Non-Local Equations in dS-spacetime

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International Conference Dedicated to the 100th Anniversary of the Birthday of V. S. Vladimirov

09. 01. 2023 - 14. 01. 2023

Outlook

- Introduction
  - $\mathfrak{Quid\ \ est\ \ Mathematical - Physics\ \ Equations}$
  - NonLocal Equations in Theoretical Physics
- Cauchy problem in de Sitter spacetime
- Space-homogenous solutions of non-linear non-local equations in dS

- Vladimirov V. S. "Equations of mathematical physics Moscow Nauka. – 1981.
- From the contents

Глава I

В. С. ВЛАДИМИРОВ

УРАВНЕНИЯ МАТЕМАТИЧЕСКОЙ ФИЗИКИ

ИЗДАНИЕ ЧЕТВЕРТОЕ,
ИСПРАВЛЕННОЕ И ДОПОЛНЕННОЕ

Допущено Министерством высшего и среднего специального образования СССР в качестве учебника для студентов физических и механико-математических специальностей высших учебных заведений

МОСКВА «НАУКА»
ГЛАВНАЯ РЕДАКЦИЯ
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1981

Local Equations

Quid est Mathematical – Physics Equations

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Глава IV

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Integral (Non-Local) Equations

NonLocal Equations in Theoretical Physics

- To eliminate UV divergencies
  - $\int d^D x \Phi(x)^4 \to \int d^D x \Phi(x+a)\Phi(x+b)\Phi(x+c)\Phi(x+d)$

$$\Phi(x+a) = e^{ia_\mu \partial_\mu}\Phi(x) = \sum_n^{\infty}(ia_\mu \partial_\mu)^n \Phi(x)$$

$$\Phi(x) = \phi_0 + \phi(x) \quad \int d^D x \Phi^4 \to \phi_0^2 \int d^D x \phi(x) \sum_n^{\infty}\sum_k^{\infty}(ia_\mu \partial_\mu)^n (ib_\mu \partial_\mu)^k \phi(x)$$

  - Field Theory on Lattices
- Effective Actions (after integration on a part of fields) $L_{eff} = \operatorname{Tr}\log[\Box + m^2 + \phi^2]$
- **Modes truncated action in String Field Theory**

$$\mathcal{L}_{tranc.} = \Phi(\Box + m^2)\Phi + \lambda(e^{\alpha'\Box}\Phi)^4 \to \widetilde{\Phi}(\Box + m^2)e^{-2\alpha'\Box}\widetilde{\Phi} + \lambda\widetilde{\Phi}^4$$

IA, Belov, Koshelev, Medvedev, hep/th 0011117
IA, astro-ph/0410443, I.A, Koshelev, Vernov, astro-ph/0412619

- Effective Actions for "Quantum Gravity ... Non-local cosmological solutions
  ... Dimitrijevic, Dragovich, Rakic, Stankovic, hep-th/2206.13515...

VS Vladimirov papers on NonLocal equations

- V. S. Vladimirov, “Nonexistence of solutions of the p-adic strings”, Theoret. and Math. Phys., 174:2 (2013), 178–185
- V. S. Vladimirov, “Solutions of p-adic string equations”, Theoret. and Math. Phys., 167:2 (2011), 539–546
- V. S. Vladimirov, “On the equations for p-adic closed and open strings”, P-Adic Numbers, Ultrametric Analysis, and Applications, 1:1 (2009), 79–87
- V. S. Vladimirov, “On Nonlinear Equations of p-adic Strings for Scalar Tachyon Fields”, Proc. Steklov Inst. Math., 265 (2009), 242–261
- V. S. Vladimirov, “The question of the asymptotic behavior as $|t| \to$ of boundary value problem solutions for p-adic strings”, Theoret. and Math. Phys., 157:3 (2008), 1638–1645
- V. S. Vladimirov, “The equation of the p-adic open string for the scalar tachyon field”, Izv. Math., 69:3 (2005), 487–512
- V. S. Vladimirov, Ya. I. Volovich, “Nonlinear Dynamics Equation in p-Adic String Theory”, Theoret. and Math. Phys., 138:3 (2004), 297–309

Non-locality with $\mathcal{F}(\Box)$. Cauchy problem?

