Symmetries and conservation laws of the Liouville equation

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Notation

- $\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z}_+ = \{0, 1, 2, \dots\} \subset \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$;
- $\mathbf{m} = \overline{1, m} = \{1, \dots, m\}, \quad m \in \mathbb{N}$;
- $\mathbb{I} = \{\mathrm{i} = (i^\mu) \in \mathbb{Z}_+^{\mathbf{m}} \mid i^\mu \in \mathbb{Z}_+,\ \mu \in \mathbf{m}\}$;
- $|\mathrm{i}| = i^1 + \dots + i^m$, $\mathrm{i} + (\mu) = (i^1, \dots, i^\mu + 1, \dots, i^m)$ for $\mathrm{i} = (i^1, \dots, i^m) \in \mathbb{I}$;
- $\mathrm{T} = \{t \in \mathbb{R}\}$ is the space of the time variable;
- $\mathrm{X} = \{\mathrm{x} = (x^\mu) \in \mathbb{R}^{\mathbf{m}} \mid x^\mu \in \mathbb{R},\ \mu \in \mathbf{m}\}$ is the space of the spatial variables;
- $\mathrm{U} = \{u \in \mathbb{R}\}$ is the space of the dependent variable, one may consider $u = \phi(t, \mathrm{x})$;
- $\mathbf{U} = \{\mathbf{u} = (u_\mathrm{i}) \in \mathbb{R}_\mathbb{I} \mid u_\mathrm{i} \in \mathbb{R},\ \mathrm{i} \in \mathbb{I}\}$, $u = u_0$, is the space of the differential variables,
  one may consider $u_\mathrm{i} = (\partial_{x^1})^{i^1} \dots (\partial_{x^m})^{i^m} \phi(t, \mathrm{x})$.

The main functional algebras are:

- $\Phi = \mathcal{C}^\infty(\mathrm{T} \times \mathrm{X}; \mathbb{R})$ is the algebra of all smooth real-valued functions on the space $\mathrm{T} \times \mathrm{X}$;
- $\mathcal{F} = \mathcal{C}^\infty(\mathrm{T} \times \mathrm{X} \times \mathrm{U}; \mathbb{R})$ is the algebra of all smooth real-valued functions on the space $\mathrm{T} \times \mathrm{X} \times \mathrm{U}$;
- $\mathcal{A} = \mathcal{C}^\infty_{\mathrm{fin}}(\mathrm{T} \times \mathrm{X} \times \mathbf{U}; \mathbb{R})$ is the algebra of smooth real-valued functions of finite orders on the space $\mathrm{T} \times \mathrm{X} \times \mathbf{U}$.

Remark

Note, the number $r \in \mathbb{Z}_+$ is called the *order* of a function $f(t, \mathrm{x}, \mathbf{u}) \in \mathcal{A}$, $\operatorname{ord} f = r$, if the partial derivative $\partial_{u_\mathrm{i}} f \neq 0$ for some index $\mathrm{i} \in \mathbb{I}$, $|\mathrm{i}| = r$, while the partial derivatives $\partial_{u_\mathrm{i}} f = 0$ for all indexes $\mathrm{i} \in \mathbb{I}$, $|\mathrm{i}| > r$.

The main linear operators are:

- $\mathrm{v} : \mathrm{X} \to \mathrm{X},\ \ \mathrm{x} = (x^\mu) \mapsto \mathrm{v} = v^\mu(\mathrm{x})$,
  is a given vector field, governing the dynamics of the model under study;
- $L = \partial_t + \partial_{\mathrm{x}} \circ \mathrm{v} : \Phi \to \Phi$,
  $\phi(t, \mathrm{x}) \mapsto L\phi(t, \mathrm{x}) = \partial_t \phi(t, \mathrm{x}) + \partial_{x^\mu}(v^\mu(\mathrm{x})\phi(t, \mathrm{x}))$,
  is the Liouville operator;
- $\frac{d}{dt} = \partial_t + \mathrm{v} \cdot \partial_{\mathrm{x}} : \Phi \to \Phi$,
  $\phi(t, \mathrm{x}) \mapsto \frac{d\phi}{dt}(t, \mathrm{x}) = \partial_t \phi(t, \mathrm{x}) + v^\mu \partial_{x^\mu} \phi(t, \mathrm{x})$,
  is the time derivation along the vector field v;

