On Stationary Non-Equilibrium States in Linear Hamiltonian Systems

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Brief description of problem

Consider a dynamical system:

$$\dot{Y}(t) = \mathcal{A}(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0.$$

Initial data $Y_0 \in \mathcal{E}$ (phase space)

$$W(t) : Y_0 \to Y(t), \quad t \in \mathbb{R}.$$

Let μ_0 be a Borel probability measure on $\mathcal{E}$.

Def. μ_t, $t \in \mathbb{R}$, is the distribution of $Y(t)$, $\mu_t(B) = \mu_0(W^{-1}(t)B)$, $B \in \mathcal{B}(\mathcal{E})$.

The objective is to prove *the weak convergence of measures μ_t* in the space $\mathcal{E}$ to a limiting measure μ_∞ as $t \to \infty$.

By definition, this means that $\int\limits_{\mathcal{E}} f(Y)\,\mu_t(dY) \to \int\limits_{\mathcal{E}} f(Y)\,\mu_\infty(dY)$ as $t \to \infty$

for any continuous bounded functional f in $\mathcal{E}$.

The model

Hamiltonian: $H(\varphi,\pi,u,v) = H_F(\varphi,\pi) + H_L(u,v) + H_I(\varphi,u)$, where

$$H_F(\varphi,\pi) := \frac{1}{2}\int\limits_{\mathbb{R}^d} \left(|\nabla\varphi(x)|^2 + |\pi(x)|^2 + m_0^2|\varphi(x)|^2\right)\, dx,$$

$$H_L(u,v) = \sum_{k\in\mathbb{Z}^d}\Big(\sum_{k'\in\mathbb{Z}^d} u(k)\cdot V(k-k')u(k') + |v(k)|^2\Big),$$

$$H_I(\varphi,u) := \sum_{k\in\mathbb{Z}^d}\int\limits_{\mathbb{R}^d} R(x-k)\cdot u(k)\varphi(x)\, dx.$$

Here $m_0>0$, $\varphi(\cdot)\in\mathbb{R}$, $u(\cdot),v(\cdot)\in\mathbb{R}^n$, $R(\cdot)\in\mathbb{R}^n$, $V(k)\in\mathbb{R}^n\times\mathbb{R}^n$, $d,n\geq 1$.

The given system can be considered as the description of the motion of electrons (so-called *Bloch electrons*) in the periodic medium which is generated by the ionic cores, i.e., $\varphi(x,t)$ describes the motion of electron field, $u(k,t)$ is the (small) displacements of the ionic cores from their equilibrium positions.
Understanding of this motion is one of the central problem of solid state physics.

Cauchy problem

$$\begin{cases} \dot\varphi(x,t) = \pi(x,t), \quad x\in\mathbb{R}^d, \quad t\in\mathbb{R}, \\ \dot\pi(x,t) = (\Delta - m_0^2)\varphi(x,t) - \sum\limits_{k'\in\mathbb{Z}^d} u(k',t)\cdot R(x-k'), \\ \dot u(k,t) = v(k,t), \quad k\in\mathbb{Z}^d, \\ \dot v(k,t) = -\sum\limits_{k'\in\mathbb{Z}^d} V(k-k')u(k',t) - \int_{\mathbb{R}^d} R(x'-k)\varphi(x',t)\,dx'. \end{cases} \tag{1}$$

Initial Data: $$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \pi(x,0) = \pi_0(x), \quad x\in\mathbb{R}^d, \\ u(k,0) = u_0(k), \quad v(k,0) = v_0(k), \quad k\in\mathbb{Z}^d. \end{cases} \tag{2}$$

Def. $\widetilde{V}(\theta) := F_{k\to\theta}[V(k)] = \sum\limits_{k\in\mathbb{Z}^d} e^{ik\cdot\theta}V(k), \quad \theta\in\mathbb{T}^d \equiv \mathbb{R}^d/(2\pi\mathbb{Z})^d$ (d-torus)

Conditions on the interaction matrix $V\in\mathbb{R}^n\times\mathbb{R}^n$:

