Some results from the general theory of boundary value problems for PDEs

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0. What will be discussed below:

My communication is devoted to some questions of a general theory of boundary value problems for partial differential equations and contains a description of the basic constructions of the theory and subsequent results of the author. The available methods for studying boundary value problems are closely related to the type of equation and are not suitable for a more or less general or non-classical formulations, but the general theory of boundary value problems for PDEs considers equations regardless of the type of equation. Basic questions that it asks are:

What is a boundary value problem for a general linear differential equation?

Which boundary value problem is well-posed and for which equations there is at least one well-posed boundary value problem?

How to describe the set of all well-posed boundary value problems?

How to come up with research methods of a general boundary value problem for some specific, maybe general PDE?

And, of course, how find solutions of such problem?

1. Constructions of general theory of boundary value problems for PDEs.

We call to mind general facts about extensions of the differential operator and boundary value problems in the domain.

Let Ω be a bounded domain with the boundary $\partial\Omega$ in the space $\mathbb{R}^n$,

$$\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha, D^\alpha = (-i\partial)^{|\alpha|}/\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}, \alpha \in \mathbb{Z}_+^n, |\alpha| = \sum_k \alpha_k$$

be a differential operation with complex coefficients from the space $C^\infty(\Omega)$,

$$\mathcal{L}^+ = \sum_{|\alpha|\le m} D^\alpha(a_\alpha^*(x)\cdot),\ a_\alpha^* = \overline{a_\alpha}$$

be the formally adjoint differential operation. The closing of the operator, which is given on the space $C_0^\infty(\Omega)$ by means of the operation $\mathcal{L}$, in the norm of the graph

$$\|u\|_L^2 = \|u\|_{L_2(\Omega)}^2 + \|\mathcal{L}u\|_{L_2(\Omega)}^2$$

is called the minimal expansion of the operator $\mathcal{L}$ in the space $L_2(\Omega)$ or simply **the minimal operator** L_0. Its domain is $D(L_0)$.

The contraction of the operator, which is generated by the operation $\mathcal{L}$ in the space $\mathcal{D}'(\Omega)$, to the domain of the definition $D(L) = \{u \in L_2(\Omega) | \mathcal{L}u \in L_2(\Omega)\}$, $L = \mathcal{L}|_{D(L)}$ is said to be the maximal expansion of the operator $\mathcal{L}|_{C_0^\infty(\overline{\Omega})}$ or simply **the maximal operator** L. $D(L)$ is the Hilbert space with scalar product of the norm $\| \ \|_L$ as well as his close subspace $D(L_0)$. The kernel $\ker L$ is closed in the spaces $D(L)$ and $L_2(\Omega)$, the kernel $\ker L_0$ is closed in the spaces $D(L)$ and $\ker L$. Another expansion of the operator $\mathcal{L}|_{C_0^\infty(\overline{\Omega})}$, which we define $\tilde{L}$. This is the operator with the definition domain $D(\tilde{L})$, which is the closing of the space $C^\infty(\overline{\Omega})$ in the norm of the graph $\| \cdot \|_L$.
We consider the following conditions:

$$\text{the operator } L_0 : D(L_0) \to L_2(\Omega) \text{ has the continuous left-inverse;} \tag{1}$$

$$\text{the operator } L_0^+ : D(L_0^+) \to L_2(\Omega) \text{ has the continuous left-inverse;} \tag{2}$$

$$\tilde{L} = (L_0^+)^*; \tag{3}$$

$$\tilde{L}^+ = (L_0)^*. \tag{4}$$

It is well known that $L = (L_0^+)^*$ and $L^+ = (L_0)^*$, so that the conditions (3),(4) mean the equalities $D(L) = D(\tilde{L})$, $D(L^+) = D(\tilde{L}^+)$, i.e. the possibility to approximate each fuunction from $D(L)$ or $D(L^+)$ by functions from $C^\infty(\overline{\Omega})$. The conditions (1), (2) imply respectively the conditions: $\ker L_0 = 0, \ker L_0^+ = 0$.
We define **the boundary space** $C(L)$ as $D(L)/D(L_0)$.
The homogeneous linear boundary value problem is by definition the problem to find the solution $u \in D(L)$ of the relations

