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TOPOLOGICAL PHASES IN THE THEORY OF SOLID STATES

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I. INTRODUCTION

The role of topology in the solid state physics revealed first in the study of quantum Hall effect starting from the papers by Loughlin and Thouless et al. From the physical point of view the topological invariance is equivalent to the adiabatic stability. Topological methods play a leading role in the theory of topological phases characterized by having the energy gap stable under small deformations.

The topological phases are defined in the following way. Denote by G the symmetry group and consider the set Ham_G of classes of homotopy equivalence of G-symmetric Hamiltonians satisfying the gap condition mentioned above. On this set it is possible to introduce a natural stacking operation so that the set Ham_G, provided with this operation, becomes an Abelian monoid (i.e. an Abelian semigroup with neutral element). The group of invertible elements of this monoid is called the topological phase The initial ideas, lying in the base of the theory of topological phases, were formulated by Alexei Kitaev in his talks.

The family (F_d) of d-dimensional topological phases forms an Ω-spectrum. It means that this family has the property that the loop space ΩF_{d+1} is homotopy equivalent to the space F_d. This fact opens a way to wide use of algebraic topology methods for the study of topological phases. More concretely, one can associate with it the generalized cohomology theory, determined by the functor h^d, which assigns to the topological space X the set $[X, F_d]$ of classes of homotopy equivalent maps $X \to F_d$.

II. TOPOLOGICAL PHASES

We consider the quantum mechanical systems described by the Hamiltonians H invariant under the action of the symmetry group G. The Hamiltonians H are given by the selfadjoint operators in a Hilbert space $\mathcal{H}$ and the group G acts on $\mathcal{H}$ by the unitary or anti-unitary operators. Apart from the G-invariance condition we shall impose on the Hamiltonians H some other restrictions, the most important of them is the gap condition requiring that the point 0 should not belong to the spectrum of H. We shall call the G-symmetric gapped Hamiltonians admissible. It is useful to describe the properties of admissible Hamiltonians in terms of their ground states, i.e. the eigenstates with minimal energy. Such states will be also called admissible.

We introduce on the set of admissible ground states the stacking operation defined in the following way. Suppose that we are given with two admissible ground states Φ_0 and Φ_1 with associate admissible Hamiltonians H_0 and H_1, acting in the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively. The stacking of these two states is the ground state of the form

$$\Phi = \Phi_0 \otimes \Phi_1$$

associated with the Hamiltonian H, acting in the tensor product

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1.$$

The symmetry group G acts in $\mathcal{H}$ as the tensor product of representations of G in the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ and the operator H is given by the formula

$$H = H_0 \otimes I + I \otimes H_1.$$

The constructed ground state Φ and Hamiltonian H are G-symmetric and gapped if the initial ground states Φ_0, Φ_1 and Hamiltonians H_0, H_1 were of this type.

Consider next continuous deformations of admissible Hamiltonians, i.e. continuous paths of the form H_t, $0 \leq t \leq 0$, in the class of admissible Hamiltonians.

Denote by Ham_G the set of classes of homotopy equivalent admissible Hamiltonians and the corresponding ground states. The stacking operation, introduced above, can be pushed down to a binary operation on Ham_G. Denote by $[\Phi]$ the class in Ham_G containing the ground state Φ and by $[\Phi_1] + [\Phi_2]$ the stacking of the ground states $[\Phi_1]$ and $[\Phi_2]$.

This operation has the following properties:

1. associativity: for any admissible ground states $[\Phi_1]$, $[\Phi_2]$, $[\Phi_3]$ the following relation holds

$$([\Phi_1] + [\Phi_2]) + [\Phi_3] = [\Phi_1] + ([\Phi_2] + [\Phi_3]);$$

2. commutativity: for any admissible ground states $[\Phi_1]$, $[\Phi_2]$ the following equality takes place

$$[\Phi_1] + [\Phi_2] = [\Phi_2] + [\Phi_1];$$

3. existence of the neutral element: there exists a trivial ground state $[0]$ such that for any admissible ground state $[\Phi]$ we have the equality

$$[0] + [\Phi] = [\Phi] + [0] = [\Phi].$$

Note the analogy of the given construction with the definition of the semigroup $\mathrm{Vect}_s(X)$ of vector bundles over the topological space X defined up to the stable equivalence. By this analogy the stacking operation corresponds to the direct sum of the bundles while the trivial state corresponds to the trivial bundle.