- $L = \frac{d}{dt} + \operatorname{div}\mathrm{v}$, where $\operatorname{div}\mathrm{v} = \partial_{x^\mu} v^\mu(\mathrm{x})$ is the divergence of the vector field v;
- $D_\mu = \partial_{x^\mu} + u_{\mathrm{i}+(\mu)}\partial_{u_\mathrm{i}} : \mathcal{A} \to \mathcal{A}$, $g \mapsto D_\mu g = \partial_{x^\mu} g + u_{\mathrm{i}+(\mu)}\partial_{u_\mathrm{i}} g$, $\mu \in \mathbf{m}$, are the total partial spatial derivations, $[D_\mu, D_\nu] = 0$, $\mu, \nu \in \mathbf{m}$;
- $D_\mathrm{i} = (D_1)^{i^1} \circ \ldots \circ (D_m)^{i^m}, \quad \mathrm{i} = (i^1, \ldots, i^m) \in \mathbb{I}$;
- $D_t = \partial_t + \mathrm{f}_\mathrm{i}\,\partial_{u_\mathrm{i}} : \mathcal{A} \to \mathcal{A}$, $g \mapsto D_t g = \partial_t g + \mathrm{f}_\mathrm{i}\,\partial_{u_\mathrm{i}} g$, is the total time derivation, $\mathrm{f} = -D_\mu\big(v^\mu(\mathrm{x})u\big)$, $\mathrm{f}_\mathrm{i} = D_\mathrm{i}\mathrm{f}$, $\mathrm{i} \in \mathbb{I}$, $[D_\mu, D_t] = 0$, $\mu \in \mathbf{m}$;
- $\mathbf{D}_t = D_t + \mathrm{f}^* = D_t + v^\mu(\mathrm{x})\,D_\mu : \mathcal{A} \to \mathcal{A}$ is the total time derivation along the vector field v;
- $\mathbf{L} = D_t - \mathrm{f}_* = D_t + D_\mu \circ v^\mu(\mathrm{x}) : \mathcal{A} \to \mathcal{A}$ is the total Liouville operator, $\mathbf{L} = \mathbf{D}_t + \operatorname{div}\mathrm{v}$.

The Liouville equation

The Liouville equation

$$L\phi = \partial_t\phi + \partial_{x^\mu}\left(v^\mu \cdot \phi\right) = \frac{d\phi}{dt} + \operatorname{div} \mathrm{v} \cdot \phi = 0, \tag{1}$$

where $\phi(t, \mathrm{x}) \in \Phi$ is a unknown function, $\mathrm{v} : \mathrm{X} \to \mathrm{X}$ is a given vector field.

Theorem

The general solution $\phi(t, \mathrm{x}) \in \Phi$ of the Liouville equation (1) is implicitly defined by the equality

$$K(\varkappa, s^1, \ldots, s^m) = 0,$$

where K is an arbitrary function, while $\varkappa(t, \mathrm{x}, \phi), \mathrm{s} = (s^1(t, \mathrm{x}), \ldots, s^m(t, \mathrm{x}))$ are first integrals of the system of the ordinary differential equations

$$\frac{dt}{1} = \frac{dx^1}{v^1} = \cdots = \frac{dx^m}{v^m} = -\frac{d\phi}{\operatorname{div} \mathrm{v} \cdot \phi}.$$

The situation simplifies when the vector field v is divergence free, the general solution can be found in the explicit form. Indeed, if $\operatorname{div} \mathrm{v} = 0$ then the Liouville equation (1) takes the form

$$L\phi = \frac{d\phi}{dt} = \partial_t \phi + v^\mu \partial_{x^\mu} \phi = 0, \tag{2}$$

and has the general solution $\phi = \rho(s)$, where ρ is an arbitrary function, while $\mathrm{s} = (s^1(t, \mathrm{x}), \dots, s^m(t, \mathrm{x}))$ are first integrals of the shorten system

$$\frac{dt}{1} = \frac{dx^1}{v^1} = \dots = \frac{dx^m}{v^m}.$$

Symmetries

The Lie algebra of symmetries (infinitisemes simmetries, in more detail) of the Liouville equation (1) is defined as

$$\mathrm{Sym}(L) = \big\{\zeta = \mathrm{ev}_g \;\big|\; g \in \mathcal{A},\ \mathbf{L}g = D_t g + D_\mu\big(v^\mu g\big) = 0\big\} \simeq \mathrm{Ker}\,\mathbf{L}, \quad (3)$$

the evolution derivation $\mathrm{ev}_g = D_\mathrm{i} g \cdot \partial_{u_\mathrm{i}}$, the evolution Lie bracket

$$\big[\mathrm{ev}_g, \mathrm{ev}_h\big] = \mathrm{ev}_{\{g,h\}}, \quad \{g,h\} = \mathrm{ev}_g\, h - \mathrm{ev}_h\, g, \quad g, h \in \mathcal{A}.$$

