

Non-Local Equations in dS-spacetime

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of the Birthday of V. S. Vladimirov*

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Outlook

- Introduction
 - Quid est Mathematical – Physics Equations
 - NonLocal Equations in Theoretical Physics
- Cauchy problem in de Sitter spacetime
- Space-homogenous solutions of non-linear non-local equations in dS

Quid est Mathematical – Physics Equations

- Vladimirov V. S. "Equations of mathematical physics Moscow Nauka. – 1981.

В. С. ВЛАДИМИРОВ

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УРАВНЕНИЯ МАТЕМАТИЧЕСКОЙ ФИЗИКИ

ИЗДАНИЕ ЧЕТВЕРТОЕ,
ИСПРАВЛЕННОЕ И ДОПОЛНЕННОЕ

Допущено Министерством высшего
и среднего специального образования СССР
в качестве учебного
для студентов физических и технических специальностей
специальностей высших учебных заведений



МОСКВА «НАУКА»
ГЛАВНАЯ РЕДАКЦИЯ
ФИЗИКО-МАТЕМАТИЧЕСКОЙ ЛИТЕРАТУРЫ
1981

Local Equations

Quid est Mathematical – Physics Equations

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УРАВНЕНИЯ МАТЕМАТИЧЕСКОЙ ФИЗИКИ

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Integral (Non-Local) Equations

NonLocal Equations in Theoretical Physics

- To eliminate UV divergencies

$$\bullet \int d^D x \Phi(x)^4 \rightarrow \int d^D x \Phi(x+a)\Phi(x+b)\Phi(x+c)\Phi(x+d)$$

$$\Phi(x+a) = e^{ia_\mu \partial_\mu} \Phi(x) = \sum_n (ia_\mu \partial_\mu)^n \Phi(x)$$

$$\Phi(x) = \phi_0 + \phi(x) \quad \int d^D x \Phi^4 \rightarrow \phi_0^2 \int d^D x \phi(x) \sum_n \sum_k (ia_\mu \partial_\mu)^n (ib_\mu \partial_\mu)^k \phi(x)$$

- Field Theory on Lattices
- Effective Actions (after integration on a part of fields) $L_{eff} = \text{Tr} \log[\square + m^2 + \phi^2]$
- **Modes truncated action in String Field Theory**

$$\mathcal{L}_{trunc.} = \Phi(\square + m^2)\Phi + \lambda(e^{\alpha' \square} \Phi)^4 \rightarrow \tilde{\Phi}(\square + m^2)e^{-2\alpha' \square} \tilde{\Phi} + \lambda \tilde{\Phi}^4$$

IA, Belov, Koshelev, Medvedev, hep/th 0011117

IA, astro-ph/0410443, I.A, Koshelev, Vernov, astro-ph/0412619

- Effective Actions for "Quantum Gravity ... Non-local cosmological solutions
... **Dimitrijevic, Dragovich, Rakic, Stankovic, hep-th/2206.13515...**

VS Vladimirov papers on NonLocal equations

- V. S. Vladimirov, “Nonexistence of solutions of the p-adic strings”, Theoret. and Math. Phys., 174:2 (2013), 178–185
- V. S. Vladimirov, “Solutions of p-adic string equations”, Theoret. and Math. Phys., 167:2 (2011), 539–546
- V. S. Vladimirov, “On the equations for p-adic closed and open strings”, P-Adic Numbers, Ultrametric Analysis, and Applications, 1:1 (2009), 79–87
- V. S. Vladimirov, “On Nonlinear Equations of p-adic Strings for Scalar Tachyon Fields”, Proc. Steklov Inst. Math., 265 (2009), 242–261
- V. S. Vladimirov, “The question of the asymptotic behavior as $|t| \rightarrow 0$ of boundary value problem solutions for p-adic strings”, Theoret. and Math. Phys., 157:3 (2008), 1638–1645
- V. S. Vladimirov, “The equation of the p-adic open string for the scalar tachyon field”, Izv. Math., 69:3 (2005), 487–512
- V. S. Vladimirov, Ya. I. Volovich, “Nonlinear Dynamics Equation in p-Adic String Theory”, Theoret. and Math. Phys., 138:3 (2004), 297–309

Non-locality with $\mathcal{F}(\square)$. Cauchy problem? 1/3

To compare: Cauchy problem for local theories

- Cauchy problem in the Minkowski spacetime
 - Free case (including external fields)
 - Klein-Gordon
 - Semilinear
 - Klein-Gordon
- Cauchy problem in de Sitter spacetime
 - Free case (including external fields)
 - Klein-Gordon
 - Semilinear
 - Klein-Gordon
- Cauchy problem in the Friedmann (FLRW) spacetime
 - ...

