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Normalization flow

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Traditional (discrete) approach

Consider the ODE

$$\dot{x} = v(x) = Ax + O(|x|^2), \qquad x \in \mathbb{R}^m$$

near the origin. Poincaré proposed kill the terms $O(|x|^2)$ by using change of coordinates

$$x \mapsto X = x + O(|x|^2)$$

Poincaré-Dulac theory of normal forms. Roughly: the term $v_k x^k$ ($k \in \mathbb{Z}_+^m$ is a multi-index) can be killed if it is nonresonant (some condition on α and eigenvalues of A).

Traditional approach: kill nonresonant terms in v one-by-one (more precisely, degree-by-degree): the first coordinate change deals with degree 2, the second with degree 3 and so on ...

Formal aspect of the procedure is almost trivial. The main problem is the convergence of composition of these changes.

I will restrict by the situation when

- the ODE is Hamiltonian,
- 0 is a (totally) elliptic fixed point
- frequencies ω_j of small oscillations are nonresonant.

However sure that any of these assumptions can be dropped.

Following Birkhoff, we use complex variables

$$(z, \overline{z}) = (z_1, \ldots, z_n, \overline{z}, \ldots, \overline{z}_n),$$

n is the number of degrees of freedom.

Warning. Bar is not the complex conjugation. So, z_j and $\overline{z}_j$ should be regarded as independent complex variables.

In fact, the condition that $\overline{z}_j$ is complex conjugate of z_j is the reality condition although the theory is complex.

- Hamiltonian equations:

$$\dot{z} = i\partial_{\overline{z}}\widehat{H}, \quad \dot{\overline{z}} = -i\partial_z\widehat{H}, \qquad \widehat{H} = \widehat{H}(\mathbf{z}), \quad \mathbf{z} = (z, \overline{z})$$
$$\widehat{H} = \sum_{k,\overline{k}\in\mathbb{Z}_+^n} H_{k,\overline{k}} z^k \overline{z}^{\overline{k}} = \sum_{\mathbf{k}\in\mathbb{Z}_+^{2n}} H_{\mathbf{k}}\mathbf{z}^{\mathbf{k}} = H_2 + \widehat{H}_\diamond.$$

- Ellipticity:

$$H_2 = \sum \omega_j z_j \overline{z}_j, \quad \widehat{H}_\diamond = O_3(\mathbf{z}).$$

- Non-resonance condition:

$$\langle q, \omega\rangle \neq 0 \quad \text{for any } q \in \mathbb{Z}^n \setminus \{0\}.$$

So, our system is a nonlinear perturbation of a product of n independent linear oscillators ($H = H_2$) with frequency vector $\omega = (\omega_1, \ldots, \omega_n)$.

Normal form theory: all terms of the form $\mathbf{z}^{\mathbf{k}}, \quad k \neq \overline{k}$ in the Hamiltonian can be removed degree-by-degree.

The sequence of normalization coordinate changes generically diverge (Siegel 1954, see also the resent survey by Krikorian 2020). So, the complete normalization usually exists only formally.

Another (continuous) approach

Construct the coordinate change as a shift $\mathbf{z} = (z, \overline{z}) \mapsto \mathbf{z}_\delta = (z_\delta, \overline{z}_\delta)$ along solutions of the Hamiltonian system

$$z' = i\partial_{\overline{z}} F, \quad \overline{z}' = -i\partial_z F, \qquad (\cdot)' = d/d\delta.$$

Then

$$H_2(\mathbf{z}) + \widehat{H}_\diamond(\mathbf{z}) = H_2(\mathbf{z}_\delta) + H_\diamond(\mathbf{z}_\delta, \delta).$$

Differentiate in δ:

$$\partial_\delta H_\diamond = -\{F, H_2 + H_\diamond\}, \qquad H_\diamond|_{\delta=0} = \widehat{H}_\diamond,$$

where $\{\cdot, \cdot\}$ is the Poisson bracket:

$$\{F, G\} = i \sum_{j=1}^{n} (\partial_{\overline{z}_j} F \partial_{z_j} G - \partial_{z_j} F \partial_{\overline{z}_j} G).$$

Until this moment this is the Lie method (or its Hamiltonian version the Deprit-Hori method). But ...

To have a "good" Hamiltonian function F we take $F = \xi H_\diamond$, ξ is a linear operator. For any $q \in \mathbb{Z}^n$ we put

$$\sigma_q = \operatorname{sign}\langle q, \omega\rangle, \quad \mathbf{k} = (k, \overline{k}) \in \mathbb{Z}_+^{2n}, \quad \mathbf{k}' = \overline{k} - k \in \mathbb{Z}^n.$$

Then for any $H_\diamond = \sum_{|\mathbf{k}|\geqslant 3} H_\mathbf{k}\mathbf{z}^\mathbf{k}$ we have: $H_\diamond = H^- + H^0 + H^+$,

$$H^\pm = \sum_{\pm\sigma_{\mathbf{k}'}>0} H_\mathbf{k}\mathbf{z}^\mathbf{k}, \quad H^0 = \sum_{\sigma_{\mathbf{k}'}=0} H_\mathbf{k}\mathbf{z}^\mathbf{k}.$$