To compare: Cauchy problem for local theories

- Cauchy problem in the Minkowski spacetime
  - Free case (including external fields)
    - Klein-Gordon
  - Semilinear
    - Klein-Gordon
- **Cauchy problem in de Sitter spacetime**
  - Free case (including external fields)
    - Klein-Gordon
  - Semilinear
    - Klein-Gordon
- Cauchy problem in the Friedmann (FLRW) spacetime

  ...

Non-locality with $\mathcal{F}(\Box)$. Cauchy problem? 2/3
To compare: for theories with higher derivatives

- Cauchy problem in the Minkowski spacetime
  - Free case (including external fields)
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon **?**
- Cauchy problem in de Sitter spacetime
  - Free case (including external fields)
    - Klein-Gordon
    - **Higher derivatives Klein-Gordon**
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon **?**
- Cauchy problem in the Friedmann (FLRW) spacetime

  ...

Non-locality: "Cauchy problem " $\to$? 3/3

- Cauchy problem in the Minkowski spacetime

WHAT DATA SET SOLUTIONS?

  - Free case (including external fields)
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
    - **Non-Local**
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
    - **Non-Local**
- Cauchy problem in de Sitter spacetime
  - Free case (including external fields)
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
    - **Non-Local**
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
    - **Non-Local**
- Cauchy problem in the Friedmann (FLRW) spacetime

  ...

To compare: Cauchy problem for the semilinear Klein-Gordon equations in $\mathbb{M}^{1,n}$

- A large amount of work has been devoted to the Cauchy problem for the semilinear Klein-Gordon equations in $\mathbb{M}^{1,n}$, in particular for

$$u_{tt} - \Delta u + m^2 u = -V'(u), \qquad V'(u) = \lambda |u|^{\alpha} u \qquad \text{with} \quad \alpha \geq 0, \quad \lambda \geq 0$$

  Δ – the Laplace operator in $\mathbb{R}^n$

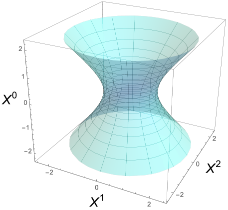
  - Conservation of energy
  - For finite energy solutions $\exists$ for $\alpha < 4/(n-1)$.
  - The existence of global weak solutions has been obtained by
    I.Segal, 1963,... J. Ginibre, G. Velo, 1985,...

De Sitter space-time. Global and planar coordinates

dS_d **is the hyperboloid** $-X_0^2 + X_1^2 + \cdots + X_d^2 = \ell^2$

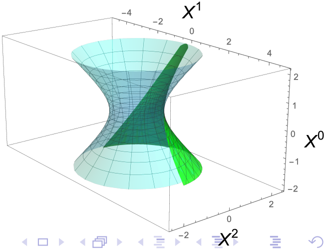
- *In the global coordinates* (τ, θ_i)
  $X^0 = \sinh\tau \quad X^i = \omega^i \cosh\tau$
  $i = 1, \ldots, d,\ \ell = 1$

  The metric on dS_d:
  $ds^2 = -d\tau^2 + (\cosh^2\tau) d\Omega^2_{d-1}$



- *In the planar coordinates:*
  $X^0 = \sinh t - \frac{1}{2} x_i x^i e^{-t} \quad X^i = x^i e^{-t}$
  $X^d = \cosh t - \frac{1}{2} x_i x^i e^{-t}, \quad i = 1, \ldots, d-1$

  $ds^2 = -dt^2 + e^{2t}\, dx^i dx_i$



Cauchy problem for the semilinear K-G eq. in dS.