Non-locality with $\mathcal{F}(\square)$. Cauchy problem? 2/3

To compare: for theories with higher derivatives

- Cauchy problem in the Minkowski spacetime
 - Free case (including external fields)
 - Klein-Gordon
 - Higher derivatives Klein-Gordon
 - Semilinear
 - Klein-Gordon
 - Higher derivatives Klein-Gordon?
- Cauchy problem in de Sitter spacetime
 - Free case (including external fields)
 - Klein-Gordon
 - **Higher derivatives Klein-Gordon**
 - Semilinear
 - Klein-Gordon
 - Higher derivatives Klein-Gordon?
- Cauchy problem in the Friedmann (FLRW) spacetime
- ...

WHAT DATA SET SOLUTIONS?

- Cauchy problem in the Minkowski spacetime

- Free case (including external fields)

- Klein-Gordon

- Higher derivatives Klein-Gordon

- Non-Local

- Semilinear

- Klein-Gordon

- Higher derivatives Klein-Gordon

- Non-Local

- Cauchy problem in de Sitter spacetime

- Free case (including external fields)

- Klein-Gordon

- Higher derivatives Klein-Gordon

- Non-Local

- Semilinear

- Klein-Gordon

- Higher derivatives Klein-Gordon

- Non-Local

- Cauchy problem in the Friedmann (FLRW) spacetime

...

To compare: Cauchy problem for the semilinear Klein-Gordon equations in $\mathbb{M}^{1,n}$

- A large amount of work has been devoted to the Cauchy problem for the semilinear Klein-Gordon equations in $\mathbb{M}^{1,n}$, in particular for

$$u_{tt} - \Delta u + m^2 u = -V'(u), \quad V'(u) = \lambda |u|^\alpha u \quad \text{with } \alpha \geq 0, \quad \lambda \geq 0$$

Δ – the Laplace operator in $\mathbb{R}^n$

- Conservation of energy
- For finite energy solutions $\exists$ for $\alpha < 4/(n-1)$.
- The existence of global weak solutions has been obtained by
I.Segal, 1963,... J. Ginibre, G. Velo, 1985,...

Cauchy problem for the semilinear K-G eq. in dS. De Sitter space-time. Global and planar coordinates

dS_d is the hyperboloid $-X_0^2 + X_1^2 + \dots + X_d^2 = \ell^2$

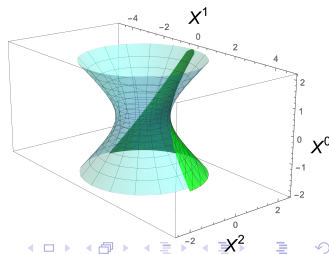
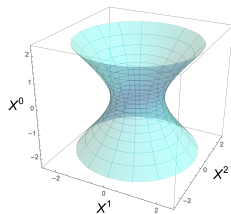
- In the global coordinates (τ, θ_i)
 $X^0 = \sinh \tau$ $X^i = \omega^i \cosh \tau$
 $i = 1, \dots, d, \ell = 1$

The metric on dS_d :

$$ds^2 = -d\tau^2 + (\cosh^2 \tau) d\Omega_{d-1}^2$$

- In the planar coordinates:
 $X^0 = \sinh t - \frac{1}{2} x_i x^i e^{-t}$ $X^i = x^i e^{-t}$
 $X^d = \cosh t - \frac{1}{2} x_i x^i e^{-t}, \quad i = 1, \dots, d-1$

$$ds^2 = -dt^2 + e^{2t} dx^i dx_i$$



Cauchy problem for the semilinear K-G eq. in dS. De Sitter space-time. Global and static coordinates

