# Discreteness+causality imply wavefunction

**Proposition**

*Given convergent series* $\sum_{j=0}^{\infty} q_j = 1$ *of non-negative real numbers* $q_j \in \mathbb{R}_{\geq 0}$ *and pairs* $(a_j, b_j)$ *of rational* $p$*-adic integers there exists an automaton* $\mathfrak{A}$ *such that* $\tilde{\Psi}_{\mathfrak{A}} = \sum_{j=0}^{\infty} q_j e^{i(a_j\rho - 2\pi p^k b_j)}$.

Let the series $\sum_{j=0}^{\infty} q_j = 1$ be s.t. $\sum_{j=0}^{\infty} \sqrt{q_j}$ converges. Then the series

$$\hat{\Psi} = \sum_{(j)} \sqrt{q_j} e^{i(a_j\rho - 2\pi p^k b_j)}$$

(where $j = 0, 1, 2, \ldots$ is enumeration of limit plots) for all $\rho, k$ can be treated as elements of Hilbert space $\ell^2$ of square-summable sequences over $\mathbb{C}$. Recall that any separable Hilbert space is metrically isomorphic to $\ell^2$; e.g., Fourier transform on the circle is such isomorphism between Hilbert space of square-integrable functions on $[0, 1] = \mathbb{I}$ and the space $\ell^2(\mathbb{Z})$ of square-summable sequences over $\mathbb{C}$ enumerated by integers.

# Discreteness+causality imply wavefunction

Let the series $\sum_{j=0}^{\infty} q_j = 1$ be s.t. $\sum_{j=0}^{\infty} \sqrt{q_j}$ converges. Then the series

$$\hat{\Psi} = \sum_{(j)} \sqrt{q_j} e^{i(a_j \rho - 2\pi p^k b_j)}$$

(where $j = 0, 1, 2, \ldots$ is enumeration of limit plots) for all $\rho, k$ can be treated as elements of Hilbert space $\ell^2$ of square-summable sequences over $\mathbb{C}$.

- As $(p^k b) \mod 1$ is a $k$-digit shift of the base-$p$ expansion of $b$, the sequence $(p^k b) \mod 1$, $k = 0, 1, 2, \ldots$, determines the orbit of the map $e^{i2\pi b} \mapsto e^{i2\pi pb}$ which is the sequence of states respective sub-automaton reaches, which, in turn, are represented by torus windings of the torus link that is the limit plot of the affine sub-automaton. The windings may also be treated as phase shifts. Substitute $t = \log_p k$; then

$$\Psi = \sum_{(j)} \sqrt{q_j} e^{i(a_j \rho - 2\pi t b_j)} \approx \hat{\Psi}, \text{ where } \rho, t \in \mathbb{R}.$$

# Discreteness+causality imply wavefunction

- One assigns to the automaton $\mathfrak{A}$ the sum

$$\tilde{\Psi}_{\mathfrak{A}}(\rho, k) = \sum_{a,b \in \mathbb{Z}_p \cap \mathbb{Q}} q_{a,b} e^{i(a\rho - 2\pi p^k b)},$$

where $q_{a,b}$ is a probability assigned to the limit plot of $z \mapsto az + b$.
- This way the automaton $\mathfrak{A}$ defines the wavefunction $\Psi$

$$\Psi = \sum_{a,b \in \mathbb{Z}_p \cap \mathbb{Q}} \sqrt{q_{a,b}} e^{i(a\rho - 2\pi t b)} \approx \hat{\Psi}_{\mathfrak{A}}, \text{ where } \rho, t \in \mathbb{R}.$$

Given a wavefunction $\Psi$, there exists an automaton $\mathfrak{A}$ such that $\Psi \approx \hat{\Psi}_{\mathfrak{A}}$ within any desirable accuracy. This can be stated rigorously in terms of $\beta$-expansions of real numbers: It can be shown that *under the finiteness assumption* the base $\beta$ is necessarily of the form $\beta = 1 + \tau$ where $0 < \tau \ll 1$; moreover, that *$\beta = \sqrt[N]{2}$ for some (large) $N \in \mathbb{N}$ (if, e.g., $\tau$ is a Planck time) and that the respective automaton is equivalent to an automaton with $N$ **binary** inputs/outputs*.