We introduce now the following important notion.

We call by the SRE(short range entangled)-state the admissible ground state which is homotopic to the trivial one in the class of admissible states.

Recalling the G-symmetry, we shall call by SPT(symmetry protected topological)-phase or G-protected topological phase the class in Ham_G such that any of its representatives is an SRE-state if we ignore the G-symmetry. In other words, if any representative of this phase can be connected to the trivial state by a continuous path if one forgets the G-symmetricity condition.

Here is another, more formal definition of topological phases. As it was pointed out before the space Ham_G, provided with the stacking operation, is an Abelian monoid. The group of invertible elements of this monoid Ham_G, which is an Abelian group with respect to stacking operation, is called the topological phase.

To describe many-particle case we need the bosonic Fock space. Such space over the Hilbert space $\mathcal{H}$ is defined as the completion

$$\mathcal{B}(\mathcal{H}) = \overline{\mathfrak{S}(\mathcal{H})} = \overline{\bigoplus_p \mathfrak{S}^p(\mathcal{H})}$$

where $\mathfrak{S}^p(\mathcal{H})$ is the subspace of p-particle states of the form

$$\mathfrak{S}^p(\mathcal{H}) = \mathrm{span}\{v_1 \otimes \ldots v_p, v_j \in \mathcal{H}\}.$$

An arbitrary linear operator $O : \mathcal{H} \to \mathcal{H}$ can be extended to a linear operator $\hat{O} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ given on monomials by the formula

$$\hat{O}(v_1 \otimes \ldots \otimes v_p) = (Ov_1) \otimes \ldots \otimes (Ov_p)$$

with subsequent extension by linearity and completion to the whole space $\mathcal{B}(\mathcal{H})$.

In terms of creation and annihilation operators this operator is written in the form

$$\hat{O} = \sum_{i,j} O_{ij} a_i^\dagger a_j. \tag{1}$$

We shall also need the fermionic Fock space $\mathcal{F}(\mathcal{H})$. A fermionic Hamiltonian H is defined similar to the bosonic case in terms of creation and annihilation operators.

Suppose now that we have a lattice $\mathcal{L}$ in $\mathbb{R}^d$, i.e. a discrete Abelian group, isomorphic to $\mathbb{Z}^d$, which acts on $\mathbb{R}^d$ by translations by vectors $\gamma \in \mathcal{L}$. Denote by G the symmetry group of the Hamiltonian. The class of admissible Hamiltonians H consists in this case of d-dimensional, local, G-symmetric, gapped selfadjoint operators acting in the Hilbert space $\mathcal{H}$ and the Fock space $\mathcal{B}(\mathcal{H})$. The admissible operators are given by the formula (1) in which the number of terms in the sum does not exceed a common constant k (locality condition). The Hamiltonian H is called bosonic if there exists a Hilbert space $\mathcal{H} = \mathcal{H}_\gamma$ associated with every $\gamma \in \mathcal{H}$.

The trivial state, called also the trivial product, is the state of the form

$$\bigotimes_{\gamma \in \mathcal{L}} \Phi_\gamma \in \bigotimes_{\gamma \in \mathcal{L}} \mathcal{H}_\gamma .$$

For any pair of such states there exists a path, connecting them in the space of trivial states. In this setting the SRE-state is the ground state of a local gapped Hamiltonian which can be connected by a path with the trivial state.

In the same terms the *d*-dimensional *G*-invariant topological phase is called the *G*-protected topological phase or SPT-phase if any of its representatives is an SRE-state if we ignore the *G*-symmetry, i.e. it can be connected by a path with the trivial state.

