On the spectrum of hierarchical Schrödinger operators

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Dedicated to the memory of V.S. Vladimirov

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This lecture is based on the project "On the spectrum of hierarchical Schrödinger operators acting on a Cantor like set" joint with A. Grigor'yan (Bielefeld University) and S. Molchanov (UNC at Charlotte). The results obtained are partially presented in two papers:

- A. Bendikov, A. Grigor'yan, S. Molchanov, On the spectrum of hierarchical Schrödinger-type operators: the case of locally bounded potentials, in: A. N. Karapetyants at al. (eds.), Operator Theory and Harmonic Analysis, Springer Proceedings in Mathematics & Statistics 358: pp. 43-90 (2021).
- A. Bendikov, A. Grigor'yan, S. Molchanov, On the spectrum of hierarchical Schrödinger-type operators: the case of potentials with local singularities, ArXiv 2006.01821v1 [math.SP] 02 June 2021, 48 pages.

1 Introduction

The concept of the hierarchical Laplacian is going back to N. Bogolubov and his school. This concept has been used by F. J. Dyson in his construction of the phase transition in **1D** ferromagnetic model with long range interaction.

- F. J. Dyson, Existence of a phase-transition in a one-dimensional Ising ferromagnet, Comm. Math. Phys., 12: 91-107, 1969.
- S. A. Molchanov, Hierarchical random matrices and operators, Application to Anderson model, in: Proc. of 6th Lucacs Symposium (1996), 179-194.

The notion of the hierarchical Laplacian acting on general ultrametric space X was developed to the high level of generality in the papers:

- A. D. Bendikov, A. A. Grigoryan, Ch. Pittet, and W. Woess, Isotropic Markov semigroups on ultrametric spaces, Russian Math. Surveys. 69:4, 589-680 (2014).
- A. D. Bendikov, Heat kernels for isotropic like Markov generators on ultrametric spaces: a Survey, p-Adic Numbers, Ultrametric Analysis and Applications, 2018, Vol. 10, No. 1, pp. 1-11

In the case $X = \mathbb{Q}_p$, the field of p-adic numbers, we would like to mention closely related works of S. Albeverio, W. Karwowski, V. S. Vladimirov, I. V. Volovich, E. I. Zelenov, and A. N. Kochubei.

Let us consider (as a simplest example) a continuous version of the Dyson dyadic model. In this example the hierarchical Schrödinger operator $H = L + V$ is realized as a perturbation of certain self-adjoint singular integral operator L acting in $L^2(0,\infty)$. Let us describe this construction in details.

The hierarchical structure is defined by the family of partitians $\{\Pi_r : r \in \mathbb{Z}\}$ of the set $X = [0,\infty)$. Each partitian Π_r is made of dyadic intervals $I = [(i-1)2^r, i2^r)$. We call r the rank of the partitian Π_r (resp. the rank of the dyadic interval I). Recal that in the Dyson model $X = \{0, 1, 2, ...\}$, $r \in \mathbb{X}$ and $I = \{k \in X : (i-1)2^r \leq k < i2^r\}$.

Any point x belongs to exactly one interval $I_r(x)$ of rank r, the one point set $\{x\} = \cap I_r(x)$, and the whole set $X = \cup I_r(x)$.

The hierarchical distance $\mathrm{d}(x,y)$ is defined as 0 if $x = y$ and for $x \neq y$ as the Lebesgues measure $|I|$ of the minimal dyadic interval I which contains both x and y. One can easily see that for all x, y, z in X,

$$\mathrm{d}(x,y) \leq \max\{\mathrm{d}(x,z), \mathrm{d}(z,y)\},$$

that is, $\mathrm{d}(x,y)$ is an *ultrametric* on X.

- The Euclidean metric $|x-y|$ and the introduced ultrametric $\mathrm{d}(x,y)$ define non-equivalent topologies. Indeed, by the very definition

  $$\mathrm{d}(x,y) \geq |x-y|, \ \forall x, y \in X,$$

  but on the other hand

  $$\mathrm{d}(1-\varepsilon, 1) = 2, \ \forall \varepsilon \in (0,1].$$

- The couple (X, d) is a complete locally compact non-compact and separable metric space. In this metric space the set $\mathcal{B}$ of all open balls coincides with the set of all dyadic intervals.
- Each open ball B in (X, d) is a closed compact set, each point $a \in B$ can be regarded as its center, any two balls either do not intersect or one is a subset of another etc. Thus (X, d) is a proper totally disconnected metric space. In particular, (X, d) is homeomorphic to the Cantor set with punctured point.
- The Borel σ-algebra generated by the ultrametric balls coincides with the classical Borel σ-algebra generated by the Euclidean metric.