V1 $\exists C,\gamma>0: \ \|V(k)\| \le Ce^{-\gamma|k|}$

V2 $V^T(-k) = V(k), \quad k\in\mathbb{Z}^d$

(V1, V2) $\Longrightarrow$ $\widetilde{V}(\theta) = \widetilde{V}^*(\theta)$ is real-analytic matrix-valued function $\theta\in\mathbb{T}^d$

V3 $\widetilde{V}(\theta) > 0 \quad \forall\,\theta\in\mathbb{T}^d$, i.e., $\exists\,\nu_0>0: \ \overline{u}\cdot\widetilde{V}(\theta)u \ge \nu_0^2|u|^2, \quad u\in\mathbb{C}^n$.

Notation: $Y_0 = (\varphi_0, u_0, \pi_0, v_0)$, $Y(t) = (Y^0(t), Y^1(t))$, where

$$Y^0(t) \equiv Y^0(\cdot, t) := (\varphi(x,t), u(k,t)), \quad Y^1(t) = \dot{Y}^0(t) = (\pi(x,t), v(k,t)).$$

In other words, $Y(\cdot, t)$, $Y^j(\cdot, t)$ are functions defined on the disjoint union $\mathbb{P}^d := \mathbb{R}^d \bigsqcup \mathbb{Z}^d$, for example, $Y^0(t) \equiv Y^0(p,t) = \begin{cases} \varphi(x,t), & p = x \in \mathbb{R}^d, \\ u(k,t), & p = k \in \mathbb{Z}^d. \end{cases}$

Then, the Cauchy problem (1)–(2) has a form

$$\dot{Y}(t) = \mathcal{A}(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \tag{3}$$

Here $\mathcal{A}(Y) = J\nabla H(Y)$, where $J = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}$, $H(Y)$ – Hamiltonian,

$$\Longrightarrow \mathcal{A} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathcal{H} & 0 \end{pmatrix}, \ \mathcal{H} = \begin{pmatrix} -\Delta + m_0^2 & S \\ S^* & \mathcal{V} \end{pmatrix}, \ \mathcal{V}u := \sum_{k' \in \mathbb{Z}^d} V(k-k')u(k'),$$

$$(Su)(x) = \sum_{k \in \mathbb{Z}^d} R(x-k)u(k), \quad (S^*\varphi)(k) = \int_{\mathbb{R}^d} R(x-k)\varphi(x)dx,$$

$$\langle \varphi, Su \rangle_{L^2(\mathbb{R}^d)} = \langle S^*\varphi, u \rangle_{[\ell^2(\mathbb{Z}^d)]^n}, \quad \varphi \in L^2(\mathbb{R}^d), \ u \in [\ell^2(\mathbb{Z}^d)]^n.$$

Note that **the dynamics of problem (3) is invariant w.r.t. shifts in $\mathbb{Z}^d$**.

Zak transform: $\varphi(x)\ (x \in \mathbb{R}^d) \to \widetilde{\varphi}_e(\theta, y)\ (\theta \in [0, 2\pi]^d,\ \ y \in \mathbb{T}_1^d)$

1) Split $x \in \mathbb{R}^d$ as $x = k + y,\ k = [x] \in \mathbb{Z}^d,\ y \in [0,1]^d$, and apply the discrete Fourier transform $k \to \theta$: $\widetilde{\varphi}(\theta, y, t) = \sum_{k \in \mathbb{Z}^d} e^{ik\cdot\theta} \varphi(k + y, t)$

2) Apply the **Zak transform** (or *Bloch–Floquet–Zak transform*):

$$\mathcal{Z}: Y(\cdot, t) \to \widetilde{Y}_e(\theta, t) \equiv \widetilde{Y}_e(\theta, y, t) = (\widetilde{\varphi}_e(\theta, y, t), \widetilde{u}(\theta, t), \widetilde{\pi}_e(\theta, y, t), \widetilde{v}(\theta, t)),$$