In more detail, the equation $\mathbf{L}g = 0$ is written as $Lg - W_\mathrm{i} \cdot \partial_{u_\mathrm{i}} g = 0$, where

$$Lg = \partial_t g + v^\mu \cdot \partial_{x^\mu} g + \mathrm{div}\,\mathrm{v} \cdot g,$$

$$W_\mathrm{i} = \sum_{\mathrm{j+k=i, j\neq 0}} \binom{\mathrm{i}}{\mathrm{k}} \partial_{x^\mathrm{j}} v^\mu \cdot u_{\mathrm{k}+(\mu)} + \sum_{\mathrm{j+k=i}} \binom{\mathrm{i}}{\mathrm{k}} \partial_{x^\mathrm{j}}\, \mathrm{div}\,\mathrm{v} \cdot u_\mathrm{k},$$

$$\partial_{x^\mathrm{j}} = (\partial_{x^1})^{j^1} \circ \ldots \circ (\partial_{x^m})^{j^m}, \quad \mathrm{j} = (j^1, \ldots, j^m) \in \mathbb{I}.$$

Proposition

The linear operator $\mathbf{L} = L - W_{\mathrm{i}} \cdot \partial_{u^{\mathrm{i}}} : \mathcal{A} \to \mathcal{A}$ *is the order-nonincreasing, i.e.,* $\operatorname{ord}(\mathbf{L}g) \leq \operatorname{ord} g$ *for any* $g \in \mathcal{A}$*.*

Theorem

The equation $\mathbf{L}g = 0$ *has the general solution* $g(t, \mathrm{x}, \mathbf{u}) \in \mathcal{A}$*, implicitly defined by the equality*

$$\mathbf{K}(\mathrm{s}, \mathbf{w}, \varkappa) = 0,$$

where $\mathbf{K}$ *is an arbitrary function of a finite order, while* $\mathrm{s}(t, x)$*,* $\mathbf{w}(t, x, \mathbf{u})$*,* $\varkappa(t, x, \mathbf{u}, g)$*, are the first integrals of the system*

$$\frac{dt}{1} = \frac{dx^{\mu}}{v^{\mu}} = -\frac{du_{\mathrm{i}}}{W_{\mathrm{i}}} = -\frac{dg}{\operatorname{div} \mathrm{v} \cdot g}, \quad \mu \in \mathbf{m},\ \mathrm{i} \in \mathbb{I} \ (\textit{without summation!}),$$

here $\mathrm{i} = (i^{\mu}) \in \mathbb{I}$*,* $\mathbf{u} = (u_{\mathrm{i}}), \mathbf{w} = (w_{\mathrm{i}}) \in \mathbb{R}_{\mathbb{I}}$*.*

Remark

Note, the functions $\mathrm{s}(t,x)$ are solutions of the Liouville equation (2).

Again, the situation simplifies when the vector field v is divergence free. Indeed, if $\operatorname{div}\mathrm{v}=0$ the the equation $\mathbf{L}g=0$ takes the form

$$\mathbf{L}g=\partial_t g+v^\mu\cdot\partial_{x^\mu}g-W_{\mathrm{i}}'\cdot\partial_{u_{\mathrm{i}}}g=0,\quad W_{\mathrm{i}}'=\sum_{\mathrm{j+k=i,j}\neq 0}\binom{\mathrm{i}}{\mathrm{k}}\partial_{x^{\mathrm{j}}}v^\mu\cdot u_{\mathrm{k}+(\mu)},$$

and has the general solution $g=\rho(\mathrm{s},\mathbf{w})$, where ρ is an arbitrary function of a finite order, while $\mathrm{s}(t,x)$, $\mathbf{w}(t,x,\mathbf{u})$ are the first integrals of the shorten system

$$\frac{dt}{1}=\frac{dx^\mu}{v^\mu}=-\frac{du_{\mathrm{i}}}{W_{\mathrm{i}}'}=\frac{du_0}{0},\quad \mu\in\mathbf{m},\ \mathrm{i}\in\mathbb{I}\setminus\{0\}\quad \text{(without summation!)}.$$