# Concluding remarks

There exists a wide class $\mathcal{C}_p(\mathbb{R})$ of causal functions which are continuous w.r.t. both real and $p$-adic metrics; moreover, w.r.t. the latter metric the functions are 1-Lipschitz, whence they are automata functions. Thus the functions are well defined both for 'discrete' time and for 'continuous' time. Every continuous function $[a, b] \to \mathbb{R}$ can be uniformly approximated w.r.t. real metric by functions from $\mathcal{C}_p(\mathbb{R})$; every automaton function of an automaton having a finite number of states can be uniformly approximated w.r.t. $p$-adic metric by $\mathcal{C}_p(\mathbb{R})$- functions.

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This implies:

- It is impossible for an observer who processes observational data w.r.t. real metric to determine whether the data are values of a causal function from $\mathcal{C}_p(\mathbb{R})$ for some $p$ or not, whether the data are 'purely random' or the data are produced by some 'strictly deterministic' procedure.
- The observer's conclusions solely depend on the observer's 'free will', i.e., on the metric (real or $p$-adic) the observer chooses to process the observational data.

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A finite automaton function maps $\mathbb{Z}_p \cap \mathbb{Q} \to \mathbb{Z}_p \cap \mathbb{Q}$; this implies the following properties of an automaton-based model of a (generally, open) quantum system:

- Pure states correspond to finite affine sub-automata; mixed states correspond to states of the automaton which do not belong to finite affine sub-automata. Collapse of wavefunction means that the automaton reaches a state from a finite affine sub-automata and can be observed. Due to the (inevitably non-zero) observational error, an observer can never determine what is $p$.

# Concluding remarks

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A finite automaton function maps $\mathbb{Z}_p \cap \mathbb{Q} \to \mathbb{Z}_p \cap \mathbb{Q}$; this implies the following properties of an automaton-based model of a (generally, open) quantum system:

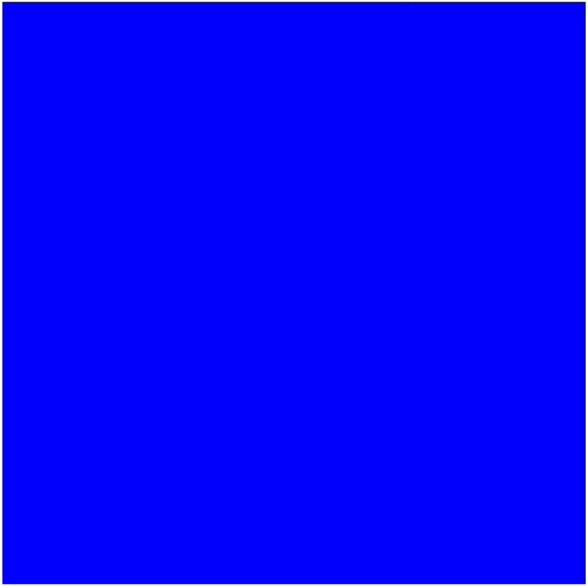
- The 'automaton wavefunction' $\hat{\Psi}_{\mathfrak{A}}$ introduced in the talk can be treated as a 'discrete-time' approximation of a wavefunction $\Psi$.
- The model is more informational rather than physical: It describes how observational data are processed w.r.t. the metric (the $p$-adic or real) which an observer can choose freely.