The hierarchical Laplacian Let $\mathcal{D}$ be the set of all compactly supported locally constant functions. Let $\varkappa \in]0,1[$ be a fixed parameter.

The hierarchical Laplacian L is introduced as sum of (minus) Markov generators L_r of pure jump processes [1]

$$(Lf)(x) = \sum_{r=-\infty}^{+\infty} \underbrace{(1-\varkappa)\varkappa^r \left(f(x) - \frac{1}{|I_r(x)|} \int\limits_{I_r(x)} f dl \right)}_{(L_r f)(x)}, \ \forall f \in \mathcal{D}.$$

As each *elementary Laplacian* L_r can be written in the form

$$L_r f(x) = \int\limits_0^\infty \left(f(x) - f(y)\right) J_r(x,y) dy,$$

$$J_r(x,y)dy = \underbrace{(1-\kappa)\kappa^{r-1}}_{\lambda_r(x)} \cdot \underbrace{\mathbf{1}_{I_r(x)}(y)/|I_r(x)|dy}_{\mathcal{U}_r(x,dy)}$$

the operator L can be represented as a hypersingular integral operator

$$(Lf)(x) = \int_0^\infty (f(x) - f(y)) J(x,y) dy,$$

$$J(x,y) = \frac{\kappa^{-1} - 1}{1 - \kappa/2} \cdot \frac{1}{\mathrm{d}(x,y)^{1+\alpha}}, \ \alpha = \log_2 \frac{1}{\kappa}.$$

[1] A Markov process is called a pure jump process if, starting from any point x, it has all sample paths constant except for isolated jumps, and right-continuous.

The basic data which defines the process are (i) a function $0 < \lambda(x) < \infty$, and (ii) a Markov kernel $\mathcal{U}(x, dy)$ satisfying $\mathcal{U}(x, \{x\}) = 0$. Intuitively a particle starting from x remains there for an exponentialy distributed time with parameter $\lambda(x)$ at which time it "jumps" to a new position x' according to distribution $\mathcal{U}(x, \cdot)$ etc.

The spectrum of L To each dyadic interval $I = [(i-1)2^r, i2^r)$ we associate the Haar function

$$\mathcal{X}_I(x) = \begin{cases} 2^{-r/2} & \text{if } \; x \in [(i-1)2^r, (i-1/2)2^r) \\ -2^{-r/2} & \text{if } \; x \in [(i-1/2)2^r, i2^r) \\ 0 & \text{if } \; x \notin I \end{cases}.$$

The Haar function $\mathcal{X}_I$ is an eigenfunction of the operator L subject to the eigenvalue $\lambda(I) = \varkappa^r$,

$$L\mathcal{X}_I = \varkappa^r \mathcal{X}_I.$$

It is easy to see that each eigenvalue $\varkappa^r$ has infinite multiplicity.

The set $\{\mathcal{X}_I : I \in \mathcal{B}\}$ is a complete orthonormal basis in $L^2(0,\infty)$. In particular, L is essentially self-adjoint operator having a pure point spectrum

$$Spec(L) = \{\varkappa^r : r \in \mathbb{Z}\} \cup \{0\}.$$

The heat kernel of L The operator L generates a symmetric Markov semigroup $(e^{-tL})_{t>0}$. The semigroup $(e^{-tL})_{t>0}$ admits a heat kernel $p(t,x,y)$ (the integral kernel of the operator e^{-tL}) which can be represented in the form

$$p(t,x,y) = t \int_0^{1/\mathrm{d}_*(x,y)} e^{-t\tau} N(\tau) d\tau,$$

where $\mathrm{d}_*(x,y)$ is an ultrametric and $N(\tau)$ is the so-called spectral distribution function intrinsically related to L. We refer interested reader to the paper

- A. D. Bendikov, A. A. Grigoryan, Ch. Pittet, and W. Woess, Isotropic Markov semigroups on ultrametric spaces, Russian Math. Surveys. 69:4, 589-680 (2014).