$\widetilde{\varphi}_e(\theta, y, t) := e^{iy\cdot\theta} \widetilde{\varphi}(\theta, y, t)$ (periodic in $y \in \mathbb{T}_1^d \equiv \mathbb{R}^d/\mathbb{Z}^d$ and quasi-periodic in θ)

Then, the Cauchy problem (3) is reduced to the *Bloch problem on the torus* ($y \in \mathbb{T}_1^d$) with the parameter θ (so-called *Bloch momentum*), $\theta \in [0, 2\pi]^d$:

$$\dot{\widetilde{Y}}_e(\theta, t) = \widetilde{\mathcal{A}}(\theta)(\widetilde{Y}_e(\theta, t)), \quad t \in \mathbb{R}, \qquad \widetilde{Y}_e(\theta, 0) = \widetilde{Y}_{0,e}(\theta),$$

where $\widetilde{\mathcal{A}}(\theta) = \begin{pmatrix} 0 & 1 \\ -\widetilde{\mathcal{H}}(\theta) & 0 \end{pmatrix}$, $\widetilde{\mathcal{H}}(\theta)$ is a self-adjoint operator in $L^2(\mathbb{T}_1^d) \otimes \mathbb{C}^n$ with the discrete positive spectrum, since we assume that $\widetilde{\mathcal{H}}(\theta) > 0$ for $\theta \in [0, 2\pi]^d$.

$$\widetilde{\mathcal{H}}(\theta) = \mathcal{Z}\mathcal{H}\mathcal{Z}^{-1} = \begin{pmatrix} (i\nabla_y + \theta)^2 + m_0^2 & \widetilde{S}(\theta) \\ \widetilde{S}^*(\theta) & \widetilde{V}(\theta) \end{pmatrix}, \text{ where } \mathcal{Z} \text{ is the Zak transform,}$$

$$(\widetilde{S}(\theta)\widetilde{u}) = \widetilde{R}_e(\theta, y) \cdot \widetilde{u}(\theta), \quad (\widetilde{S}^*(\theta)\widetilde{\psi}_e(\theta, \cdot)) = \int_{\mathbb{T}_1^d} \overline{\widetilde{R}_e(\theta, y)}\, \widetilde{\psi}_e(\theta, y)\, dy.$$

Conditions on the coupling function $R(\cdot) \in \mathbb{R}^n$:

(R1) $R \in C^\infty(\mathbb{R}^d)$, $\quad |R(x)| \le \bar{R} \exp(-\gamma|x|)$ with some $\gamma > 0$ and $\bar{R} < \infty$.

(R2) (*hyperbolicity of the problem*) The operator $\widetilde{\mathcal{H}}(\theta) > 0$ for $\theta \in [0, 2\pi]^d$.

This is equivalent to the uniform bound

$$\left(\widetilde{Y}_e^0, \widetilde{\mathcal{H}}(\theta)\widetilde{Y}_e^0\right) \ge \kappa^2 \left\|\widetilde{Y}_e^0\right\|^2_{H^1(\mathbb{T}_1^d)\oplus\mathbb{C}^n} \quad \text{for} \quad \widetilde{Y}_e^0 \in H^1(\mathbb{T}_1^d) \oplus \mathbb{C}^n,$$

where a constant $\kappa > 0$, $\quad (\cdot, \cdot)$ stands for the scalar product in $H^0(\mathbb{T}_1^d) \oplus \mathbb{C}^n$.

Condition (R2) holds for functions R satisfying condition (R1) with $\bar{R}\gamma^{-d} \ll 1$.

Phase space

- The weighted Sobolev spaces $H^s_\alpha(\mathbb{R}^d)$: $\|\varphi\|_{s,\alpha} \equiv \|\langle x\rangle^\alpha \Lambda^s \varphi\|_{L^2(\mathbb{R}^d)} < \infty$, $\langle x\rangle := \sqrt{|x|^2+1}$, $s, \alpha \in \mathbb{R}$.