$$Lu = f,\ \Gamma u \in B, \tag{5}$$

where $\Gamma : D(L) \to C(L)$ is the map of the factorization, B is a linear set in $C(L)$. The boundary value condition $\Gamma u \in B$ generates the subspace $D(L_B) = \Gamma^{-1}(B)$ of the space $D(L)$ and the operator L_B, which is the contraction of the operator L on the space $D(L_B)$ and which is **an expansion of the operator** L_0. This operator L_B is closed if and only if the linear space B is closed in $C(L)$ or the space L_B is closed in $D(L)$ [2]. The boundary value problem (5) is called **correct** (or **well-posed**) and the operator L_B is called **solvable expansion** of the operator L_0 if the operator $L_B : D(L_B) \to L_2(\Omega)$ has a two-sided inverse operator.

Statement 1. (M.Yo.Vishik). *There exist a solvable expansion of the operator L_0 and there exists a well-posed boundary value problem for the equation $Lu = f$ if and only if the conditions (1) and (2) are fulfilled.*

Note that the same is thrue for the equation $L^+ u = f$.
We consider the following conditions too:

$$\text{the operator } L : D(L) \to L_2(\Omega) \text{ is surjective;} \tag{6}$$

$$\text{the operator } L^+ : D(L^+) \to L_2(\Omega) \text{ is surjective;} \tag{7}$$

$$\text{the operator } L_0 : D(L_0) \to L_2(\Omega) \text{ is normally solvable.} \tag{8}$$

Statement 2. (well known). *The Vishik condition (1) is equivalent to the condition (7) and the Vishik condition (2) is equivalent to the condition (6).*

We collect the introduced objects and morphisms into a diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & D(L_0) & \xrightarrow{L_0} & \mathrm{Im}\ L_0 & \to & 0 \\
 & & \downarrow & & \downarrow i_0 & & \downarrow i_{\mathrm{Im}} & & \\
0 & \to & \ker L & \xrightarrow{i_L} & D(L) & \xrightarrow{L} & H & \to & 0 \quad (\mathrm{D}) \\
 & & \downarrow \Gamma_{\ker} & & \downarrow \Gamma & & \downarrow \Gamma_{\mathrm{Im}} & & \\
0 & \to & C(\ker L) & \xrightarrow{i_C} & C(L) & \xrightarrow{L_C} & \ker L^+ & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Statement 3. *The diagram (D) is commutative, its rows and columns are exact.*

The meaning of the diagram is that **the maximum operator** L **decomposes into the direct sum of its inner part – the minimum operator** L_0 **and the boundary part – the operator** L_C:

$$L = L_0 \oplus L_C.$$

From here we immediately obtain

Statement 4. (M.Yo.Vishik) *Under conditions (1) and (2):*
1). There is the decomposition in the direct sum of subspaces

$$D(L) = D(L_0) \oplus \ker L \oplus W,$$

where $L : W \to \ker L^+ -$ *isomorphism.*
2). Each well-posed boundary value problem for the equation $Lu = f$ *generates and is generated by a linear continuous operator* $Q : \ker L^+ \to \ker L$.

This operator Q determines the subspace $B \subset C(L)$ by the formula $B := \{(v, w) \in C(L) = \ker L \oplus \ker L^+ \mid v = Qw\}$. Vice versa, if a subspace $B \subset C(L)$ is given, which defines a solvable expansion L_B, then there is a continuous inverse operator L_B^{-1}, then the composition of the restriction $L_B^{-1}|_{\ker L^+}$ and the projector $P_{C(\ker L)}$ from the direct sum on its term $C(\ker L)$ will give us the operator Q. In particular, Vishik's famous description of all well-posed boundary value problems for the Poisson equation ([1]) follows directly from this statement 4.