De Sitter space-time. Global and static coordinates

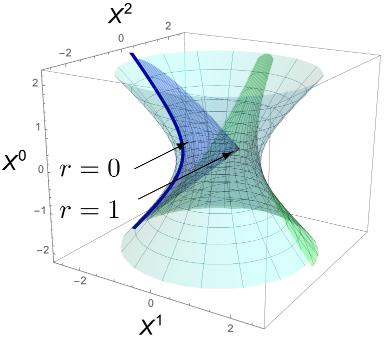
- *In the static coordinates, d=2:*

$$X^0 = \sqrt{1-r^2}\,\sinh\mathfrak{t}$$
$$X^i = \sqrt{1-r^2}\,\cosh\mathfrak{t}$$
$$X^2 = r$$

$$ds^2 = -(1-r^2)d\mathfrak{t}^2 + \frac{dr^2}{1-r^2}$$

- Static vs plane coordinates (d=2)

$$r = xe^t \quad \mathfrak{t} = t - \frac{1}{2}\ln(-1 + x^2 e^2 t)$$



Cauchy problem for the semilinear K-G eq. in dS.
De Sitter space-time. Global and static coordinates

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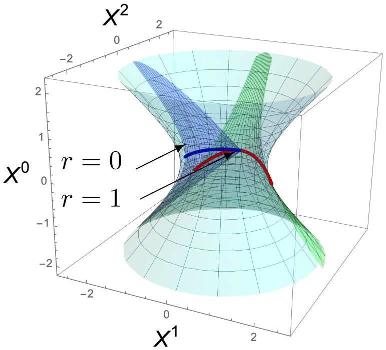
- Static vs plane coordinates (d=2)

$$r = xe^t \quad \mathfrak{t} = t - \frac{1}{2}\ln(-1+x^2e^{2t})$$

- Initial data at $t=0$ (red line) vs initial data at $\mathfrak{t}=0$ (blue line)



Klein-Gordon equations in dS

Semilinear KG eq. : $(\Box - m^2)\phi \equiv \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}g^{\mu\nu}\frac{\partial\phi}{\partial x^\nu}\right) - m^2\phi = V'(\phi)$

dS the planar coordinates: $ds^2 = -dt^2 + e^{2t}\, dx^i dx_i, \quad i = 1, 2, ...n = d-1$

$$\mathcal{S}u = -e^{\frac{n}{2}t}V'(e^{-\frac{n}{2}t}u)$$

$$\mathcal{S}u \equiv (\partial^2_{tt} - e^{-2t}\triangle + M^2)u, \quad u = e^{-\frac{n}{2}t}\phi, \quad M^2 = m^2 - \frac{n^2}{4}$$

Two cases

- $M^2 = m^2 - \frac{n^2}{4} > 0$
- $M^2 = m^2 - \frac{n^2}{4} < 0, \qquad \bar{M}^2 = \frac{n^2}{4} - m^2 > 0$

The Cauchy problem for the Klein-Gordon equation in dS

- The fundamental solutions $\mathcal{E} = \mathcal{E}_{\pm}(x,t;x_0,t_0)$ are solutions of

  $$\mathcal{S}E_{\pm} = \delta(x-x_0,t-t_0), \qquad \mathcal{S} \equiv (\partial^2_{tt} - e^{-2t}\triangle + M^2)$$

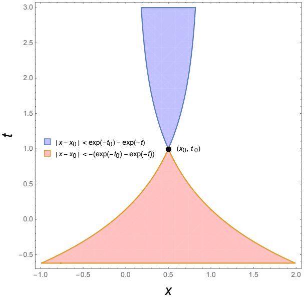
  with supports in the forward/backward light cone

  $$D_{\pm}(x_0,t_0) := \left\{(x,t) \in \mathbb{R}^{n+1}\,;\, |x-x_0| \leq \pm(e^{-t_0}-e^{-t})\right\}$$

- The tachyonic fundamental solutions $\mathcal{E}_t = \mathcal{E}_{\pm,t}(x,t;x_0,t_0)$ are solutions of

  $$\mathcal{S}_t E_{\pm,t} = \delta(x-x_0,t-t_0), \qquad \mathcal{S}_t \equiv (\partial^2_{tt} - e^{-2t}\triangle - \bar{M}^2)$$

  with supports in $D_{\pm}(x_0,t_0)$



$$\mathcal{E}_{+}(x,t;0,t_0) := E(x,t;0,t_0) \quad \text{in } D_{+}(0,t_0), 0 \qquad \text{elsewhere}, \qquad \mathcal{E}_{-}(x,t;0,t_0) := E(x,$$

Since function $E = E(x,t;0,t_0)$ is smooth in $D_{\pm}(0,t_0)$ and is locally integrable, it follows that $\mathcal{E}_{+}(x,t;0,\ t_0)$ and $\mathcal{E}_{-}(x,t;0,\ t_0)$ are distributions whose supports are in $D_{+}(0,t_0)$ and $D_{-}(0,t_0)$, respectively.