  Here $\Lambda^s\varphi := F^{-1}_{\xi\to x}(\langle\xi\rangle^s \widehat{\varphi}(\xi))$, $\quad \widehat{\varphi}(\xi) := \int_{\mathbb{R}^d} e^{i\,x\cdot\xi}\varphi(x)\,dx$

- $\ell^2_\alpha = \left\{ u(k) \in \mathbb{R}^n : \ \|u\|^2_\alpha \equiv \sum\limits_{k\in\mathbb{Z}^d} \langle k\rangle^{2\alpha} |u(k)|^2 < \infty \right\}$, $\ \alpha \in \mathbb{R}$.

- $Y = (\varphi, u, \pi, v) \in \mathcal{E}^s_\alpha \equiv H^{1+s}_\alpha(\mathbb{R}^d) \oplus \ell^2_\alpha \oplus H^s_\alpha(\mathbb{R}^d) \oplus \ell^2_\alpha$ with finite norm $\|Y\|^2_{s,\alpha} = \|\varphi\|^2_{1+s,\alpha} + \|u\|^2_\alpha + \|\pi\|^2_{s,\alpha} + \|v\|^2_\alpha < \infty$.

Def. $Y_0 = (\varphi_0, u_0, \pi_0, v_0) \in \mathcal{E} \equiv \mathcal{E}^0_\alpha$ with $\quad \alpha < -d/2$.

Lemma. *For any $Y_0 \in \mathcal{E}$ there exists a unique solution $Y(t) = W(t)Y_0 \in C(\mathbb{R}, \mathcal{E})$ to the Cauchy problem (3). Moreover,* $\sup\limits_{|t|\le T} \|W(t)Y_0\|_{0,\alpha} \le C(T)\|Y_0\|_{0,\alpha}$.

Assume that the initial data $Y_0 = (Y_0^0, Y_0^1)$, $Y_0^0 \equiv (\varphi_0, u_0)$, $Y_0^1 \equiv (\pi_0, v_0)$ is a random measurable function with values in $\mathcal{E}$, μ_0 denotes the distribution of Y_0.

Conditions on the initial measure μ_0:

1. $$\mathbb{E}(Y_0(p)) \equiv \int_{\mathcal{E}} Y_0(p)\,\mu_0(dY_0) = 0, \quad p \in \mathbb{P}^d := \mathbb{R}^d \cup \mathbb{Z}^d.$$

2. μ_0 has a finite mean energy density,
$\mathbb{E}(|\nabla\varphi_0(x)|^2 + |\varphi_0(x)|^2 + |\pi_0(x)|^2 + |u_0(k)|^2 + |v_0(k)|^2) \le C < \infty.$

3. $$Q_0^{ij}(p,p') := \mathbb{E}(Y_0^i(p) \otimes Y_0^j(p')) = \begin{cases} Q_-^{ij}(p,p') & \text{for} \quad p_1, p_1' < -a, \\ Q_+^{ij}(p,p') & \text{for} \quad p_1, p_1' > a, \end{cases} \qquad a > 0,$$

   where $Q_\pm^{ij}(p,p')$, $i,j = 0,1$, are the correlation functions of some translation invariant w.r.t. shifts in $\mathbb{Z}^d$ measures $\mu_\pm$, $p = (p_1, \ldots, p_d)$, $p' = (p_1', \ldots, p_d')$

   In particular, $Q_\pm^{ij}(p+k, p'+k) = Q_\pm^{ij}(p,p') \quad \forall\, k \in \mathbb{Z}^d$.

4. μ_0 satisfies a *mixing condition*. This means roughly that $Y_0(p)$ and $Y_0(p')$ are asymptotically independent as $|p - p'| \to \infty$.

Rosenblatt mixing condition

Let $\sigma(\mathcal{A}) := \sigma\{Y_0 \in \mathcal{E} : \ \mathrm{supp}\, Y_0 \subset \mathcal{A}\}, \ \ \mathcal{A} \subset \mathbb{P}^d$.