2. Operator classes with basic conditions (1)-(4). We start with the example of the operator, which have no correct boundary value problems. Since $\ker \mathcal{L}|_{C_0^\infty(\Omega)} \subset \ker L_0$, the conditions

$$\ker \mathcal{L}|_{C_0^\infty(\Omega)} = \ker \mathcal{L}^+|_{C_0^\infty(\Omega)} = 0 \qquad (ker)$$

are necessary for the fulfilment of the conditions (1),(2). The statement of the theorem 13.6.15 from Hörmander's book [4], which belong to A.Plis, give the example of the elliptic operator, for which the condition (ker) is not fulfilled. The following is correct.

Proposition 1. *There exists an elliptic operator $\mathcal{L}$ of order 4 with C^∞-coefficients in $\mathbb{R}^3$, which have none well-posed boundary value problem.*

Remind that the differential operator $\mathcal{L} = \mathcal{L}(x, D)$ has the constant strenght in the domain $\overline{\Omega}$, if

$$\forall x \in \overline{\Omega}, \forall y \in \overline{\Omega}, \exists C > 0, \forall \xi \in \mathbb{R}^n, \tilde{l}(x,\xi)/\tilde{l}(y,\xi) \le C,$$

where $\tilde{l}(x,\xi)^2 = \sum\limits_{|\alpha|\le m} |D_\xi^\alpha \mathcal{L}(x,\xi)|^2$, and that a operator with constant coefficients $P_1(D)$ is weaker than the same $P_2(D)$, if $\tilde{P}_1(\xi)/\tilde{P}_2(\xi) \le C$.

Conditions (1),(2) are satisfied by the following types of operators

Statement 5. *Let Ω be a bounded domain Ω with the smooth boundary. The Vishik conditions (1),(2) are satisfied and generalized Dirichlet problem for the equation is well-posed and the generalized Neumann problem for the same equation is normally correct if the operator $\mathcal{L}$ is one of indicated below:*

1) *$\mathcal{L}$ is a scalar operator with constant coefficients;*

2) *$\mathcal{L}$ is an operator of the real principal type ($\nabla_\xi P_0(\xi) \neq 0$) of the form*

$$\mathcal{L} = P_0(D) + \sum_{j=1}^{N} c_j(x) P_j(D), \tag{9}$$

where P_j are operators of orders less then $m = \deg P_0$;

3) *$\mathcal{L}$ is an operator of the constant force of the form* (9) *with analytical in the domain $\Omega' \supset \overline{\Omega}$ coefficients, where P_j are operators with constant coefficients of force less then of the operator P_0,*

4) *$\mathcal{L}$ is a matrix operator with constant coefficients satisfying the condition of Panejach-Fuglede.*

5) *$\mathcal{L}$ is a matrix operator, properly elliptic by Douglis-Nirenberg.*

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Statement 6. *As for conditions (3), (4), they are satisfied:*

1) for every scalar linear uniformly elliptic operator in any domain with the cone condition;

2) for a hypoelliptic operator with constant coefficients in any domain;

3) for any scalar differential operator with constant coefficients in a domain with property T by L. Hörmander: Ω is a bounded domain and there exists a finite cover of Ω by open sets O_i such that $\forall i = 1, ..., N, \forall \epsilon > 0, \exists t \in \mathbb{R}^n : |t| < \epsilon, \bar{\Omega} \cap O_i + t \in \Omega$.

4) for the same operator with a possible addition in the form of any differential operator with smooth coefficients of the first order in a domain "normal" in the sense of A.A. Dezin.

5) for any scalar differential operator of real principal type in any domain compactly embedded in the original domain Ω.

3. Traces of L_2-solution on the boundary. Let us for general operator $L = L(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ consider the Green's formula

$$\int_\Omega (Lu\,\overline{v} - u\,\overline{L^+ v})\,dx = \sum_{q=0}^{m-1} \int_{\partial\Omega} L_{(m-q-1)} u\,\overline{\partial_\nu^q v}\,ds, \qquad (10)$$

where L^+ is the formally adjoint operator, $L_{(j)}\,u$ is the linear differential expression of the order j, $j = 0, ..., m-1$. The expressions $L_{(j)}\,u$ are generated by transferring derivatives and which for smooth function u one can count up if functions $\psi_j = \partial u/\partial\nu|_{\partial\Omega}$ are known. These L-traces $L_{(j)}\,u$ exist for any function from the domain of maximal expansion $D(\tilde{L})$:

Statement 7. *For any function $u \in D(\tilde{L})$ exist there generalized functions $L_{(j)}\,u \in H^{-j-1/2}(\partial\Omega)$ such that the Green's formula (10) is fulfilled for $u \in D(\tilde{L})$, $v \in H^m(\Omega)$.*

Let's call these functions $L_{(j)}\,u \in H^{-j-1/2}(\partial\Omega)$ **L-traces of function u on the boundary** $\partial\Omega$. The L-traces help characterize the space $D(L_0)$:

Statement 8. *Each element $u \in D(\tilde{L})$ has trivial L-traces $L_k u = 0,\ k = 0, 1, ..., m-1$ if and only if the function u belongs to $D(L_0)$.*

L-traces help also characterize a general boundary value problem:

Statement 9. *Under conditions (2),(4) any boundary value problem $\Gamma u \in B \subset C(L)$ for the equation $Lu = f$ in the space $L_2(\Omega)$ can be written as*

$$B_i^0 L_{(0)} u + B_i^1 L_{(1)} u + ... + B_i^{l-1} L_{(l-1)} u = 0, \quad i = 1, ..., l \quad (11)$$

where $B_i^k : H^{-k-1/2}(\partial\Omega) \to H^{-i-1/2}(\partial\Omega)$ are some linear continuous operators.

L-traces help also to obtain propositions on boundary properties of solutions, on comparison of differential operators, on fundamental solutions. For example, the statements are true:

Statement 10. *An operator $\mathcal{P} = P_0(D) + \sum_{j=1}^{N} c_j(x) P_j(D)$ of constant force with analytical coefficients has a fundamental solution in the domain Ω if $\left(\sum_{\alpha \le m} |D_\xi^\alpha P_0(\xi)|^2\right)^{-1/2} \in L_2(\mathbb{R}^n)$.*

Here a fundamental solution for the equation with smooth coefficients is a function $\mathcal{E}_y(x)$, $x, y \in \Omega$, for which $\mathcal{L}_x\mathcal{E}_y(x) = \delta(y - x)$.

4. Generalized solutions. On the path above, it is possible to build the theory of generalized solutions to the Dirichlet, Neumann, other boundary value problems for equations of the form

$$\mathcal{L}^+\mathcal{L}u = f, \tag{12}$$

note that the Poisson equation $-\Delta u = f$ can be written as $-div\nabla u = f$, $-div = \nabla^*$, and also the theory for equations

$$\mathcal{L}^+A\mathcal{L}u = f$$

with a linear or nonlinear operator $A : L_2(\Omega) \to L_2(\Omega)$.
And on this basis I gave a description of the way to find the unique solution of the general boundary value problem (5) for the equation $Lu = f \in L_2(\Omega)$. In particular, it is prove to be that **the generalized Dirichlet problem for equation (12) is well-posed if and only if the Vishik condition (1) is fulfilled.**

Besides, it was possible to find a criterion of well-posedness of a general differential boundary value problem

$$B_j u = \sum_{|\beta_j| \leq m_j} b_{j\beta}(x) D^\beta u = g_j, \; j = 1, ..., \kappa. \tag{13}$$

for nonproper elliptic equation $Lu = f$ in bounded domain $\Omega \in \mathbb{R}^2$. I call in mind that a boundary value problem (13) for proper elliptic equation $Lu = f$ has Fredfolm property iff it satisfies the condition by Ya.B. Lopatinsky, which is named sometimes the covering condition.

It is proved that for an improper elliptic equation $Lu = f$ in a bounded domain $\Omega \in \mathbb{R}^2$ differential boundary value problem (13) satisfies the Fredvolm condition if and only if two conditions of Ya.B. Lopatinsky are fulfilled: one for equation (12) and one more equation $\mathcal{L}\mathcal{L}^+ v = g$ with the corresponding boundary value problems generated by problem (13).