The Cauchy problem for the Klein-Gordon equation in dS

Fundamental solutions $\mathcal{E}$ and $\mathcal{E}_t$ have been used to represent solutions of the Cauchy problems

$$(*) \qquad u_{tt} - e^{-2t} \triangle u + M^2 u = f, \quad u(x,0) = 0, \quad \partial_t u(x,0) = 0, \quad t > 0, x \in \mathbb{R}$$

$$(**) \qquad u_{tt} - e^{-2t} \triangle u - \bar{M}^2 u = f, \quad u(x,0) = 0, \quad \partial_t u(x,0) = 0 \quad t > 0, x \in \mathbb{R}$$

(*/**) are strictly hyperbolic $\Rightarrow$ the Cauchy problems (*/**) are well-posed

The Cauchy problem for the Klein-Gordon equation in dS, , $M^2 > 0$

- For $M^2 > 0$

$$u(x,t) = \int_0^t db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy \, \frac{f(y,b)}{(4e^{b+t})^{iM}} \frac{F\left(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^{-b}-e^{-t})^2-(x-y)^2}{(e^{-b}+e^{-t})^2-(x-y)^2}\right)}{\left((e^{-t}+e^{-b})^2-(x-y)^2\right)^{\frac{1}{2}+iM}} (\#)$$

where $F\big(a,b;c;\zeta\big)$ is the hypergeometric function ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$

Theorem I. *Assume that the function f is continuous along with its all second order derivatives, and that for every fixed t it has a compact support, $supp f(\cdot,t) \subset \mathbb{R}$. Then the function $u = u(x,t)$ defined by (#) is a C^2-solution to the Cauchy problem for the equation (*) with vanishing initial data.*

K. Yagdjian, A. Galstian, CMP, 285 (2009) 293

The Cauchy problem for the Klein-Gordon equation in dS, $M^2 > 0$

Theorem II. *The solution $u = u(x,t)$ of the Cauchy problem*

$$(*') \quad u_{tt} - e^{-2t} u_{xx} + M^2 u = 0 \quad M^2 > 0 \qquad u(x,0) = \varphi_0(x) \quad u_t(x,0) = \varphi_1(x)$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$ can be represented as

$$u(x,t) = \frac{1}{2} e^{\frac{t}{2}} \Big[\varphi_0(x+1-e^{-t}) + \varphi_0(x-1+e^{-t})\Big] + \int_0^{1-e^{-t}} \Big[\varphi_0(x-z) + \varphi_0(x+z)\Big] K_0(z,t)\, dz$$

$$+ \int_0^{1-e^{-t}} \Big[\varphi_1(x-z) + \varphi_1(x+z)\Big] K_1(z,t) dz\,,$$

where the kernels $K_0(z,t)$ and $K_1(z,t)$ are defined by

$$K_0(z,t) = -\left[\frac{\partial}{\partial b} E(z,t;0,b)\right]_{b=0}, \qquad K_1(z,t) = E(z,t;0,0)$$

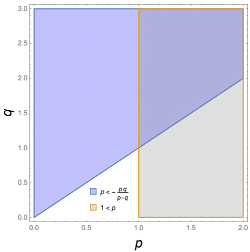
$$E(x,t;x_0,t_0) = (4e^{-t_0-t})^{iM} \frac{F\left(\frac{1}{2}+iM, \frac{1}{2}+iM; 1; \frac{(e^{-t_0}-e^{-t})^2-(x-x_0)^2}{(e^{-t_0}+e^{-t})^2-(x-x_0)^2}\right)}{\left((e^{-t}+e^{-t_0})^2-(x-x_0)^2\right)^{\frac{1}{2}+iM}}$$