- In the static coordinates, $d=2$:

$$X^0 = \sqrt{1 - r^2} \sinh t$$

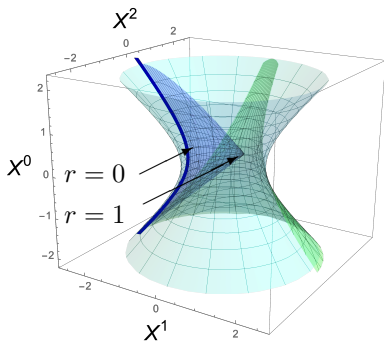
$$X^i = \sqrt{1 - r^2} \cosh t$$

$$X^2 = r$$

$$ds^2 = -(1 - r^2)dt^2 + \frac{dr^2}{1 - r^2}$$

- Static vs plane coordinates ($d=2$)

$$r = xe^t \quad t = t - \frac{1}{2} \ln(-1 + x^2 e^{2t})$$



Cauchy problem for the semilinear K-G eq. in dS. De Sitter space-time. Global and static coordinates

- In the static coordinates, $d=2$:

$$X^0 = \sqrt{1-r^2} \sinh t$$

$$X^i = \sqrt{1-r^2} \cosh t$$

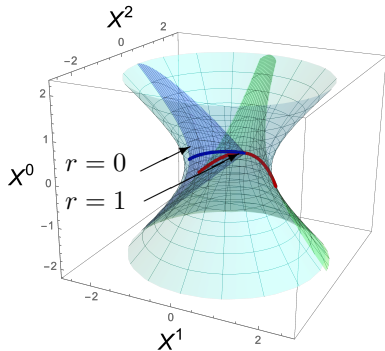
$$X^2 = r$$

$$ds^2 = -(1-r^2)dt^2 + \frac{dr^2}{1-r^2}$$

- Static vs plane coordinates ($d=2$)

$$r = xe^t \quad t = t - \frac{1}{2} \ln(-1 + x^2 e^{2t})$$

- Initial data at $t = 0$ (red line) vs
initial data at $t = 0$ (blue line)



Klein-Gordon equations in dS

Semilinear KG eq. : $(\square - m^2)\phi \equiv \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} \left(\sqrt{|g|} g^{\mu\nu} \frac{\partial \phi}{\partial x^\nu} \right) - m^2 \phi = V'(\phi)$

dS the planar coordinates: $ds^2 = -dt^2 + e^{2t} dx^i dx_i$, $i = 1, 2, \dots, n = d - 1$

$$\mathcal{S}u = -e^{\frac{n}{2}t} V'(e^{-\frac{n}{2}t} u)$$

$$\mathcal{S}u \equiv (\partial_{tt}^2 - e^{-2t} \Delta + M^2)u, \quad u = e^{-\frac{n}{2}t} \phi, \quad M^2 = m^2 - \frac{n^2}{4}$$

Two cases

- $M^2 = m^2 - \frac{n^2}{4} > 0$
- $M^2 = m^2 - \frac{n^2}{4} < 0, \quad \bar{M}^2 = \frac{n^2}{4} - m^2 > 0$

The Cauchy problem for the Klein-Gordon equation in dS

- The fundamental solutions $\mathcal{E} = \mathcal{E}_{\pm}(x, t; x_0, t_0)$ are solutions of

$$\mathcal{S}E_{\pm} = \delta(x - x_0, t - t_0), \quad \mathcal{S} \equiv (\partial_{tt}^2 - e^{-2t} \Delta + M^2)$$

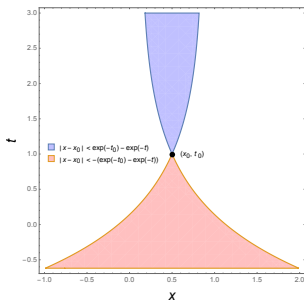
with supports in the forward/backward light cone

$$D_{\pm}(x_0, t_0) := \{(x, t) \in \mathbb{R}^{n+1}; |x - x_0| \leq \pm(e^{-t_0} - e^{-t})\}$$