The Rosenblatt mixing coefficient of a probability measure μ_0 on $\mathcal{E}$ is

$$\alpha(r) \equiv \sup_{\mathcal{A},\mathcal{B}\subset\mathbb{P}^d:\rho(\mathcal{A},\mathcal{B})\geq r} \ \sup_{A\in\sigma(\mathcal{A}),B\in\sigma(\mathcal{B})} |\mu_0(A\cap B) - \mu_0(A)\mu_0(B)|.$$

Def. *A measure μ_0 satisfies the strong Rosenblatt mixing condition* [a] *if the mixing coefficient $\alpha(r) \to 0$ as $r \to \infty$.*

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[a] Ibragimov I.A., Linnik Yu.V., *Independent and Stationary Sequences of Random Variables*.
Rosenblatt, M.A., *A central limit theorem and a strong mixing condition* (1956).
Bulinskii A.V., *Limit theorems under weak dependence conditions*. MSU Press (1989) [in Russian]
Bradley R.C., *Basic properties of strong mixing conditions*, Probability Surveys (2005)

Assume that μ_0 satisfies the strong Rosenblatt mixing condition, and

$\int_0^{+\infty} r^{d-1}\alpha^p(r)\,dr < \infty$ with $\ p := \min(\delta/(2+\delta), 1/2)$.

$\exists\,\delta > 0 : \ \ \int |Y_0(p)|^{2+\delta}\,\mu_0(Y_0) \leq C < \infty.$

Main result

Def. μ_t is the Borel probability measure in $\mathcal{E}^0_\alpha$ given the distribution of the random solution $Y(t)$, $\mu_t(B) = \mu_0(W(-t)B)$, $B \in \mathcal{B}(\mathcal{E}^0_\alpha)$, $t \in \mathbb{R}$.

Theorem. (i) The measures μ_t weakly converge as $t \to \infty$ in the space $\mathcal{E}^s_\beta$ for any $s < 0$, $\beta < \alpha < -d/2$.
The limiting measure μ_∞ is a Gaussian measure concentrated in $\mathcal{E}^0_\alpha$,
$\widehat{\mu}_\infty(Z) \equiv \int e^{i\langle Y,Z\rangle} \mu_\infty(dY) = e^{-1/2\langle Q_\infty Z,Z\rangle}$, $\forall Z \in [C^\infty_0(\mathbb{R}^d) \oplus C_0(\mathbb{Z}^d)]^2$.

(ii) The correlation matrices of μ_t converge to a limit,

$$Q_t(p,p') \equiv \int \Big(Y(p) \otimes Y(p')\Big)\, \mu_t(dY) \to Q_\infty(p,p'), \quad t \to \infty, \quad p,p' \in \mathbb{P}^d.$$

(iii) The measure μ_∞ is stationary in time, i.e., $[W(t)]^* \mu_\infty = \mu_\infty$, $t \in \mathbb{R}$.
The group $W(t)$ satisfies a mixing condition w.r.t. μ_∞, i.e., $\forall f, g \in L^2(\mathcal{E}^0_\alpha, \mu_\infty)$,

$$\int f(W(t)Y) g(Y)\, \mu_\infty(dY) \to \int f(Y)\, \mu_\infty(dY) \int g(Y)\, \mu_\infty(dY), \quad t \to \infty.$$

The initial measures μ_0 in $\mathcal{E}_\alpha^s$, $s, \alpha < -d/2$

Let $Y_0 \in \mathcal{E}_\alpha^s$ and $Y_0(p) = \left\{ \begin{array}{lll} Y_-(p) & \text{for} & p_1 < -a \\ Y_+(p) & \text{for} & p_1 > a \end{array} \right. \Bigg|\ a > 0,\ \ p = (p_1, \dots, p_d)$,

where distributions of Y_- and Y_+ are independent Gaussian translation invariant w.r.t. shifts in $\mathbb{Z}^d$ measures $\mu_\pm$. Then the correlation matrix of μ_0 is

$Q_0(p, p') = \left\{ \begin{array}{lll} Q_-(p, p') & \text{for} & p_1, p_1' < -a, \\ Q_+(p, p') & \text{for} & p_1, p_1' > a, \end{array} \right.$ where $Q_\pm$ – cor. matrices of $\mu_\pm$.