# Publications

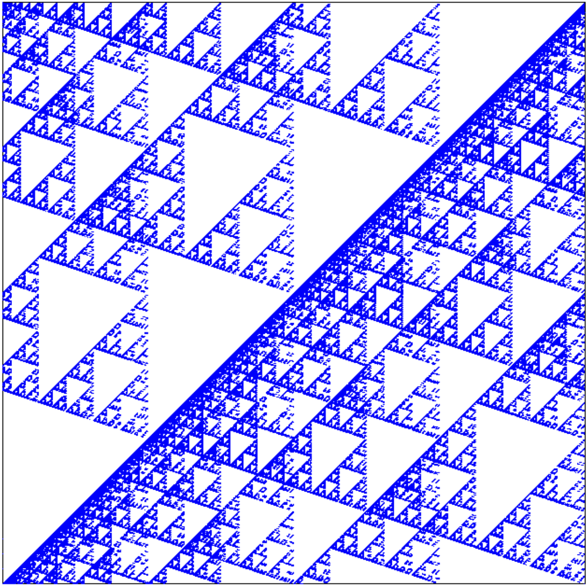

1. Vladimir Anashin. Toward the (non-cellular) automata interpretation of quantum mechanics: Volovich postulates as a roadmap. *International Journal of Modern Physics A*, Vol. 37, No. 20–21, 2243003 (2022)
2. Vladimir Anashin. Discreteness causes waves. *Facta Universitatis*, Ser. *Physics, Chemistry and Technology*, Vol. 14, No. 3, 143–196 (2016).
3. Anashin V.S. Quantization causes waves: Smooth finitely computable functions are affine. $p$*-Adic Numbers, Ultrametric Analysis, and Applications*, Vol. 7, No. 3, 169–227 (2015)
4. Vladimir Anashin. The Non-Archimedean Theory of Discrete Systems, *Mathematics in Computer Science*, Vol. 6, No. 4, . 375–393 (2012)


Website: https://www.researchgate.net/profile/Vladimir-Anashin

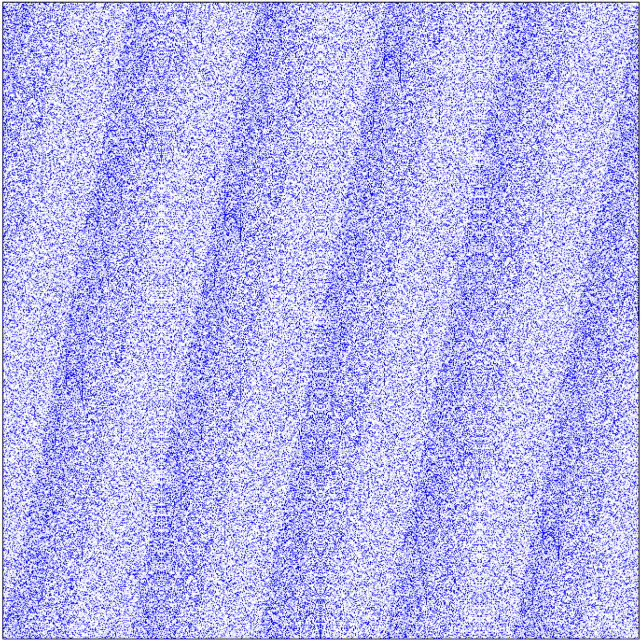
Thank you!



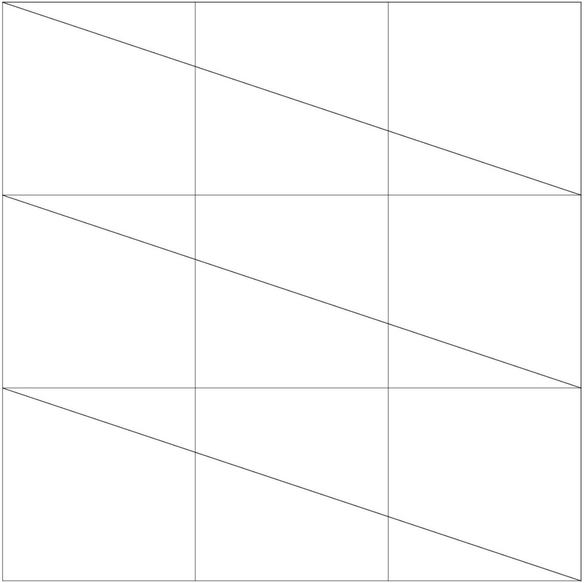
$p = 2$: $f(x) = 1 + x + 4x^2$; $\alpha(f) = 1$.