- The tachyonic fundamental solutions $\mathcal{E}_t = \mathcal{E}_{\pm, t}(x, t; x_0, t_0)$ are solutions of

$$\mathcal{S}_t E_{\pm, t} = \delta(x - x_0, t - t_0), \quad \mathcal{S}_t \equiv (\partial_{tt}^2 - e^{-2t} \Delta - \bar{M}^2)$$

with supports in $D_{\pm}(x_0, t_0)$



$$\mathcal{E}_+(x, t; 0, t_0) := E(x, t; 0, t_0) \quad \text{in } D_+(0, t_0), 0$$

$$\text{elsewhere,} \quad \mathcal{E}_-(x, t; 0, t_0) := E(x, t; 0, t_0)$$

Since function $E = E(x, t; 0, t_0)$ is smooth in $D_{\pm}(0, t_0)$ and is locally integrable, it follows that $\mathcal{E}_+(x, t; 0, t_0)$ and $\mathcal{E}_-(x, t; 0, t_0)$ are distributions whose supports are in $D_+(0, t_0)$ and $D_-(0, t_0)$, respectively.

The Cauchy problem for the Klein-Gordon equation in dS

Fundamental solutions $\mathcal{E}$ and $\mathcal{E}_t$ have been used to represent solutions of the Cauchy problems

$$(*) \quad u_{tt} - e^{-2t} \Delta u + M^2 u = f, \quad u(x, 0) = 0, \quad \partial_t u(x, 0) = 0, \quad t > 0, x \in \mathbb{R}$$

$$(**) \quad u_{tt} - e^{-2t} \Delta u - \bar{M}^2 u = f, \quad u(x, 0) = 0, \quad \partial_t u(x, 0) = 0 \quad t > 0, x \in \mathbb{R}$$

(*/**) are strictly hyperbolic $\Rightarrow$ the Cauchy problems (*/**) are well-posed

The Cauchy problem for the Klein-Gordon equation in dS, , $M^2 > 0$

- For $M^2 > 0$

$$u(x, t) = \int_0^t db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy \frac{f(y, b)}{(4e^{b+t})^{iM}} \frac{F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{-b}-e^{-t})^2 - (x-y)^2}{(e^{-b}+e^{-t})^2 - (x-y)^2}\right)}{\left((e^{-t} + e^{-b})^2 - (x-y)^2\right)^{\frac{1}{2} + iM}} (\#)$$

where $F(a, b; c; \zeta)$ is the hypergeometric function ${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$

Theorem I. *Assume that the function f is continuous along with its all second order derivatives, and that for every fixed t it has a compact support, $\text{supp} f(\cdot, t) \subset \mathbb{R}$. Then the function $u = u(x, t)$ defined by (#) is a C^2 -solution to the Cauchy problem for the equation (*) with vanishing initial data.*

K. Yagdjian, A. Galstian, CMP, 285 (2009) 293

The Cauchy problem for the Klein-Gordon equation in dS, $M^2 > 0$

Theorem II. *The solution $u = u(x, t)$ of the Cauchy problem*

$$(*)' \quad u_{tt} - e^{-2t}u_{xx} + M^2u = 0 \quad M^2 > 0 \quad u(x, 0) = \varphi_0(x) \quad u_t(x, 0) = \varphi_1(x)$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$ can be represented as

$$u(x, t) = \frac{1}{2}e^{\frac{t}{2}} \left[\varphi_0(x + 1 - e^{-t}) + \varphi_0(x - 1 + e^{-t}) \right] + \int_0^{1-e^{-t}} [\varphi_0(x - z) + \varphi_0(x + z)] K_0(z, t) dz \\ + \int_0^{1-e^{-t}} [\varphi_1(x - z) + \varphi_1(x + z)] K_1(z, t) dz,$$

where the kernels $K_0(z, t)$ and $K_1(z, t)$ are defined by

$$K_0(z, t) = - \left[\frac{\partial}{\partial b} E(z, t; 0, b) \right]_{b=0}, \quad K_1(z, t) = E(z, t; 0, 0)$$