We define

$$\xi H_\diamond = i(H^- - H^+).$$

We obtain the IVP for the "averaging system" in $\mathcal{F}$

$$\partial_\delta H_\diamond = v_0(H_\diamond) + v_1(H_\diamond) + v_2(H_\diamond), \qquad H_\diamond|_{\delta=0} = \widehat{H}_\diamond,$$

$$v_0 = -i\{H^- - H^+, H_2\}, \ \ v_1 = -i\{H^- - H^+, H_0\}, \ \ v_0 = -2i\{H^-, H^+\}.$$

Why such ξ? Informal explanation. Dropping the nonlinearities v_1 and v_2, we obtain:

$$\partial_\delta H_{\mathbf{k}} = -|\langle \mathbf{k}', \omega\rangle| H_{\mathbf{k}}, \qquad H_{\mathbf{k}}|_{\delta=0} = \widehat{H}_{\mathbf{k}}.$$

This system is easily solved:

$$H_{\mathbf{k}}(\delta) = e^{-|\langle \mathbf{k}', \omega\rangle|\delta} \widehat{H}_{\mathbf{k}}.$$

All non-resonant terms $H_{\mathbf{k}}$, $\mathbf{k}' \neq 0$ tend to zero. Non-uniformly: small divisors ...

Existence of a solution

(a) Formal aspect. Let $\mathcal{F}$ be the space of formal Hamiltonians:

$$H_\diamond \in \mathcal{F} \quad \text{iff} \quad H_\diamond = \sum_{|\mathbf{k}|\geqslant 3} H_{\mathbf{k}} \mathbf{z}^{\mathbf{k}},$$

formal power series.

We endow $\mathcal{F}$ with the product (Tikhonov) topology: we say that the sequence $F^{(1)}, F^{(2)}, \ldots \in \mathcal{F}$ converges if $F^{(1)}_{\mathbf{k}}, F^{(2)}_{\mathbf{k}}, \ldots \in \mathbb{C}$ converges for any index $\mathbf{k}$.

Theorem 1. Suppose $\widehat{H}_\diamond \in \mathcal{F}$. Then $H(\cdot, \delta) \in \mathcal{F}$ exists for any $\delta \geqslant 0$ and

$$\lim_{\delta \to +\infty} H_\diamond \in \mathcal{N} := \{F \in \mathcal{F} : F = F(z_1\overline{z}_1, \ldots, z_n\overline{z}_n)\}.$$

Sketch of the proof.

(1) Put $\mathcal{H}_{\mathbf{k}} = e^{-|\langle \mathbf{k}', \omega \rangle|\delta} H_{\mathbf{k}}$.

(2) Write the IVP in the variables $\mathcal{H}_{\mathbf{k}}$.

(3) Nilpotent structure of the system:

$$\partial_\delta \mathcal{H}_{\mathbf{k}} = \text{ a quadratic expression depending only on } \mathcal{H}_{\mathbf{m}},\ |\mathbf{m}| < |\mathbf{k}|$$

In particular, if $|\mathbf{k}| = 3$ then $\partial_\delta \mathcal{H}_{\mathbf{k}} = 0$.
Hence equations of this system are easily solved inductively.

(b) Analytic aspect. If the series $F \in \mathcal{F}$ converges somewhere outside the origin then it converges in the polydisk

$$D_\rho = \{\mathbf{z} \in \mathbb{C}^{2n} : |z_j| \leqslant \rho,\ |\overline{z}_j| \leqslant \rho\}.$$

Let $\mathcal{A}^\rho$ denote the corresponding subspace in $\mathcal{F}$.
We put $\mathcal{A} = \cup_{\rho>0}\mathcal{A}^\rho$. Scale of Banach spaces with norms

$$\|F\|_\rho = \sup_{D_\rho} |F|.$$

Theorem 2. Suppose $\widehat{H}_\diamond \in \mathcal{A}$. Then $H(\cdot,\delta) \in \mathcal{A}$ for all $\delta \geqslant 0$.

Looks too positive and optimistic. A more detailed version:

Theorem $2'$. Suppose $\widehat{H}_\diamond \in \mathcal{A}^\rho$. Then $H(\cdot,\delta) \in \mathcal{A}^{r(\delta)}$ for all $\delta \geqslant 0$, where

$$r(\delta) > \frac{c}{1+\delta}.$$

The proof is based on the majorant method.

Further things to discuss

1. Algebraic structure of the averaging system. A series of algebraic properties of solutions.

For example: if

$$\widehat{H}_{\mathbf{k}} \neq 0 \quad \Rightarrow \quad |\langle \mathbf{k}', \omega \rangle| \leqslant N$$

then for any $\delta \geqslant 0$

$$H_{\mathbf{k}}(\cdot, \delta) \neq 0 \quad \Rightarrow \quad |\langle \mathbf{k}', \omega \rangle| \leqslant N.$$

Seems, no analogs in the traditional approach.

2. Examples when the averaging system has an explicit solution.

One example: $\widehat{H}_{\mathbf{k}} \neq 0 \quad \Rightarrow \quad |\langle \mathbf{k}', \omega \rangle| \leqslant N$.

Contradicts to reality although very instructive.

3. Situations when $H(\cdot, \delta) \in \mathcal{A}^\rho$ ($\rho > 0$ fixed) for all $\delta \geqslant 0$.

1 DOF ?

Completely integrable systems (an analog of the Ito-Vey theorem) ?

4. A more precise estimate for $r(\delta)$ in Theorem $2'$.

5. Study manifolds asymptotic to $N \in \mathcal{N}$.

6. Partial normalization.

7 ...