In particular, $p(t,x,y)$ is a continuous (and even locally Lipshitz continuous) function in the d-topology but it is discontinuous function in the Euclidean topology.

As a consequence of the representation formula the function $p(t,x,y)$ can be estimated as follows

$$p(t,x,y) \asymp \frac{t}{[t^{1/\alpha} + \mathrm{d}(x,y)]^{1+\alpha}}.$$

Another consequence of the representation formula is that the function $p(t,x,x)$ does not depend on x and can be written in the form

$$p(t,x,x) = t^{-1/\alpha} \mathcal{A}(\log_2 t),$$

where $\mathcal{A}(\tau)$ is a continuous non-constant α-periodic function. In particular, in contrary to the classical case (symmetric stable densities in $\mathbb{R}^1$), the function $t \to p(t,x,x)$ does not vary regularly.

The Taibleson-Vladimirov multiplier It is remarkable that the hierarchical Laplacian L introduced above can be identified with the Taibleson-Vladimirov multiplier $\mathfrak{D}^{\alpha}, \alpha > 0$, acting in $L^2(\mathbb{Q}_2)$, where $\mathbb{Q}_2$ is the field of 2-adic numbers,

$$\widehat{\mathfrak{D}^{\alpha} f}(\zeta) = \|\zeta\|_2^{\alpha}\, \widehat{f}(\zeta).$$

In particular, $-\mathfrak{D}^{\alpha}$ is a symmetric α-stable Lévy generator acting on the Abelian group $\mathbb{Q}_2$. On the set $\mathcal{D}$ of locally constant functions having compact supports it can be represented in the form

$$\mathfrak{D}^{\alpha} f(x) = -\frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{\|x-y\|_p^{1+\alpha}} dm(y), \tag{1.1}$$

where $\Gamma_p(z) = (1-p^{z-1})(1-p^{-z})^{-1}$ is the p-adic Gamma-function introduced by V.S. Vladimirov. The heat kernel $p_{\alpha}(t,x,y)$ of the operator $\mathfrak{D}^{\alpha}$ can be estimated as follows

$$p_{\alpha}(t,x,y) \asymp \frac{t}{[t^{1/\alpha} + \|x-y\|_2]^{1+\alpha}}.$$

2 Schrödinger operators

In what follows we consider Taibleson-Vladimirov multiplier $\mathfrak{D}^{\alpha}, \alpha > 0$, acting in $L^2(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is the ring of p-adic numbers equipped with normed Haar measure.

The case of locally bounded potentials Let V be a locally bounded measurable function and $V : u \to V \cdot u$ a multiplicator. The operator $H = \mathfrak{D}^{\alpha} + V$ with domain $\mathcal{D}$ is a densly defined symmetric operator acting in the Hilbert space $L^2(\mathbb{Q}_p)$.

Theorem 2.1 *The following properties hold true:*

1. The operator H is essentially self-adjoint.

2. If $V(x) \to +\infty$ as $x \to \infty$, then the self-adjoint operator H has a compact resolvent. (Thus, its spectrum is discrete).

3. If $V(x) \to 0$ as $x \to \infty$, then the essential spectrum of H coincides with the spectrum of $\mathfrak{D}^{\alpha}$. (Thus, the spectrum of H is pure point and the negative part of the spectrum consists of isolated eigenvalues of finite multiplicity).

Remark 2.2 *In the case of Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^n$ the statement about essential self-adjointness of H does not hold in such a great generality. Indeed, in the case of Schrödinger operator*

$$H\psi = -\psi'' + V \cdot \psi,\ \psi \in C_c^{\infty}(0,\infty),$$

with $V(x) = -x^\gamma$*,* $\gamma > 2$*, there is* continuum *of self-adjoint extensions of* H*.*

Furthermore, due to S. Kotani the spectrum of the operator H *may contain non-trivial absolutely continuous and singular continuous parts.*[2]

The case of potentials with local singularities If we are interested in potentials with local singularities, such as $V(x) = b\,\|x\|_p^{-\beta}$, $b \in \mathbb{R}$, $x \in \mathbb{Q}_p$, then certain local conditions on the potential V are necessary in order to prove that the quadratic form

$$Q(u,u) := Q_{\mathfrak{D}^\alpha}(u,u) + Q_V(u,u), \tag{2.2}$$

defined on the set

$$dom(Q) := dom(Q_{\mathfrak{D}^\alpha}) \cap dom(Q_V)$$

is a densly defined closed and bounded below quadratic form and whence it is associated to a bounded below self-adjoint operator H. It is customary to write $H = \mathfrak{D}^\alpha + V$, but it must be remembered that this is a quadratic form sum and not an operator sum as in the previous subsection.