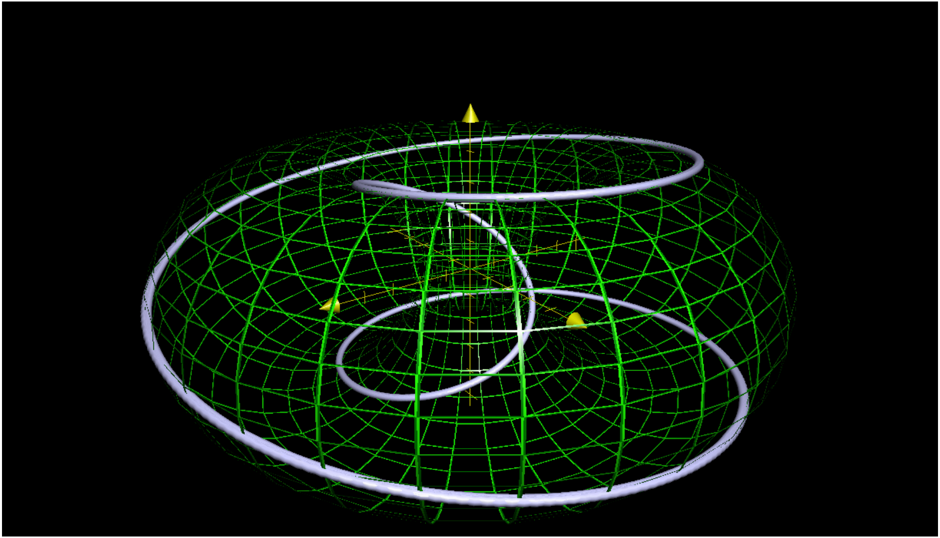
$\alpha(f) = 0.$



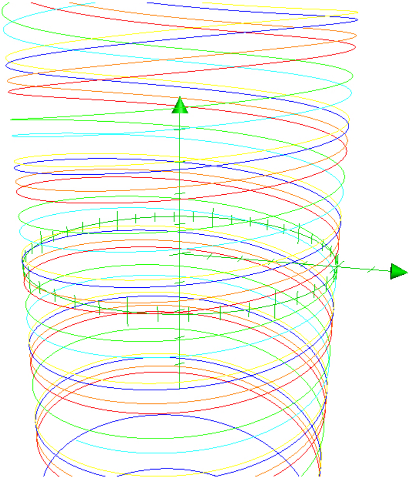
$\alpha(f) = 0.$



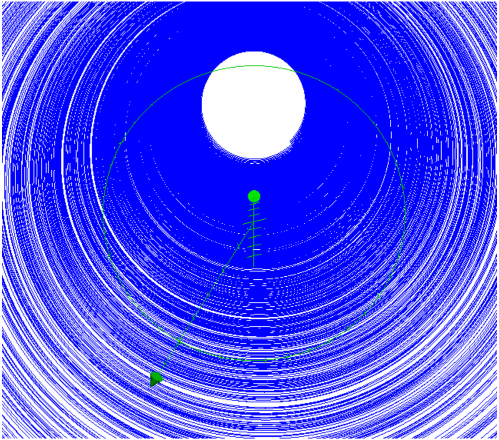
The limit plot $P\hat{\mathcal{P}}^2(f)$ of the automaton function $f(z) = -\frac{1}{3}z$, $(z \in \mathbb{Z}_2)$ in the unit square. Limit plots of the functions $f(z) = -\frac{1}{3}z - 1, f(z) = -\frac{1}{3}z - \frac{1}{3}, f(z) = -\frac{1}{3}z - \frac{2}{3}$ coincide.



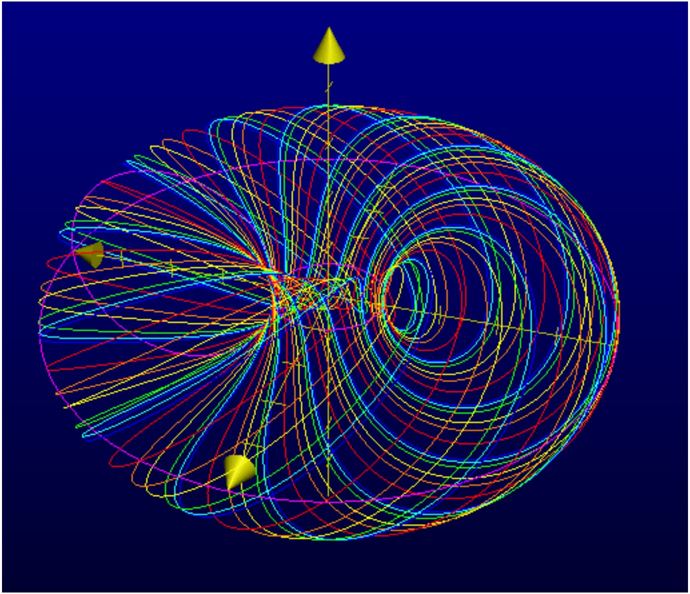
Limit plot of the automaton whose automaton function is $f\colon z \mapsto -\frac{1}{3}z$, $(z \in \mathbb{Z}_2)$, on the torus.