Conservation laws

The linear space of conservation laws for the Liouville equation $L\phi = 0$ is defined as the factor-space

$$\begin{aligned} \mathrm{CL}(L) &= \{(\rho, \mathrm{J}) \in \mathcal{A} \times \mathcal{A}^{\mathbf{m}} \mid D_t\rho + \mathrm{Div}\,\mathrm{J} = 0\} \Big/ \{\text{trivial currents}\}, \\ &= \{\rho \in \mathcal{A} \mid D_t\rho \in \mathrm{Div}\,\mathcal{A}^{\mathbf{m}}\} \Big/ \{\rho \in \mathrm{Ker}\,\delta_u\} \\ &= \{\rho \in \mathcal{A} \mid \delta_u D_t\rho = 0\} \Big/ \{\delta_u\rho = 0\}, \end{aligned} \tag{4}$$

where $\mathrm{Div}\,\mathrm{J} = D_\mu J^\mu$, $\mathrm{J} = (J^\mu) \in \mathcal{A}^{\mathbf{m}}$, while the variational derivative

$$\delta_u : \mathcal{A} \to \mathcal{A}, \quad \rho \mapsto \delta_u\rho = (-D)_{\mathrm{i}}\big(\partial_{u_{\mathrm{i}}}\rho\big).$$

Theorem

There is defined the isomorphism of the linear spaces

$$\delta_u : \mathrm{CL}(L) \simeq \{\chi \in \operatorname{Ker} \mathbf{D}_t \mid \chi_* = \chi^*\}, \quad [\rho] \mapsto \chi = \delta_u \rho, \tag{5}$$

where $[\rho] = \rho + \operatorname{Ker} \delta_u$, $\chi \in \mathcal{A}$,

- $\mathbf{D}_t \chi = \left(\partial_t + v^\mu \cdot \partial_{x^\mu} - W'_{\mathrm{i}} \cdot \partial_{u_{\mathrm{i}}}\right)\chi, \quad W'_{\mathrm{i}} = \sum_{\mathrm{j}+\mathrm{k}=\mathrm{i},\mathrm{j}\neq 0} \binom{\mathrm{i}}{\mathrm{k}} \partial_{x^{\mathrm{j}}} v^\mu \cdot u_{\mathrm{k}+(\mu)}$;
- $\chi_* : \mathcal{A} \to \mathcal{A}, \quad g \mapsto \chi_* g = \partial_{u_{\mathrm{i}}} \chi \cdot D_{\mathrm{i}} g$;
- $\chi^* : \mathcal{A} \to \mathcal{A}, \quad g \mapsto \chi^* g = (-D)_{\mathrm{i}} \left(g \cdot \partial_{u_{\mathrm{i}}} \chi\right)$.

Note, the linear operator $\mathbf{D}_t : \mathcal{A} \to \mathcal{A}$ is the order-nonincreasing.

Theorem

The equation $\mathbf{D}_t\chi = 0$ *has the general solution* $\chi = R(\mathrm{s}, \mathbf{w})$, *where* R *is an arbitrary function of a finite order, while* $\mathrm{s}(t, x)$, $\mathbf{w}(t, x, \mathbf{u})$ *are the first integrals of the system*

$$\frac{dt}{1} = \frac{dx^\mu}{v^\mu} = -\frac{du_{\mathrm{i}}}{W'_{\mathrm{i}}} = \frac{du_0}{0}, \quad \mu \in \mathbf{m},\ \mathrm{i} \in \mathbb{I} \setminus \{0\} \quad \textit{(without summation!)},$$

here $\mathrm{i} = (i^\mu) \in \mathbb{I}$, $\mathbf{u} = (u_{\mathrm{i}}), \mathbf{w} = (w_{\mathrm{i}}) \in \mathbb{R}_{\mathbb{I}}$.

Remark

Note, $\mathbf{D}_t = \mathbf{L}|_{\mathrm{div}\,\mathrm{v}=0}$.

Examples.

Example

$\mathrm{x} = (x^1, x^2) \in \mathrm{X} = \mathbb{R}^2, \quad x^1 = q, \quad x^2 = p, \quad H = \frac{p^2}{2m},$
$\mathrm{v} = (v^1, v^2), \quad v^1 = \partial_{x^2} H = \frac{p}{m}, \quad v^2 = -\partial_{x^1} H = 0, \qquad \operatorname{div} \mathrm{v} = 0;$

$$L = \frac{d}{dt} = \partial_t + \frac{p}{m}\partial_q, \quad \mathbf{L} = \mathbf{D}_t = L - W_{j,k} \cdot \partial_{u_{j,k}}, \quad W_{j,k} = \frac{k}{m} u_{j+1,k-1}.$$