Theorem 2.3 *If* $0 \leq V \in L^1_{loc}(\mathbb{Q}_p)$*, then the quadratic form (2.2) is a regular Dirichlet form. In particular, it is the form of a non-negative self-adjoint operator* H*,*

$$Q(u,u) = (H^{1/2}u, H^{1/2}u),\ \forall u \in dom(Q),$$

and the set $\mathcal{D}$ *is a core for* Q*.*

Remark 2.4 *It is clear that Theorem 2.3 can be extended for those* V *which are bounded below and in* $L^1_{loc}(\mathbb{Q}_\text{p})$ *by simply adding a large enough positive constant. If, however, we are interested in* V *with negative local singularities, then stronger local conditions on* V *are necessary in order to be able to prove that the form* Q *is closed.*

Definition 2.5 *Let* $p \geq 1$ *be fixed. We say that a potential* V *lies in* $L^p + L^\infty$ *if one can write* $V = V' + V''$ *where* $V' \in L^p(X,m)$ *and* $V'' \in L^\infty(X,m)$ *. This decomposition is not unique, and, if it is possible at all, then one can arrange for* $\|V'\|_p$ *to be as small as one chooses.*

[2] Whether this result holds true in the setting of the hierarchical Schrödinger operator $H = \mathfrak{D}^\alpha + V$ is an interesting open at present writing question.

Theorem 2.6 *Consider quadratic form* $Q = Q_{\mathfrak{D}^\alpha} + Q_V$ *and assume that* $V \in L^p + L^\infty$ *for some* $p > 1/\alpha$. *Then:*

1. Q *is a densly defined closed and bounded below form whence it is associated with a bounded below self-adjoint operator* H.

2. If $2 \leq 1/\alpha < p$ *then* $dom(H) = dom(\mathfrak{D}^\alpha)$. *The same is true if* $1/\alpha < 2$ *and* $p = 2$.

Remark 2.7 *Clearly the above theorem applies well for the operator*

$$H = \mathfrak{D}^\alpha + b \left\| \cdot \right\|_p^{-\beta}, \ \beta < \alpha < 1.$$

The case $\beta = \alpha < 1$ *is more delicate. In this case we get the result assuming that* $b \geq b_*$ *a critical value which we define below, see Theorem 2.8.*

The positive spectrum We present a crteria for the Schrödinger operator $H = \mathfrak{D}^\alpha + V$ to have spectrum $Spec(H) \subset [0, \infty)$. We assume that the quadratic form $Q = Q_{\mathfrak{D}^\alpha} + Q_V$ is *a densely defined closed and bounded below quadratic form having* $\mathcal{D}$ *as a core, and* H *is a bounded below self-adjoint operator associated with* Q*, i.e.*

$$Q(u, u) = (Hu, u), \ \forall u \in dom(H).$$

The classical Hardy inequality in $\mathbb{R}^n$, $n \geq 3$, reads as follows

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{\|x\|^2} dx \leq \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx.$$

Its p-adic version which we use in our analysis has the form

$$\left(\Gamma_p \left(\frac{1+\alpha}{2} \right) \right)^2 \int_{\mathbb{Q}_p} \frac{|f(x)|^2}{\|x\|_p^\alpha} dx \leq Q_{\mathfrak{D}^\alpha}(f, f),$$

where

$$\Gamma_p(z) = (1 - p^{z-1})(1 - p^{-z})^{-1}$$

is the p-adic Gamma-function. The function $\Gamma_p(z)$ is related to our analysis becase of the following identity

$$\mathfrak{D}^\alpha \|x\|_p^\beta = \frac{\Gamma_p(\beta + 1)}{\Gamma_p(\beta + 1 - \alpha)} \|x\|_p^{\beta - \alpha}, \ \forall \beta \neq \alpha,$$

where the left-hand side of this identity we understand in the sence of distributions.