Now I need to introduce some notations. Recall that the symmetry group G acts on the Hilbert space $\mathcal{H}$ by the unitary or anti-unitary transformations. It is convenient to introduce the homomorphism $\phi : G \to \mathbb{Z}_2 = \{\pm 1\}$ indicating that for $\phi(g) = +1$ the element $g \in G$ acts on $\mathcal{H}$ as a unitary operator while for $\phi(g) = -1$ it acts as a anti-unitary operator. Apart from that the group G may contain the symmetry with respect to time inversion, given by the homomorphism $T : G \to \{\pm 1\}$, and the symmetry with respect to charge conjugation, given by the homomorphism $C : G \to \{\pm 1\}$.

We generalize now the initial problem statement by including a C^*-algebra $\mathcal{A}$ into it. We shall consider the pairs $(G, \mathcal{A})$ in which the action of G on the algebra $\mathcal{A}$ is given by the homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ into the group of linear $*$-automorphisms of the algebra $\mathcal{A}$. A covariant representation of the pair $(G, \mathcal{A})$ is a non-degenerate $*$-representation of the algebra $\mathcal{A}$ by bounded linear operators in the Hilbert space $\mathcal{H}$ given by the homomorphism θ.

Suppose now that the algebra $\mathcal{A}$ is graded, i.e. $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ where $\mathcal{A}_0$, $\mathcal{A}_1$ are selfadjoint closed subspaces satisfying the relations

$$\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{(i+j)(\mathrm{mod}\,2)}.$$

Denote by $\mathrm{Aut}(\mathcal{A})$ the group of even $*$-automorphisms of the algebra $\mathcal{A}$, i.e. $*$-automorphisms of the algebra $\mathcal{A}$ preserving the decomposition $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$.

Then a graded covariant representation of the system $(G, \mathcal{A}, c)$, where c is the homomorphism $G \to \{\pm 1\}$, is the graded $*$-representation of the algebra $\mathcal{A}$ in the graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ satisfying the condition that $\theta(g)$ is an even operator for $c(g) = +1$ and an odd operator for $c(g) = -1$.

III. SPECTRA AND GENERALIZED COHOMOLOGY THEORIES

By definition the Ω-spectrum is the family of pointed topological spaces (T_n), $n \in \mathbb{Z}$, having the following property: for any $n \in \mathbb{Z}$ the pointed topological spaces

$$T_n \sim \Omega T_{n+1}$$

are homotopy equivalent where ΩT_{n+1} is the loop space of the topological space T_{n+1}.

With every Ω-spectrum we can associate a generalized cohomology theory determined by the contravariant functor h^n. This functor assigns to any pair of pointed topological spaces (X, Y) with $Y \subset X$ the Abelian group

$$h^n(X, Y) = [(X, Y), (T_n, *)]$$

where on the right stands the set of homotopy classes of continuous maps $(X, Y) \to (T_n, *)$ sending Y to the marked point $*$.

To take into account the action of the symmetry group G suppose that it acts on the pair (X, Y) by a continuous homeomorphism φ. In this case we can introduce the G-invariant generalized cohomology theory given by the functor

$$h^n_G(X, Y) = h^n(EG \times_G X, EG \times_G Y)$$

where $EG \to BG$ is the classifying bundle over the classifying space BG, and $EG \times_G X$ denotes the quotient $(EG \times X)/G$. In particular, for $X = *$ we get

$$h^n_G(*) = h^n(BG).$$

Denote by F_d the space of d-dimensional SRE-states. Then for such spaces we have the property

$$F_d \sim \Omega F_{d+1} \quad \text{for } d \geq 0.$$

If this property is fulfilled for $d \geq 0$ then it allows to define by induction the spaces F_d for all $d \in \mathbb{Z}$. So the family of the spaces $(F_d)_{d \in \mathbb{Z}}$ will form an Ω-spectrum.

Here what is known about the homotopy groups of the spaces F_d. The group $\pi_0(F_d)$ classifies the d-dimensional SPT-phases without symmetry. In lower dimensions this group is equal to

$$\pi_0(F_0) = 0, \ \pi_0(F_1) = 0, \ \pi_0(F_2) = \mathbb{Z}, \ \pi_0(F_3) = 0$$

(the group $\mathbb{Z}$ in dimension 2 is generated by the so called E_8-phase).