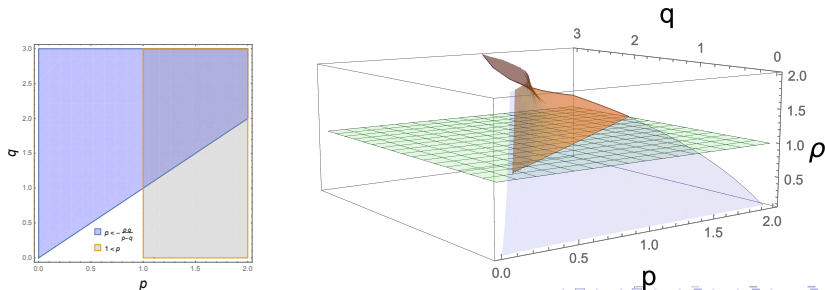
$$E(x, t; x_0, t_0) = (4e^{-t_0-t})^{iM} \frac{F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{-t_0}-e^{-t})^2 - (x-x_0)^2}{(e^{-t_0}+e^{-t})^2 - (x-x_0)^2}\right)}{\left((e^{-t} + e^{-t_0})^2 - (x - x_0)^2\right)^{\frac{1}{2} + iM}}$$

The Klein-Gordon Cauchy problem, $L^p - L^q$ Estimates, $n = 1, \quad M^2 > 0$

Theorem III. For every function $f \in C^2(\mathbb{R} \times [0, \infty))$ such that $f(\cdot, t) \in C_0^\infty(\mathbb{R}_x)$ the solution $u = u(x, t)$ of the Cauchy problem with zero initial data satisfies inequality

$$\|u(x, t)\|_{L^q(\mathbb{R}_x)} \leq C_M e^{t(1-1/\rho)} \int_0^t (1+t-b)(e^{t-b}-1)^{1/\rho} (e^{t-b}+1)^{-1} \|f(x, b)\|_{L^p(\mathbb{R}_x)} db$$

for all $t > 0$, where $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\rho < 2$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$.



The Klein-Gordon Cauchy problem, $L^p - L^q$ Estimates, $n = 1, \quad M^2 > 0$

Theorem IV. *Let $u = u(x, t)$ be a solution of the Cauchy problem*

$$u_{tt} - e^{-2t}u_{xx} + M^2u = 0, \quad u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x),$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$. If $\rho \in (1, 2)$, then

$$\begin{aligned} \|u(x, t)\|_{L^q(\mathbb{R}_x)} &\leq e^{\frac{t}{2}} \|\varphi_0(x)\|_{L^q(\mathbb{R}_x)} + C_\rho(1+t)(e^t - 1)^{\frac{1}{\rho}} e^{t[\frac{1}{2} - \frac{1}{\rho}]} \|\varphi_0(x)\|_{L^p(\mathbb{R}_x)} \\ &\quad + C_\rho(1+t)(e^t - 1)^{\frac{1}{\rho}} e^{-\frac{t}{\rho}} \|\varphi_1(x)\|_{L^p(\mathbb{R}_x)} \end{aligned}$$

for all $t \in (0, \infty)$. Here $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$.

To compare: $L^p - L^q$ Estimates for Wave eq. in $\mathbb{M}^{1,n}$

Cauchy problem for the wave equation $\mathbb{M}^{1,n}$

$$v_{tt} - \Delta v = 0, \quad v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0$$

with $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ one has the following $L^p - L^q$ decay estimate

$$\|(-\Delta)^{-s} v(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq C t^{2s - n(\frac{1}{p} - \frac{1}{q})} \|\varphi\|_{L^p(\mathbb{R}^n)} \quad \text{for all } t > 0,$$

under assumptions

$$s \geq 0, \quad 1 < p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad \frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p} - \frac{1}{q}\right).$$

The Cauchy problem for the Klein-Gordon operator in dS, $\bar{M}^2 > 0$

- For $\bar{M}^2 > 0$

$$(**) \quad u_{tt} - e^{-2t} \Delta u - \bar{M}^2 u = f, \quad u(x, 0) = 0, \quad \partial_t u(x, 0) = 0 \quad t > 0, x \in \mathbb{R}$$

$$u = \int_0^t db \int_{x-(e^{-b}-e^{-t})}^{x+e^{-b}-e^{-t}} dy \frac{f(y, b)}{(4e^{-b-t})^M} \frac{F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b}-e^{-t})^2 - (x-y)^2}{(e^{-b}+e^{-t})^2 - (x-y)^2}\right)}{\left((e^{-t}+e^{-b})^2 - (x-y)^2\right)^{\frac{1}{2}-M}}$$

where $F(a, b; c; \zeta)$ is the hypergeometric function ${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$