Theorem 2.8 *Assume that* $0 < \alpha < 1$ *and that the following inequality*

$$V_-(x) \le \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \|x\|_p^{-\alpha}$$

holds almost everywhere, then

$$Q(u,u) \ge 0, \forall u \in \mathcal{D}.$$

In particular, since $\mathcal{D}$ *is a core for* Q*,*

$$Spec(\mathfrak{D}^\alpha + V) \subseteq [0,\infty).$$

The negative spectrum Next we discuss several results about the negative part of the spectrum of the operator $H = \mathfrak{D}^\alpha + V$.

Theorem 2.9 *Let* $V \in L^p(\mathbb{Q}_\mathrm{p})$ *for some* $p > 1/\alpha$*. Then the following propertes hold:*

1. The operator $H = \mathfrak{D}^\alpha + V$ *has essential spectrum equals to the spectrum of the operator* $\mathfrak{D}^\alpha$*.*

2. In particular, if H *has any negative spectrum, then it consists of a sequence of negative eigenvalues of finite multiplicity. If this sequence is infinite then it converges to zero.*

3. Suppose that there exists an open set $U \subset X$ *on which* V *is negative. If* E_λ *is the bottom of the spectrum of the operator* $H_\lambda = \mathfrak{D}^\alpha + \lambda V$*, then* $E_\lambda \le 0$ *for all* $\lambda \ge 0$ *and* $\lim_{\lambda\to\infty} E_\lambda = -\infty$*.*

The following example shows that the crucial issue for the existence of negative eigenvalues in Theorem 2.9 for all $\lambda > 0$ is the rate at which the potential $V(x)$ converges to 0 as $\|x\|_\mathrm{p} \to \infty$.

Example 2.10 *Let* $0 < \alpha < 1$ *and* $H_\lambda = \mathfrak{D}^\alpha + \lambda V$ *where*

$$V(x) = -(\|x\|_\mathrm{p} + 1)^{-\beta}$$

for some $0 < \beta < 1$ *and* $\lambda > 0$*. We have:*

1. If $\beta \ge \alpha$ *then Theorem 2.8 and Theorem 2.9 are applicable and thus there exists a positive threshold for the existence of negative eigenvalues of* H_λ*.*

2. If $\beta > \alpha$ *then the number of negative eigenvalues of* H_λ *counted with their multiplicity can be estimated as follows*

$$Neg(H_\lambda) \le c(\alpha,\beta)\lambda^{1/\alpha}.$$

3. If $0 < \beta < \alpha$ *then* H_λ *has non-empty negative spectrum for all* $\lambda > 0$*.*

3 Rank one perturbations

In this section we assume that the ultrametric measure space (X, d, m) is *countably infinite and homogeneous*. Let L be a homogeneous hierarchical Laplacian on X and H_σ a rank one perturbation of the operator L,

$$H_\sigma f(x) = Lf(x) - \sigma(f, \delta_a)\delta_a(x).$$

Let $\mathcal{R}_V(\lambda, x, y) = (H_\sigma - \lambda I)^{-1}\delta_y(x)$ be the resolvent kernel of the operator H_σ. To prove that the spectrum of the operator $H_\sigma = L - \sigma\delta_a$ is pure point for all σ, we use the Krein like identity

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}(\lambda, x, y) + \frac{\sigma\mathcal{R}(\lambda, x, a)\mathcal{R}(\lambda, a, y)}{1 - \sigma\mathcal{R}(\lambda, a, a)}.$$

Here $\mathcal{R}(\lambda, x, y) = (L - \lambda I)^{-1}\delta_y(x)$ is the resolvent kernel of the operator L. It can be computed using the spectral resolution formula, in particular,

$$\mathcal{R}(\lambda, a, a) = \sum_{k=1}^{\infty} \frac{A_k}{\lambda_k - \lambda}, \quad A_k = \left(\frac{1}{m(B_{k-1})} - \frac{1}{m(B_k)}\right).$$

Theorem 3.1 *$Spec(H_\sigma)$ is pure point for all σ, it consists of at most one negative eigenvalue and countably many positive eigenvalues.*