The equation $\mathbf{D}_t \chi = 0$ has the general solution $\chi = R(r, s, \mathbf{w})$, where $R(r, s, \mathbf{w})$ is an arbitrary function of a finite order, while $r(t, q, p)$, $s(t, q, p)$, $\mathbf{w}(t, q, p, \mathbf{u})$ are the first integrals of the system

$$\frac{dt}{1} = \frac{m\,dq}{p} = \frac{dp}{0} = -\frac{du_{j,k}}{W_{j,k}}, \quad j, k \in \mathbb{Z}_+ \quad \text{(without summation!)},$$

here $\mathrm{i} = (j, k) \in \mathbb{I} = \mathbb{Z}_+ \times \mathbb{Z}_+$, $\mathbf{u} = (u_{j,k})$, $\mathbf{w} = (w_{j,k})$.

Proposition

In the above settings, the first integrals $r, s, \mathbf{w}$ *are:*

- $r(t,q,p) = q - \tau p, \quad s(t,q,p) = p \quad \tau = \frac{t}{m}$;
- $w_{j,k}(t,q,p,\mathbf{u}) = \sum_{0\le n\le k} \binom{k}{n} u_{j+n,k-n}\tau^n$.

Corollary

The Lie algebra $\mathrm{Sym}(L) \simeq \mathrm{Ker}\,\mathbf{D}_t$.

Corollary

The linear space $\mathrm{CL}(L) \simeq \{\chi \in \mathrm{Ker}\,\mathbf{D}_t \mid \chi_* = \chi^*\}$. *Note,*

$$r_* = r^* = 0, \quad s_* = s^* = 0, \quad (w_{j,k})_* = (-1)^{j+k}(w_{j,k})^*.$$

Remark

One may say that $\mathrm{CL}(L) \subset \mathrm{Sym}(L)$.

Example

$\mathrm{x} = (x^1, x^2) \in \mathrm{X} = \mathbb{R}^2, \quad x^1 = q, \quad x^2 = p, \quad H = \frac{p^2}{2m} + \frac{\varkappa q^2}{2}.$
$\partial_{x^2} H = \frac{p}{m}, \quad \partial_{x^1} H = \varkappa q, \quad \mathrm{v} = (\frac{p}{m}, -\varkappa q), \quad \operatorname{div} \mathrm{v} = 0.$

$$L = \frac{d}{dt} = \partial_t + \frac{p}{m}\partial_q - \varkappa q\, \partial_p, \quad \mathbf{L} = \mathbf{D}_t = L - W_{j,k}\, \partial_{u_{j,k}},$$

where $\mathrm{i} = (j,k) \in \mathbb{I} = \mathbb{Z}_+ \times \mathbb{Z}_+$, $W_{j,k} = -j\varkappa u_{j-1,k+1} + \frac{k}{m} u_{j+1,k-1}$, $j,k \in \mathbb{Z}_+$. The equation $\mathbf{D}_t \chi = 0$ has the general solution $\chi = R(r, s, \mathbf{w})$, where R is an arbitrary function of a finite order, while $r(t,q,p)$, $s(t,q,p)$, $\mathbf{w}(t,q,p,\mathbf{u})$ are the first integrals of the system

$$\frac{dt}{1} = \frac{m\, dq}{p} = -\frac{dp}{\varkappa q} = -\frac{du_{j,k}}{W_{j,k}}, \quad j,k \in \mathbb{Z}_+ \quad \text{(without summation!)},$$

$\mathrm{i} = (j,k) \in \mathbb{I} = \mathbb{Z}_+ \times \mathbb{Z}_+$, $\mathbf{u} = (u_{j,k})$, $\mathbf{w} = (w_{j,k})$.

In the above setting, the first integrals r, s are

- $r = q\cos\omega t - p\frac{1}{m\omega}\sin\omega t, \quad s = qm\omega\sin\omega t + p\cos\omega t;$
- $\omega = \sqrt{\frac{\varkappa}{m}}, \quad m\omega = \sqrt{m\varkappa}.$

The first integrals $w_{j,k}$ are defined by the equations

$$-j\varkappa u_{j-1,k+1} + \dot{u}_{j,k} + \frac{k}{m}u_{j+1,k-1} = 0, \quad j,k \in \mathbb{Z}_+, \quad \dot{u} = \frac{du(t)}{dt}.$$