The one-dimensional integral

$$\begin{aligned} \frac{1}{M} \sinh(M(t-b)) &= \int_{-(e^{-b}-e^{-t})}^{e^{-b}-e^{-t}} (4e^{-b-t})^{-M} \left((e^{-t}+e^{-b})^2 - z^2\right)^{-\frac{1}{2}+M} \\ &\quad \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b}-e^{-t})^2 - z^2}{(e^{-b}+e^{-t})^2 - z^2}\right) dz; \end{aligned}$$

The Klein-Gordon Cauchy problem, $L^p - L^q$ Estimates, tachyon

Theorem V. *The solution $\Phi = \Phi(x, t)$ of the Cauchy problem*

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi \pm m^2\Phi = 0, \quad \Phi(x, 0) = \varphi_0(x), \quad \Phi_t(x, 0) = \varphi_1(x),$$

satisfies the following $L^p - L^q$ estimate

$$\begin{aligned} \|(-\Delta)^{-s}\Phi(x, t)\|_{L^q(\mathbb{R}^n)} &\leq C_{M,n,p,q,s} (1 - e^{-t})^{2s-n(\frac{1}{p}-\frac{1}{q})} \mathbf{e}^{(M-\frac{n}{2})t} \\ &\quad \times \left\{ \|\varphi_0\|_{L^p(\mathbb{R}^n)} + (1 - e^{-t}) \|\varphi_1\|_{L^p(\mathbb{R}^n)} \right\} \end{aligned}$$

for all $t \in (0, \infty)$ under conditions

- $1 < p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \frac{1}{2}(n+1) \left(\frac{1}{p} - \frac{1}{q} \right) \leq 2s \leq n \left(\frac{1}{p} - \frac{1}{q} \right) < 2s + 1.$
- $M = \sqrt{\frac{n^2}{4} - m^2}$ and $m < \sqrt{n^2 - 1}/2$ for (+)
- $M = \sqrt{\frac{n^2}{4} + m^2}$ for (-)

K.Yagdjian, J.Math.Appl. 396 (2012) 323

- Cauchy problem in the Minkowski spacetime
 - Free case (including external fields)
 - Higher derivatives Klein-Gordon

$$(\square - m_1^2) \dots (\square - m_n^2) \phi = J$$
 - Klein-Gordon
 - Auxiliary fields: $\{\psi_1, \dots, \psi_{n-1}\}$

$$(\square - m_1^2) \Psi_1 = J, (\square - m_2^2) \Psi_2 = \Psi_1,$$

$$\dots \text{ Pais, Uhlenbeck, 1950}$$
 - Semilinear
 - Klein-Gordon
 - Higher derivatives Klein-Gordon

$$?$$
- Cauchy problem in de Sitter spacetime
 - Free case (including external fields)
 - Klein-Gordon $\rightarrow$
 - Higher derivatives Klein-Gordon

$$\text{Auxiliary fields as in } M^{1,d-1}$$
 - Semilinear
 - Klein-Gordon
 - Higher derivatives Klein-Gordon

$$?$$
- Cauchy problem in the Friedmann (FLRW) spacetime

The Cauchy problem for equations with higher derivatives kinetic operators

$$F(\square)\Phi(x) = J(x) \quad x \in \mathbb{M}^{(1,n)}$$

A.Pais, G.Uhlenbeck, Field theories with non-localized action, Phys.Rev.,1950

- $F(z)$ is a polynomial on z
- $F(\square)$ via the Fourier transform $\varphi(x) \rightarrow \tilde{\varphi}(\xi)$,

$$\begin{aligned} F(\square)\varphi(x) &= \frac{1}{(2\pi)^n} \int F(-\xi^2) \tilde{\varphi}(\xi) e^{-ix\xi} d^n \xi \\ \tilde{\varphi}(\xi) &= \int \varphi(x) e^{i\xi x} d^n x, \end{aligned}$$