1. If $\sigma > 0$, then H_σ has precisely one negative eigenvalue

$$\lambda_-^\sigma < 0 < ... < \lambda_{k+1} < \lambda_k^\sigma < \lambda_k < ... < \lambda_2 < \lambda_1^\sigma < \lambda_1.$$

if and only if one of the following two conditions holds

- *the Markov semigroup $(e^{-tL})_{t>0}$ is recurrent, i.e. $\mathcal{R}(0, a, a) = \infty$,*
- *the Markov semigroup $(e^{-tL})_{t>0}$ is transient, i.e. $\mathcal{R}(0, a, a) < \infty$, and*

$$\mathcal{R}(0, a, a) > 1/\sigma.$$

2. If $\sigma < 0$, then all eigenvalues of H_σ are positive

$$0 < ... < \lambda_{k+1} < \lambda_k^\sigma < \lambda_k < ... < \lambda_2 < \lambda_1^\sigma < \lambda_1 < \lambda_+^\sigma.$$

The eigenvalues λ_k are eigenvalues of the operator L. All λ_k have infinite multiplicity and compactly supported eigenfunctions, the eigenfunctions of the operator L whose supports do not contain a.

The eigenvalue λ_k^σ (resp. λ_-^σ, λ_+^σ) is the unique solution of the equation

$$\mathcal{R}(\lambda, a, a) = 1/\sigma$$

in the interval $]\lambda_{k+1}, \lambda_k[$ (resp. $]-\infty, 0[$, $]\lambda_1, +\infty[$). Each λ_k^σ (resp. λ_-^σ, λ_+^σ) has multiplicity one and non-compactly supported eigenfunction

$$\psi_k(x) = \mathcal{R}(\lambda_k^\sigma, x, a) \ (resp. \psi_-(x) = \mathcal{R}(\lambda_-^\sigma, x, a), \psi_+(x) = \mathcal{R}(\lambda_+^\sigma, x, a)).$$

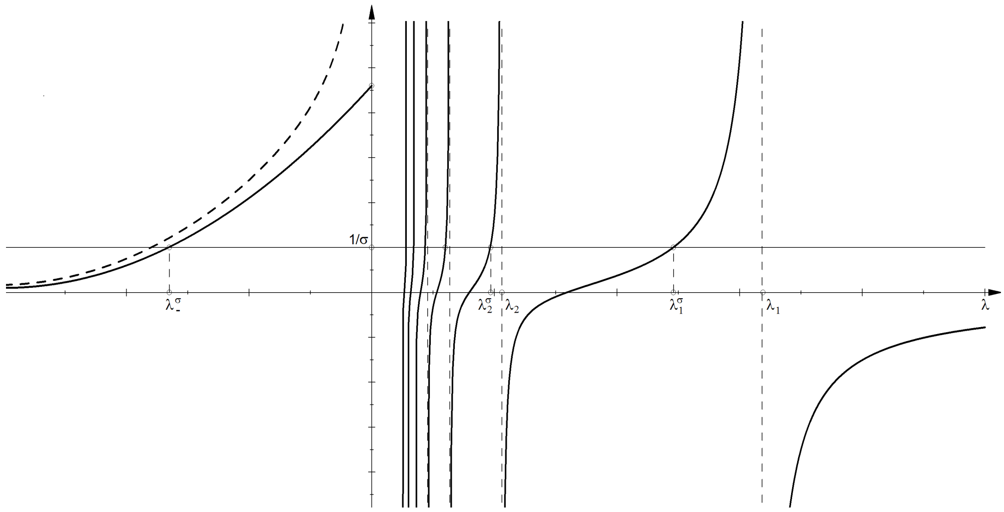

Figure 1: The roots $\{\lambda_*^\sigma\}$ of the equation $\mathcal{R}(\lambda, a, a) = 1/\sigma$. The dashed graph corresponds to a recurrent case, the solid graph – to the transient case.