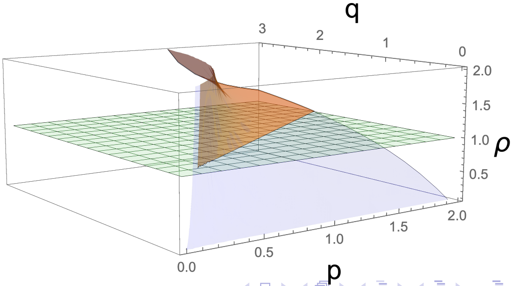
The Klein-Gordon Cauchy problem, $L^p - L^q$ Estimates, $n = 1, \quad M^2 > 0$

Theorem III. *For every function* $f \in C^2(\mathbb{R} \times [0, \infty))$ *such that* $f(\cdot, t) \in C_0^\infty(\mathbb{R}_x)$ *the solution* $u = u(x, t)$ *of the Cauchy problem with zero initial data satisfies inequality*

$$\|u(x,t)\|_{L^q(\mathbb{R}_x)} \leq C_M e^{t(1-1/\rho)} \int_0^t (1+t-b)(e^{t-b}-1)^{1/\rho}(e^{t-b}+1)^{-1} \|f(x,b)\|_{L^p(\mathbb{R}_x)}\, db$$

for all $t > 0$, *where* $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\rho < 2$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$.





The Klein-Gordon Cauchy problem, $L^p - L^q$ Estimates, $n = 1, \quad M^2 > 0$

Theorem IV. *Let $u = u(x,t)$ be a solution of the Cauchy problem*

$$u_{tt} - e^{-2t} u_{xx} + M^2 u = 0\,, \qquad u(x,0) = \varphi_0(x)\,, \qquad u_t(x,0) = \varphi_1(x)\,,$$

with φ_0, $\varphi_1 \in C_0^\infty(\mathbb{R})$. If $\rho \in (1,2)$, then

$$\|u(x,t)\|_{L^q(\mathbb{R}_x)} \le e^{\frac{t}{2}} \|\varphi_0(x)\|_{L^q(\mathbb{R}_x)} + C_\rho (1+t)(e^t - 1)^{\frac{1}{\rho}} e^{t[\frac{1}{2} - \frac{1}{\rho}]} \|\varphi_0(x)\|_{L^p(\mathbb{R}_x)}$$

$$+ C_\rho (1+t)(e^t - 1)^{\frac{1}{\rho}} e^{-\frac{t}{\rho}} \|\varphi_1(x)\|_{L^p(\mathbb{R}_x)}$$

for all $t \in (0,\infty)$. Here $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$.

To compare: $L^p - L^q$ Estimates for Wave eq. in $\mathbb{M}^{1,n}$

Cauchy problem for the wave equation $\mathbb{M}^{1,n}$

$$v_{tt} - \triangle v = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0$$

with $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ one has the following $L^p - L^q$ decay estimate

$$\|(-\triangle)^{-s} v(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq C t^{2s - n(\frac{1}{p} - \frac{1}{q})} \|\varphi\|_{L^p(\mathbb{R}^n)} \quad \text{for all} \quad t > 0,$$

under assumptions

$s \geq 0$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\qquad \frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) \leq 2s \leq n(\frac{1}{p} - \frac{1}{q})$.

The Cauchy problem for the Klein-Gordon operator in dS, $\bar{M}^2 > 0$

- For $\bar{M}^2 > 0$

$$(**) \qquad u_{tt} - e^{-2t}\triangle u - \bar{M}^2 u = f, \quad u(x,0) = 0, \quad \partial_t u(x,0) = 0 \quad t > 0,\, x \in \mathbb{R}$$

$$u = \int_0^t db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy\, \frac{f(y,b)}{(4e^{-b-t})^M} \frac{F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^{-b}-e^{-t})^2-(x-y)^2}{(e^{-b}+e^{-t})^2-(x-y)^2}\right)}{\left((e^{-t}+e^{-b})^2-(x-y)^2\right)^{\frac{1}{2}-M}}$$

where $F\big(a,b;c;\zeta\big)$ is the hypergeometric function ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$