Theorem. *The measures μ_t weakly converge to a limiting measure μ_∞ as $t \to \infty$ in the spaces $\mathcal{E}_\alpha^s$, $s, \alpha < -d/2$.*

Example. Let $\mu_\pm = g_{\beta_\pm}$ be Gibbs measures, corresponding to different positive temperatures $T_- \neq T_+$. Formally, they are defined by the rule

$$g_{\beta_\pm}(dY) = \frac{1}{Z_\pm} e^{-\beta_\pm H(Y)} \prod_{p \in \mathbb{P}^d} dY(p), \qquad \beta_\pm = T_\pm^{-1}, \quad T_\pm > 0,$$

where $H(Y)$ is the Hamiltonian of the system, $Z_\pm$ is normalization factor.

Gibbs measures

Def. $\mathrm{E}^s_\alpha := H^s_\alpha(\mathbb{R}^d) \oplus \ell^2_\alpha(\mathbb{Z}^d)$. Then $\mathcal{E}^s_\alpha = \mathrm{E}^{s+1}_\alpha \otimes \mathrm{E}^s_\alpha$, $s, \alpha < -d/2$.

For $\beta > 0$, the Gibbs measures $g_\beta(dY)$ are Borel probability measures of a form $g_\beta(dY) = g^0_\beta(dY^0) \times g^1_\beta(dY^1)$ in $\mathcal{E}^s_\alpha = \mathrm{E}^{s+1}_\alpha \otimes \mathrm{E}^s_\alpha$, where $g^0_\beta(dY^0)$ and $g^1_\beta(dY^1)$ are Gaussian Borel probability measures in E^{s+1}_α and E^s_α with characteristic functionals of the form

$$\widehat{g}^0_\beta(Z) = \int \exp\{i\langle Y^0, Z\rangle\}\, g^0_\beta(dY^0) = \exp\left\{-\frac{1}{2\beta}\langle \mathcal{H}^{-1} Z, Z\rangle\right\}$$
$$\widehat{g}^1_\beta(Z) = \int \exp\{i\langle Y^1, Z\rangle\}\, g^1_\beta(dY^1) = \exp\left\{-\frac{1}{2\beta}\langle Z, Z\rangle\right\}$$

$\forall Z \in C^\infty_0(\mathbb{R}^d) \oplus [C_0(\mathbb{Z}^d)]^n$.

By the Minlos theorem, the measures g^0_β and g^1_β exist in E^{s+1}_α and E^s_α, resp., since

$$\int \|Y^0\|^2_{s+1,\alpha}\, g^0_\beta(dY^0) < \infty, \quad \int \|Y^1\|^2_{s,\alpha}\, g^1_\beta(dY^1) < \infty \quad \text{for} \quad s, \alpha < -d/2.$$

Hence, the measures g_β exist in $\mathcal{E}^s_\alpha$, $s, \alpha < -d/2$.

- $Y_\pm$ – independent random functions in the probability spaces $(\mathcal{E}^s_\alpha, g_{\beta_\pm})$, $g_{\beta_\pm}$ – Gibbs measures, $\beta_\pm = T_\pm^{-1} > 0$, $T_\pm$ – temperatures.
- $\mu_0 \equiv g_0$ – the distribution of the random function Y_0 of the form

$$Y_0(p) = \left\{ \begin{array}{ll} Y_+(p) & \text{for } p_1 > a \\ Y_-(p) & \text{for } p_1 < -a \end{array} \right| \quad p = (p_1, \dots, p_d), \quad \text{with some } a > 0.$$

- g_t – distribution of the random solution $W(t)Y_0$, $t \in \mathbb{R}$.

Theorem. *(i)* $g_t \rightharpoondown g_\infty$ *as* $t \to \infty$ *in* $\mathcal{E}^s_\alpha$, $s, \alpha < -d/2$.