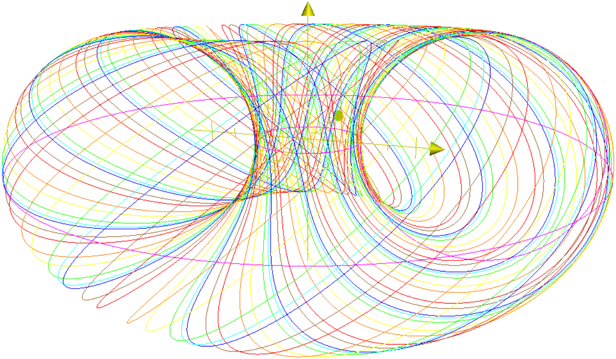
$f(z) = \frac{11}{5}z + \frac{1}{21}$, $z \in \mathbb{Z}_2$. $\mathrm{mult}_2\, 21 = 6$.
Representation by solenoid.

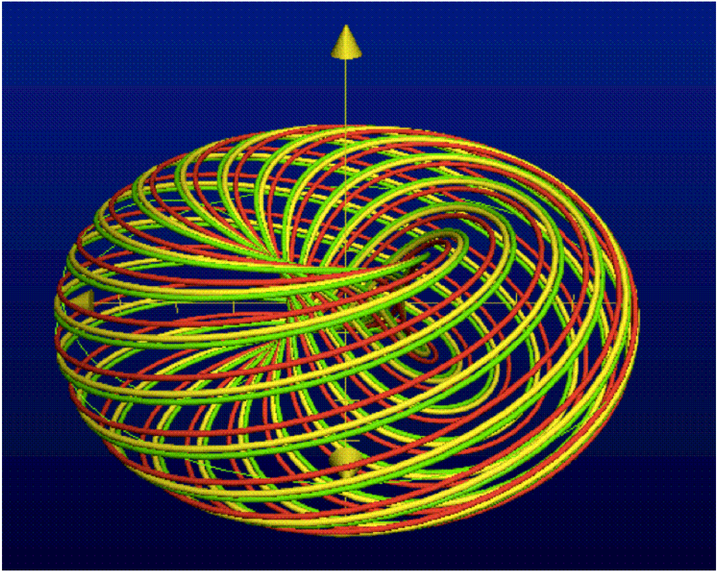


The solenoid limit plot of an automaton having 3 minimal finite affine sub-automata



$f(z) = \frac{11}{5}z + \frac{1}{21}$, $z \in \mathbb{Z}_2$. $\operatorname{mult}_2 21 = 6$. Representation by torus link, 6 windings.





$f(z) = \frac{11}{15}z + \frac{1}{21}$, $p = 2$.

As $\gcd(15, 21) = 3$, the multiplicative order of $2$ modulo $\frac{21}{\gcd(15,21)} = \frac{21}{3} = 7$ is $3$; this is the number of windings in the torus link.

Figure: Limit plot of the constant function $f(z) = \frac{2}{7}$ ($z \in \mathbb{Z}_2$), in $\mathbb{R}^2$

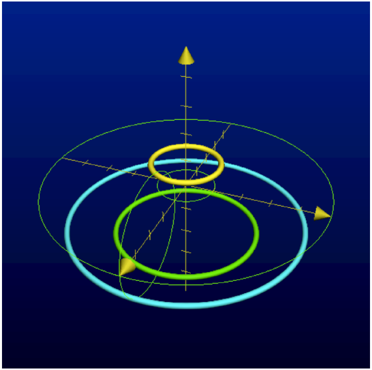


Figure: Limit plot of the same function on the torus $\mathbb{T}^2$