The one-dimensional integral

$$\begin{aligned} \frac{1}{M}\sinh(M(t-b)) \quad &= \quad \int_{-(e^{-b}-e^{-t})}^{e^{-b}-e^{-t}} (4e^{-b-t})^{-M} \left((e^{-t}+e^{-b})^2 - z^2\right)^{-\frac{1}{2}+M} \\ &\times F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^{-b}-e^{-t})^2 - z^2}{(e^{-b}+e^{-t})^2 - z^2}\right) dz; \end{aligned}$$

The Klein-Gordon Cauchy problem, $L^p - L^q$ Estimates, tachyon

Theorem V. *The solution* $\Phi = \Phi(x,t)$ *of the Cauchy problem*

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi \pm m^2\Phi = 0, \quad \Phi(x,0) = \varphi_0(x), \quad \Phi_t(x,0) = \varphi_1(x),$$

satisfies the following $L^p - L^q$ *estimate*

$$\left\|(-\Delta)^{-s}\Phi(x,t)\right\|_{L^q(\mathbb{R}^n)} \leq C_{M,n,p,q,s}\left(1 - e^{-t}\right)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \mathbf{e}^{\left(M-\frac{n}{2}\right)t}$$

$$\times \left\{\|\varphi_0\|_{L^p(\mathbb{R}^n)} + \left(1 - e^{-t}\right)\|\varphi_1\|_{L^p(\mathbb{R}^n)}\right\}$$

for all $t \in (0,\infty)$ *under conditions*

- $1 < p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s + 1.$
- $M = \sqrt{\frac{n^2}{4} - m^2}$ and $m < \sqrt{n^2 - 1}/2$ for $(+)$
- $M = \sqrt{\frac{n^2}{4} + m^2}$ for (-)

K.Yagdjian, J.Math.Appl. 396 (2012) 323

- Cauchy problem in the Minkowski spacetime
  - Free case (including external fields)
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
      $(\Box - m_1^2)...(\Box - m_n^2)\phi = J$
      Auxiliary fields: $\{\psi_1, ...\psi_{n-1}\}$
      $(\Box - m_1^2)\Psi_1 = J$, $(\Box - m_2^2)\Psi_2 = \Psi_1$, ... Pais, Uhlenbeck, 1950
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
      ?
- Cauchy problem in de Sitter spacetime
  - Free case (including external fields)
    - Klein-Gordon $\to$
    - Higher derivatives Klein-Gordon
      Auxiliary fields as in $M^{1,d-1}$
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
      ?
- Cauchy problem in the Friedmann (FLRW) spacetime

...

The Cauchy problem for equations with higher derivatives kinetic operators

$$F(\Box)\Phi(x) = J(x) \quad x \in \mathbb{M}^{(1,n)}$$

A.Pais, G.Uhlenbeck, Field theories with non-localized action, Phys.Rev.,1950

- $F(z)$ is a polynomial on z
- $F(\Box)$ via the Fourier transform $\varphi(x) \to \tilde{\varphi}(\xi)$,

$$\begin{aligned} F(\Box)\varphi(x) &= \frac{1}{(2\pi)^n}\int F(-\xi^2)\tilde{\varphi}(\xi)\, e^{-ix\xi} d^n\xi \\ \tilde{\varphi}(\xi) &= \int \varphi(x) e^{i\xi x} d^n x, \end{aligned}$$

- This definition is natural is SFT, where all expressions came from calculations in the momentum space
- $\Box \to -\partial_t^2, \quad -\infty < t < \infty$

$e^{\partial_t^2}$ and Fock method (diffusion equation)

V.A. Fock, *The eigen-time in classical and quantum mechanics*, **1937**

- To give the mathematical meaning to the operator $e^{\partial_t^2}$ one can use an auxiliary function of two variables

$$\Psi(\tau,t) = e^{\tau\partial_t^2}\varphi(t)$$

satisfying the heat (diffusion) equation

$$\begin{array}{rcll} \text{Eq.}: \quad (\partial_\tau - \partial_t^2)\Psi(t,\tau) & = & 0, & (*) \\ \text{Initial condition}: \quad \Psi(t,\tau)|_{\tau=0} & = & \varphi(t). & (**) \end{array}$$