Note that the condition $F_d \sim \Omega F_{d+1}$ implies that

$$\pi_k(F_d) \cong \pi_{k+1}(F_{d+1}).$$

The space F_0 is identified with the infinite-dimensional projective space

$$F_0 = \mathbb{CP}^\infty$$

and the other spaces F_d of lower dimensions are described in terms of the Eilenberg–Mac Lane spaces $K(\mathbb{Z}, n)$ as

$$F_1 = K(\mathbb{Z}, 3),\ F_2 = K(\mathbb{Z}, 4) \times \mathbb{Z},\ F_3 = K(\mathbb{Z}, 5) \times \mathrm{U}(1).$$

Consider the application of the introduced notions to fermionic systems with the so called hourglass symmetry. These are the symmetry groups including the charge conjugation symmetry U(1), time reversion symmetry T with $T^2 = -1$ and glide symmetry given by the composition of the translation to half-period with reflection.

As an example of systems with glide symmetry we can take the three-dimensional system in which the planes with constant coordinate $x \in \mathbb{Z}$ are occupied by the two-dimensional systems (quantum spin Hall insulators), and the planes with constant coordinate $x \in \mathbb{Z} + 1/2$ are occupied with their mirror reflections. The obtained system is invariant under the glide given by the map: $(x, y, z) \mapsto (x + \frac{1}{2}, -y, z)$. We call this procedure the alternating fibers construction.

The constructed system may be described in terms of the diagram

$$\mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$$

connecting topological insulators in two and three dimensions. In dimension 2 the generator of the first group $\mathbb{Z}_2$ is the quantum spin Hall insulator (QSH-phase). We can assign to it the three-dimensional system, described above, corresponding to the group $\mathbb{Z}_4$ and having the hourglass symmetry. Transition from the group $\mathbb{Z}_4$ to the second group $\mathbb{Z}_2$ is done by "forgetting" the glide symmetry.

IV. CONNECTION WITH K-THEORY

Consider the Hamiltonians H acting in the Hilbert space $\mathcal{H}$ and satisfying gap condition. Denote by Γ the spectral flattening $\operatorname{sgn} H$ of the Hamiltonian H. In other words, Γ is the grading operator, belonging to the same phase as H, with the spectrum consisting of two points $\{+1, -1\}$. The space of grading operators Γ, acting in the Hilbert space $\mathcal{H}$, is denoted by $\mathrm{Grad}(\mathcal{H})$.

Two grading operators Γ_1, Γ_2 are called homotopic if they can be connected by the continuous path inside $\mathrm{Grad}(\mathcal{H})$. The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ with $\Gamma_1, \Gamma_2 \in \mathrm{Grad}(\mathcal{H})$ is called the ordered difference between the grading operators Γ_1, Γ_2 or the corresponding Hamiltonians H_1, H_2. If Γ_1 is homotopic to Γ_2 in this triple we call such triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ trivial.

We can extend the definition given above by including into consideration the symmetry group G. Namely, denote by $\mathcal{A}$ the C^*-algebra on which the group G acts by the representation $\alpha : G \to \operatorname{Aut} \mathcal{A}$. Let W be a finitely generated $\mathcal{A}$-module and $\operatorname{Grad}_{\mathcal{A}}(W)$ denotes the space of $\mathcal{A}$-compatible grading operators acting in W. The above definitions, related to $\operatorname{Grad}(\mathcal{H})$, immediately extend to the case $\operatorname{Grad}_{\mathcal{A}}(W)$. The direct sum operation provides $\operatorname{Grad}_{\mathcal{A}}(W)$ with the structure of the Abelian monoid.