5. Applicaions to investigation of specific tasks. Another important application of L-traces is the study boundary value problems based on the connection condition solution traces. Indeed, traces of a smooth solution of, for example, the equation $Lu = 0$ can not arbitrary and they are somehow connected to each other. If we consider the Cauchy problem

$$Lu = 0, \quad u|_{\partial\Omega} = \psi_0, u'_\nu|_{\partial\Omega} = \psi_1, ..., u_\nu^{(m-1)}|_{\partial\Omega} = \psi_{m-1}.$$

Consider for smooth functions u and v the Green's formula

$$\int\limits_\Omega (Lu\,\overline{v} - u\,\overline{L^+ v})\;dx = \sum_{q=0}^{m-1} \int\limits_{\partial\Omega} \; L_{(m-q-1)} u \;\overline{\partial_\nu^q v}\; ds,$$

where L^+ is a formally adjoint operator, $L_{(j)}\,u$ is a linear differential expression of the order j, which one can count up if functions ψ_j are known. This Green's formula can be extanded on the case $u \in D(\tilde{L})$, $v \in H^m(\Omega)$. These L-traces $L_{(j)}\,u$ exist for any function from the domain of maximal expansion $D(\tilde{L})$. It allows us to continue construct a general theory of boundary value problems.

One can obtain a necessary condition of the connection of the traces ψ_j of the solution $u \in H^m(\Omega)$ from the Green's formula Let us take the functions u as a solution of the equation $Lu = f$ and v as a solution of the equation $L^+ v = 0$. Then the integral from the left vanishes and we have condition: $\forall v \in \ker L^+$

$$\sum_{q=0}^{m-1} \int_{\partial\Omega} L_{(m-q-1)} u \, \overline{\partial_\nu^q v} \, ds = 0. \tag{14}$$

This condition proves to be necessary and sufficient if the domain $D(L)$ of the maximal operator L is the closing of the space $C^\infty(\overline{\Omega})$ in the norm of the graph (i.e. the above condition (3)). The last is fulfilled, in particular, for any operator with constant coefficients in each good domain. The obtained condition allows us to study some general well-posed and ill-posed boundary value problems. The application of this methods gives, in particular, the following results.

Boundary value problems for the equation

$$L(D)\,u = (b^1\cdot\nabla)\,(b^2\cdot\nabla)\,u = 0 \qquad (15)$$

with constant complex vectors b^1, b^2 in a bounded plane domain $\Omega \in \mathbb{R}^2$ are closely connected with the following moments problem $(j = 1, 2)$, where for given domain Ω, two vectors $\tilde{b}^1, \tilde{b}^2$ and two given sequvences μ_1^N, μ_2^N one find the function α:

$$\forall N \in \mathbf{Z}_+, \int\limits_{\partial\Omega} \alpha(s)\,(\widetilde{b}^j\cdot x(s))^N\, ds = \mu_j^N,\ \widetilde{b}^j = (-b_2^j, b_1^j). \qquad (16)$$

The uniqueness problem for this moment problem is: Is there a non-trivial function α such that for $j = 1, 2;\ \forall N \in \mathbf{Z}_+$,

$$\int\limits_{\partial\Omega} \alpha(x)\,(\widetilde{b}^j\cdot x(s))^N\, ds = 0 \text{ for given vectors } \widetilde{b}^1,\ \widetilde{b}^2 \in \mathbb{C}^2 \qquad (17)$$

It is proved to be the problem (17) is equivalent to the problem of uniqueness solution of some class of boundary value problems in Ω for the equation (15).

The following fact takes place

Statement 11. *Let $m \geq k \geq 3$ and let we have three sets of statements:*

1_m*) The homogeneous moment problem (17) has a nontrivial solution* $\alpha \in H^{m-3/2}(\partial\Omega)$.

2_k*) The Dirichlet problem* $u|_{\partial\Omega} = 0$ *for the equation* (15) *has a nontrivial solution* $u \in H^k(\Omega)$.

3_k*) The Neumann problem* $u'_{\nu_*}|_{\partial\Omega} = 0$ *for the equation* (15) *has a nonconstant solution* $u \in H^k(\Omega)$.