4 Sparse potentials

Here we continue to assume that the ultrametric measure space (X, d, m) is *countably infinite and homogeneous.* Analysis of the finite dimensional perturbations $H = L - \sum_{i=1}^{n} \sigma_i \delta_{a_i}$ indicates that in the case of increasing distances between locations $\{a_i\}$ of the *bumps* $V_i = -\sigma_i \delta_{a_i}$ their contributions to the spectrum are close to the union of the contributions of the individual bumps V_i (each bump contributes one eigenvalue in each gap $(\lambda_{k+1}, \lambda_k)$ of the spectrum of the operator L). The development of this idea leads to consideration of the class of *sparse potentials*

$$V = -\sum_{i=1}^{\infty} \sigma_i \delta_{a_i}$$

where distances between locations $\{a_i : i = 1, 2, ...\}$ form an increasing to infinity sequence. In the classical spectral theory this idea goes back to D. B. Pearson, S. Molchanov, and to A. Kiselev, J. Last, S. and B. Simon.

Notation. Let us set

- $\mathcal{I}_*$ is the set of limit points of the sequence $\{\sigma_i\}$.
- $1/\mathcal{I}_* := \{1/\sigma_* : \sigma_* \in \mathcal{I}_*\}$.

- $\mathcal{R}^{-1}(1/\mathcal{I}_*) := \{\lambda : \mathcal{R}(\lambda, a, a) \in 1/\mathcal{I}_*\}$.

Theorem 4.1 *Assume that $\alpha < \sigma_i < \beta$ for some $\alpha, \beta > 0$ and that*

$$\lim_{n\to\infty} \sup_{i\geq n} \sum_{j\geq n:\ j\neq i} \frac{1}{\mathrm{d}(a_i, a_j)} = 0, \tag{4.3}$$

then

$$Spec_{ess}(H) = Spec(L) \cup \mathcal{R}^{-1}(1/\mathcal{I}_*). \tag{4.4}$$

5 Spectral localization

Theorem 4.1 does not contain any information about absolutely continuous and singular continuous parts of $Spec(H)$. Under more restrictive assumption these two parts of $Spec(H)$ are indeed *empty sets*, i.e. $Spec(H)$ is *pure point*. Moreover, the eigenfunctions of H decay at infinity exponentially - this is the so-called *spectral localization property*. We refere to papers [1], [4], [5], [6] and references therein.

6 Green function estimates

We consider the hierarchical Schrödinger operator $H = \mathfrak{D}^\alpha + b\,\|x\|_p^{-\alpha}$, where we assume that $0 < \alpha < 1$, $b \geq b_*$ and

$$b_* := -\{\Gamma_p\left((1+\alpha)/2\right)\}^2.$$

Under these conditions the equation $Hu = v$ has unique solution u which can be represented in the form

$$u(x) = \int g_H(x,y)v(y)dm(y).$$

The kernel $g_H(x,y)$ is a continuous strictly positive function which is bounded outside the diagonal set Δ and $g_H|_\Delta = +\infty$. We call $g_H(x,y)$ the Green function defined by the operator $H = \mathfrak{D}^\alpha + b\,\|x\|_p^{-\alpha}$. Remember that the equation $\mathfrak{D}^\alpha u = \nu$ also admits similar representation

$$u(x) = \int g_{\mathfrak{D}^\alpha}(x,y)v(y)dm(y),$$

$$g_{\mathfrak{D}^\alpha}(x,y) = \frac{1}{\Gamma_p(\alpha)} \frac{1}{\|x-y\|_p^{1-\alpha}}.$$

The function $g_H(x,y)$ does not admit such elementary form as $g_{\mathfrak{D}^\alpha}(x,y)$.Our aim here is to compare the Green functions $g_H(x,y)$ and $g_{\mathfrak{D}^\alpha}(x,y)$. We obtain the following result

Theorem 6.1 *For any $b \geq b_*$ there exists unique $\frac{\alpha-1}{2} \leq \beta < \alpha$ such that*

$$\frac{g_H(x,y)}{g_{\mathfrak{D}^\alpha}(x,y)} \asymp \left(\frac{\|x\|_p}{\|y\|_p} \wedge \frac{\|y\|_p}{\|x\|_p}\right)^\beta.$$

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[2] A. Bendikov, A. Grigor'yan, S. Molchanov, On the spectrum of hierarchical Schroedinger type operators: the case of potentials with local singularities, ArXiv 2006.01821v1 [math.SP] 2 June 2021.

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Thank you for your attention!