Simple but tedious calculations give:
(0) $j + k = 0$

- $u_{0,0} = w_{0,0}, \quad w_{0,0} = u_{0,0};$

(1) $j + k = 1$

- $u_{0,1} = w_{0,1}\cos\omega t + w_{1,0}\sin\omega t;$
- $u_{1,0} = m\omega\big(w_{0,1}\sin\omega t - w_{1,0}\cos\omega t\big);$

hence

- $w_{0,1} = u_{0,1}\cos\omega t + \frac{1}{m\omega}u_{1,0}\sin\omega t;$
- $w_{1,0} = u_{0,1}\sin\omega t - \frac{1}{m\omega}u_{1,0}\cos\omega t;$

(2) $j+k=2$

- $u_{0,2} = w_{0,2} + w_{1,1}\cos 2\omega t + w_{2,0}\sin 2\omega t$;
- $u_{1,1} = m\omega\big(w_{1,1}\sin 2\omega t - w_{2,0}\cos 2\omega t\big)$;
- $u_{2,0} = (m\omega)^2\big(w_{0,2} - w_{1,1}\cos 2\omega t - w_{2,0}\sin 2\omega t\big)$;

hence

- $w_{0,2} = \frac{1}{2}\big(u_{0,2} + \frac{1}{(m\omega)^2}u_{2,0}\big)$;
- $w_{1,1} = \frac{1}{2}\big(u_{0,2} - \frac{1}{(m\omega)^2}u_{2,0}\big)\cos 2\omega t + \frac{1}{m\omega}u_{1,1}\sin 2\omega t$;
- $w_{2,0} = \frac{1}{2}\big((u_{0,2} - \frac{1}{(m\omega)^2}u_{2,0}\big)\sin 2\omega t - \frac{1}{m\omega}u_{1,1}\cos 2\omega t$;

(3) $j+k=3$

- $u_{0,3} = w_{0,3}\cos\omega t + w_{1,2}\sin\omega t + w_{2,1}\cos 3\omega t + w_{3,0}\sin 3\omega t;$
- $u_{1,2} = \frac{m\omega}{3}\big(w_{0,3}\sin\omega t - w_{1,2}\cos\omega t + 3w_{2,1}\sin 3\omega t - 3w_{3,0}\cos 3\omega t\big);$
- $u_{2,1} = \frac{(m\omega)^2}{3}\big(w_{0,3}\cos\omega t + w_{1,2}\sin\omega t - 3w_{2,1}\cos 3\omega t - 3w_{3,0}\sin 3\omega t\big);$
- $u_{3,0} = (m\omega)^3\big(w_{0,3}\sin\omega t - w_{1,2}\cos\omega t - w_{2,1}\sin 3\omega t + w_{3,0}\cos 3\omega t\big);$

hence

- $w_{0,3} = \frac{3}{4}\big(u_{0,3} + \frac{1}{(m\omega)^2}u_{2,1}\big)\cos\omega t + \frac{3}{4m\omega}\big(u_{1,2} + \frac{1}{(m\omega)^2}u_{3,0}\big)\sin\omega t;$
- $w_{1,2} = \frac{3}{4}\big(u_{0,3} + \frac{1}{(m\omega)^2}u_{2,1}\big)\sin\omega t - \frac{3}{4m\omega}\big(u_{1,2} + \frac{1}{(m\omega)^2}u_{3,0}\big)\cos\omega t;$
- $w_{2,1} = \frac{1}{4}\big(u_{0,3} - \frac{3}{(m\omega)^2}u_{2,1}\big)\cos 3\omega t + \frac{1}{4m\omega}\big(3u_{1,2} - \frac{1}{(m\omega)^2}u_{3,0}\big)\sin 3\omega t;$
- $w_{3,0} = \frac{1}{4}\big(u_{0,3} - \frac{3}{(m\omega)^2}u_{2,1}\big)\sin 3\omega t - \frac{1}{4m\omega}\big(3u_{1,2} - \frac{1}{(m\omega)^2}u_{3,0}\big)\cos 3\omega t;$

and so on, for $j+k=4,5,\ldots$.

Again, the statements from the previous example are true. Namely:

- the Lie algebra $\mathrm{Sym}(L) \simeq \mathrm{Ker}\, \mathbf{D}_t$;
- the linear space $\mathrm{CL}(L) \simeq \{\chi \in \mathrm{Ker}\, \mathbf{D}_t \mid \chi_* = \chi^*\}$;
- $r_* = r^* = 0, \quad s_* = s^* = 0, \quad (w_{j,k})_* = (-1)^{j+k}(w_{j,k})^*$.

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THANK YOU