(ii) g_∞ *is a Gaussian measure in* $\mathcal{E}^s_\alpha$ *with the correlation matrix* $Q_\infty(p, p')$*:*

$$Q_\infty(k + r, k' + r') = q_\infty(k - k', r, r') \equiv (q^{ij}_\infty(k - k', r, r'))_{i,j=0,1}, \quad k \in \mathbb{Z}^d.$$

In the Zak transform, the correlation operators $\tilde{q}^{ij}_\infty(\theta) = \mathrm{Op}(\tilde{q}^{ij}_\infty(\theta, r, r'))$ *are*

$$\tilde{q}^{11}_\infty(\theta) = \widetilde{\mathcal{H}}(\theta)\tilde{q}^{00}_\infty(\theta) = \mathbf{T}_+ \mathrm{I}, \qquad \mathbf{T}_\pm := \frac{1}{2}(T_+ \pm T_-),$$

$$\tilde{q}^{01}_\infty(\theta) = -\tilde{q}^{10}_\infty(\theta) = \mathbf{T}_- \sum_{\sigma=1}^{+\infty} i\,\omega_\sigma^{-1}(\theta)\,\mathrm{sign}(\partial_1\omega_\sigma(\theta))\,\Pi_\sigma(\theta),$$

$\omega_\sigma(\theta)$ *– eigenvalues of* $\sqrt{\widetilde{\mathcal{H}}(\theta)}$, Π_σ *– corresponding projectors*

Theorem *The limiting mean energy current density is*

$$\mathbf{J}_\infty = -C(T_+ - T_-, 0, \ldots, 0) \quad \textit{with a constant} \ \ C > 0,$$

$$C := \frac{1}{2}\sum_{\sigma=1}^{\infty}\frac{1}{(2\pi)^d}\int_{[0,2\pi]^d}\left|\frac{\partial\omega_\sigma}{\partial\theta_1}(\theta)\right|d\theta, \ \ \omega_\sigma(\theta) - \textit{eigenvalues of } \sqrt{\widetilde{\mathcal{H}}(\theta)}.$$

i.e., the energy current flows from the "hot" to "cold" reservoir, that corresponds to Second law of thermodynamics.

Thus, we prove that there are stationary non-equilibrium states (i.e., probability limiting measures μ_∞) in which there is a nonzero heat flux in our model.

Let for simplicity, the coupling function $R \equiv 0$. Then

1). *For the Klein–Gordon field, we have ultraviolet divergence of the limiting mean density of energy current:* $\mathbf{J}_\infty = -C(T_+ - T_-, 0, \ldots, 0)$, $C = +\infty$.

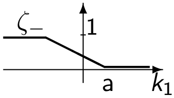
2). *For the harmonic crystal, the limiting mean energy current density is*

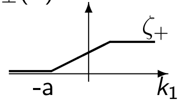
$$\mathbf{J}_\infty = -C(T_+ - T_-, 0, \ldots, 0) \quad \text{with a constant} \ \ C > 0.$$

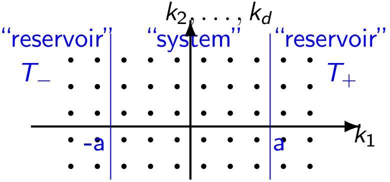
Harmonic crystal: "System + 2 heat reservoirs"

$Y_0(k) = \zeta_-(k_1)Y_-(k) + \zeta_+(k_1)Y_+(k)$

$\mu_\pm$ – distributions of $Y_\pm(k)$







$$Q_0(k,k') = \begin{cases} q_-(k-k'), & k_1, k_1' < -a, \\ q_+(k-k'), & k_1, k_1' > a, \end{cases} \quad k = (k_1, \dots, k_d) \in \mathbb{Z}^d, \ a > 0,$$

where $q_\pm(k)$ – correlation matrices of translation invariant measures $\mu_\pm$.

Let $\mu_\pm = g_{T_\pm}$ be *Gibbs* measures, corresp. to temperatures $T_\pm > 0$.