- Action of $e^{\partial_t^2}$ on a function $\varphi(t)$ is defined by means of the solution to (*) with the boundary condition (**) as follows

$$e^{\partial_t^2}\varphi(t) \equiv \Psi(t,\tau)|_{\tau=1} \qquad (***)$$

Diffusion equation on $\mathbb{R}$ and $\mathbb{R}_+$

$$\begin{array}{rcll} \text{Eq. :} \quad (\partial_\tau - \partial_t^2)\Psi(t,\tau) & = & 0, & (*) \\ \text{Initial condition :} \quad \Psi(t,\tau)|_{\tau=0} & = & \varphi(t). & (**) \end{array}$$

- Now it is important to distinguish two different cases:
  - i) whole real line $-\infty < t < \infty$,
  - ii) half-line $0 < t < \infty$.
- In quantum field theory in Minkowski space-time the time variable t runs from $-\infty$ to ∞.
- In cosmology there is a reason to restrict the time variable to the half axis only, $t > 0$, since in the Friedman cosmology there is a cosmological singularity at $t = 0$
- In the case of the whole real axis under suitable assumptions there is a unique solution of the problem (*), (**).

Dirichlet's initial-boundary value problem.

- On a half-line one has to add an extra boundary condition at $t = 0$
- Mixed initial-boundary value problem:

$$\begin{cases} \frac{\partial}{\partial\tau}\Psi_D(t,\tau) = \frac{\partial^2}{\partial t^2}\Psi_D(t,\tau), \quad t>0, \quad \tau>0, \\ \Psi_D(t,0) = \varphi(t), \qquad\qquad (***) \\ \Psi_D(0,\tau) = \mu(\tau) \qquad \textbf{Dirichlet's Daemon} \end{cases}$$

I.A, I.Volovich, ***Cosmological Daemon*****, hep-th/1103.0273**

- This is the Dirichlet's problem for the heat equation on the half-line.
- The function $\mu(\tau)$ will be called the Dirichlet daemon function. It describes the Dirichlet boundary conditions at $t = 0$.
- The solution of (***) is given by

$$\begin{aligned} \Psi_D(t,\tau) &= \frac{1}{\sqrt{4\pi\tau}}\int_0^\infty \varphi(t')\,[e^{-\frac{(t-t')^2}{4\tau}} - e^{-\frac{(t+t')^2}{4\tau}}]dt' \\ &+ \frac{t}{\sqrt{4\pi}}\int_0^\tau \frac{\mu(\tau')}{(\tau-\tau')^{3/2}}e^{-\frac{t^2}{4(\tau-\tau')}}\,d\tau' \end{aligned}$$

Non-locality: "Cauchy problem " $\to$? 3/3

WHAT DATA SET SOLUTIONS?

- Cauchy problem in the Minkowski spacetime
  - Free case (including external fields)
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
    - **Non-Local**
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
    - **Non-Local**
- Cauchy problem in de Sitter spacetime
  - Free case (including external fields)
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
    - **Non-Local**
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
    - **Non-Local**
- Cauchy problem in the Friedmann (FLRW) spacetime

  ...

Non-locality and Boundary Problems

- Cauchy problem in the Minkowski spacetime
  - Free case (including external fields)
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
- Cauchy problem in de Sitter spacetime
  - Free case (including external fields)
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
  - Semilinear
    - Klein-Gordon
    - Higher derivatives Klein-Gordon
- Cauchy problem in the Friedmann (FLRW) spacetime

We have to set Boundary Conditions

- **Non-Local space-homogeneous**
- **Space-homogeneous**
- **Particular solutions**
- **Non-Local**

Conclusion

- Cauchy problem in de Sitter spacetime (in plane coordinate) similar to Cauchy problem in Minkowski (including higher derivatives cases)
- dS acts as a "tachyon"$m^2 \to m^2 - n^2/4$
- Theories with $e^{\Box}$ require boundary conditions in Minkowski as well as in de Sitter case