Tiang has proposed the following definition of K-functor. Denote by $\mathbf{K_0}(\mathcal{A})$ the quotient of the monoid $\mathrm{Grad}_{\mathcal{A}}(W)$ with respect to the equivalence relation determined by trivial triples. In more detail, the triple (W, Γ_1, Γ_2) is equivalent to the triple $(W', \Gamma'_1, \Gamma'_2)$ if there exist the trivial triples (V, Δ_1, Δ_2) и $(V', \Delta'_1, \Delta'_2)$ such that

$$(W \oplus V, \Gamma_1 \oplus \Delta_1, \Gamma_2 \oplus \Delta_2) = (W' \oplus V', \Gamma'_1 \oplus \Delta'_1, \Gamma'_2 \oplus \Delta'_2)$$

in $\mathrm{Grad}_{\mathcal{A}}(W)$. The group $\mathbf{K_0}(\mathcal{A})$ is called the group of differences of $\mathcal{A}$-compatible gapped Hamiltonians. This group is Abelian and $-[W, \Gamma_1, \Gamma_2] = [W, \Gamma_2, \Gamma_1]$. It satisfies also the condition

$$[W, \Gamma_1, \Gamma_2] + [W, \Gamma_2, \Gamma_3] = [W, \Gamma_1, \Gamma_3]$$

in $\mathbf{K_0}(\mathcal{A})$.

Let G be the symmetry group of the Hamiltonian. We shall provide it with the following homomorphisms:

1. $\phi : G \to \{\pm 1\}$ is responsible for the unitarity of the action of the element $g \in G$: this action is unitary if $\phi(g) = +1$, and it is anti-unitary if $\phi(g) = -1$;

2. $c : G \to \{\pm 1\}$ is responsible for the charge preservation: the action of $g \in G$ commutes with the Hamiltonian if $c(g) = +1$, and it anti-commutes with it if $c(g) = -1$;

3. $\tau : G \to \{\pm 1\}$ is responsible for the preservation of time direction: the action of $g \in G$ preserves the time direction if $\tau(g) = +1$, and it inverts it if $\tau(g) = -1$.

Consider a concrete example of the group G called the CT-group. It is generated by the unit and three generators T, C, S where

1. $\phi(T) = -1$, $c(T) = +1$;
2. $\phi(C) = -1$, $c(C) = -1$;
3. $\phi(S) = +1$, $c(S) = +1$.

The generators T, C and $S = CT = TC$ correspond to the symmetries of the time inversion, charge conjugation and chiral symmetry respectively. We are interested in the graded representations of the CT-group and its subgroups. We denote the operators, corresponding to the generators of the group G, by $\hat{T}$, $\hat{C}$ and $\hat{S}$ respectively. Then we have the following possibilities: $\hat{T}^2 = \pm 1$, $\hat{C}^2 = \pm 1$ and $\hat{S} = \hat{C}\hat{T} = \hat{T}\hat{C}$. The family of pairwise anti-commuting odd operators $\{\hat{C}, i\hat{C}, i\hat{C}\hat{T}\}$ generates the graded representation of the real Clifford algebra $\mathrm{Cl}_{r,s}$ where r (resp. s) is the number of the negatively (resp. positively) determined selfadjoint generators so that the representation of the full CT-group G coincides with the graded $*$-representation of the Clifford algebra $\mathrm{Cl}_{r,s}$.

In the case of the subgroup $A = \{1, C\}$ we can take for the odd generators the representation operators $\{\hat{C}, i\hat{C}\}$ with $\hat{C}^2 = \pm 1$ generating the graded representation of the Clifford algebras $\mathrm{Cl}_{0,2}$ or $\mathrm{Cl}_{2,0}$. In the case of the subgroup $A = \{1, S\}$ we have necessarily $\hat{S}^2 = +1$ so that the obtained representation coincides with the graded representation of the complex Clifford algebra $\mathbb{C}\mathrm{l}_1$. And in the case of the subgroup $A = \{1, T\}$ we have two choices for $\hat{T}^2 = \pm 1$. The family of operators $\{i, \hat{T}, i\hat{T}\hat{\Gamma}\}$, where $\hat{\Gamma}$ is the grading operator, generates the (non-graded) representation of the Clifford algebra $\mathrm{Cl}_{1,2}$ for $\hat{T}^2 = +1$ and of $\mathrm{Cl}_{3,0}$ for $\hat{T}^2 = -1$.