Then $1_m) \Rightarrow 2_{m-q})$; $1_m) \Rightarrow 3_{m-q})$; $2_m) \Rightarrow 1_m)$; $3_m) \Rightarrow 1_m)$ *with* $q = 0$ *for elliptic case,* $q = 1 + 0$ *for hyperbolic and mixed cases. (By definition, for bounded domain* $H^{k+0}(\Omega) = \bigcup_{\epsilon>0} H^{k+\epsilon}(\Omega)$*).*

If the curve $\partial\Omega$ is the unit circle then we have a criterion of indeterminacy breakdown. Find slope angles φ_1, φ_2 of characteristics ($tg\varphi_1 = b_2^1/b_1^1$ and $tg\varphi_2 = b_2^2/b_1^2$) and an angle $\varphi_0 = \varphi_1 - \varphi_2$ between them.

Statement 12. *The problem (17) has a nontrivial solution in the circle ∂K (K-unit disk) in a space $H^k(K), k \geq 2$ if and only if $\varphi_0 \in \mathbb{R}$ and*

$$\varphi_0/\pi \in \mathbb{Q}. \tag{18}$$

Therefore the last condition is the criterium that the Dirichlet problem $u|_{\partial K} = 0$ for the equation (15) has only zero solution.
Statement 13. *If the condition (18) is fulfilled then there is a denumerable set of linear independent polynomial solutions of the problem (17), a therefore the same has its place for the Dirichlet and the Neumann problems with the equation* (15).

If $M_l^j = \{\alpha \in H^l(\partial\Omega) |\ (17)\}$ are subspaces of the Sobolev space $H^l(\partial\Omega)$ then the Dirichlet problem is well-posed in elliptic case

$$(\text{i.e. } ||u||_{H^{l+1/2}(\Omega)} \leq C||\psi||_{H^l(\partial\Omega)})$$

if and only if $H^l(\partial\Omega) = M_l^1 \oplus M_l^2$. The same is valid for the Neumann problem

$$u'_{\nu_*} = \chi,$$

(u'_{ν_*} is the derivative with respect to the conormal).
If the space $H^l(\partial\Omega)$ is enclosed in the direct sum $M_p^1 \oplus M_p^2, \quad 2 < p < l$ (for nonproper elliptic and nonelliptic cases) then incorrect in the space $H^l(\partial\Omega)$ the Dirichlet and Neumann problems are correct with a reduction of smoothness in sense of the presence of estimate

$$||u||_{H^{p+1/2-q}(\Omega)} \leq C||\psi||_{H^l(\partial\Omega)} \quad (\text{or} \leq C||\chi||_{H^{l-1}(\partial\Omega)}),$$

where $q = 0$ in the elliptic case, $q = 1 + 0$ in the hyperbolic case, $q = 2 + 0$ in the case of degenerate real symbol.

Statement 14. *If Ω is the unit disk K and the equation (15) is hyperbolic, then the difference $l-p$ is characterized by the index of irrationality k of the number*

$$\varphi_0/\pi : \exists C > 0, \forall m/n \in Q, \ |\varphi_0/\pi - m/n| \geq C/n^k,$$

where $\varphi_0 = \varphi_1 - \varphi_2 \, (\tan \overline{\varphi_j} = b_j^1/b_j^2)$ *is the angle between characteristics* **(More exactly,** $l-p=k-1$**)**, *then for almost all* $\varphi_0 \in \mathbf{R}$, *for all* $\varepsilon > 0$, *for all* $\psi \in H^{l+3/2+\varepsilon}(\partial K)$ *there is a solution* $u \in H^l(K)$ *of the Dirichlet problem* $u|_{\partial K} = \psi$. *Here* $l \geq 1$.

Statement 15. If the equation (15) is hyperbolic or nonright (i.e. nonproper) elliptic, then the Dirichlet problem $u|_{\partial K} = 0$ (and the Neumann problem $u'_{\nu_*}|_{\partial K} = 0$) **has a nontrivial (nonconstant) solution** (which is a polynomial and there is a denumerable set of such linear independence polynomial solutions) in the disk K **if and only if the angle φ_0/π is a rational number.**

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