Lemma *The limiting mean energy current density is*

$$\mathbf{J}_\infty = -C(T_+ - T_-, 0, \dots, 0) \quad \textit{with a constant } C > 0,$$

$$C = \frac{1}{2}\sum_{\sigma=1}^{n} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \left|\frac{\partial \omega_\sigma}{\partial \theta_1}(\theta)\right| d\theta, \ \omega_\sigma \textit{ – eigenvalues of dispersion matrix } \sqrt{\widetilde{V}(\theta)},$$

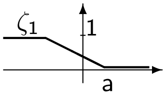
i.e., the energy current flows from the "hot" to "cold" reservoir.

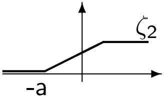
Harmonic crystal: "System + 4 heat reservoirs"

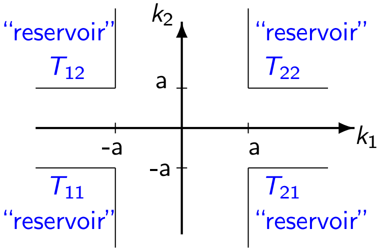
$$Y_0(x) = \sum_{i,j=1,2} \zeta_i(k_1)\zeta_j(k_2)Y_{ij}(k),$$

μ_{ij} – distributions of $Y_{ij}(k)$

μ_0 – distribution of $Y_0(k)$







Let $\mu_{ij} = g_{T_{ij}}$ be *Gibbs* measures, corresp. to temperatures $T_{ij} > 0$, $i,j = 1,2$. Then the limiting mean energy current density is

$$\mathbf{J}_\infty = -\Big(c_1\frac{T_{21}+T_{22}-T_{11}-T_{12}}{4}, c_2\frac{T_{12}+T_{22}-T_{11}-T_{21}}{4}, 0, \dots, 0\Big),$$

$$c_j = \sum_{\sigma=1}^{n} \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} \Big|\frac{\partial\omega_\sigma}{\partial\theta_j}(\theta)\Big| d\theta > 0,\ \omega_\sigma \text{ – eigenvalues of dispersion matrix } \sqrt{\widetilde{V}(\theta)}.$$

Brief survey of previous results

Harmonic crystals:

Case $d = 1$ (infinite one-dimensional chain of harmonic oscillators).

- Spohn, Lebowitz (1977) – *for initial measures which have distinct temperatures to the left and to the right*
- Boldrighini, Pellegrinotti, Triolo (1983) – *for a more general class of initial measures which satisfy a mixing condition*
- T.D. (2017) – *infinite harmonic chain in the half-line*
- T.D. (2020) – *non-homogenous chain consisting of two different semi-infinite chains coupled to a "gluing" particle (one defect)*

Case $d \geq 1$.

- Lanford, Lebowitz (1975) – *for initial measures which are absolutely continuous with respect to the canonical Gaussian measure*
- J.L. van Hemmen (1980) – *ergodic properties of the harmonic crystal*
- T.D., A. Komech, H. Spohn (2003) –*for translation-invariant initial measures*
- T.D., A. Komech, N. Mauser (2004) – *for non translation-invariant measures*
- T.D. (2008) – *for harmonic crystals in half-space with zero boundary condition*
- T.D. (2019) – *for a more general class of the initial measures*

Partial differential equations of hyperbolic type in $\mathbb{R}^d$, $d \geq 1$:

- N. Ratanov (1984), E. Kopylova (1986) – *for translation–invariant initial measures*
- T.D., A. Komech, H. Spohn (2002); T.D., A. Komech (2005, 2006);
- T.D. (2021) – *for non translation invariant measures*

Coupled systems:

- Jakšić, Pillet (1998), Rey-Bellet, Thomas (2002) – *ergodic properties of one-dimensional chain of anharmonic oscillators coupled to heat baths*
- T.D., A. Komech (2006) – *the Hamiltonian system consisting of the crystal and the scalar Klein–Gordon field* (translation invariant case)
- T.D. (2010, 2016) – *the Hamiltonian system consisting of a particle and the vector field described by the KG or wave equations with variable coefficients*
- T.D. (2022) – *the "field–crystal" system* (non translation invariant case)

Thank you for attention!