- This definition is natural in SFT, where all expressions came from calculations in the momentum space
- $\square \rightarrow -\partial_t^2, \quad -\infty < t < \infty$

$e^{\partial_t^2}$ and Fock method (diffusion equation)

V.A. Fock, *The eigen-time in classical and quantum mechanics*, 1937

- To give the mathematical meaning to the operator $e^{\partial_t^2}$ one can use an auxiliary function of two variables

$$\Psi(\tau, t) = e^{\tau \partial_t^2} \varphi(t)$$

satisfying the heat (diffusion) equation

$$\text{Eq. : } (\partial_\tau - \partial_t^2) \Psi(t, \tau) = 0, \quad (*)$$

$$\text{Initial condition : } \Psi(t, \tau)|_{\tau=0} = \varphi(t). \quad (**)$$

- Action of $e^{\partial_t^2}$ on a function $\varphi(t)$ is defined by means of the solution to (*) with the boundary condition (**) as follows

$$e^{\partial_t^2} \varphi(t) \equiv \Psi(t, \tau)|_{\tau=1} \quad (***)$$

Diffusion equation on $\mathbb{R}$ and $\mathbb{R}_+$

$$\text{Eq. : } (\partial_\tau - \partial_t^2)\Psi(t, \tau) = 0, \quad (*)$$

$$\text{Initial condition : } \Psi(t, \tau)|_{\tau=0} = \varphi(t). \quad (**)$$

- Now it is important to distinguish two different cases:
 - i) whole real line $-\infty < t < \infty$,
 - ii) half-line $0 < t < \infty$.
- In quantum field theory in Minkowski space-time the time variable t runs from $-\infty$ to ∞ .
- In cosmology there is a reason to restrict the time variable to the half axis only, $t > 0$, since in the Friedman cosmology there is a cosmological singularity at $t = 0$
- In the case of the whole real axis under suitable assumptions there is a unique solution of the problem $(*)$, $(**)$.

Dirichlet's initial-boundary value problem.

- On a half-line one has to add an extra boundary condition at $t = 0$
- Mixed initial-boundary value problem:

$$\begin{cases} \frac{\partial}{\partial \tau} \Psi_D(t, \tau) = \frac{\partial^2}{\partial t^2} \Psi_D(t, \tau), & t > 0, \quad \tau > 0, \\ \Psi_D(t, 0) = \varphi(t), & (***) \\ \Psi_D(0, \tau) = \mu(\tau) \end{cases} \quad \text{Dirichlet's Daemon}$$

I.A, I.Volovich, *Cosmological Daemon*, hep-th/1103.0273

- This is the Dirichlet's problem for the heat equation on the half-line.
- The function $\mu(\tau)$ will be called the Dirichlet daemon function. It describes the Dirichlet boundary conditions at $t = 0$.
- The solution of (***) is given by

$$\begin{aligned} \Psi_D(t, \tau) &= \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty \varphi(t') \left[e^{-\frac{(t-t')^2}{4\tau}} - e^{-\frac{(t+t')^2}{4\tau}} \right] dt' \\ &+ \frac{t}{\sqrt{4\pi}} \int_0^\tau \frac{\mu(\tau')}{(\tau - \tau')^{3/2}} e^{-\frac{t^2}{4(\tau - \tau')}} d\tau' \end{aligned}$$

WHAT DATA SET SOLUTIONS?

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- Klein-Gordon

- Higher derivatives Klein-Gordon

- Non-Local

- Semilinear

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...

Non-locality and Boundary Problems

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We have to
set **Boundary
Conditions**

- **Non-Local
space-
homoge-
neous**

- **Space-
homoge-
neous**

- **Particular
solutions**

- **Non-Local**

Conclusion

- Cauchy problem in de Sitter spacetime (in plane coordinate) similar to Cauchy problem in Minkowski (including higher derivatives cases)
- dS acts as a "tachyon" $m^2 \rightarrow m^2 - n^2/4$
- Theories with $e^\square$ require boundary conditions in Minkowski as well as in de Sitter case