Semiclassical asymptotics of the spectral function of the magnetic Schrödinger operator

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The setting

The d-dimensional space $\mathbb{R}^d$ with Riemannian metric

$$g = \sum_{j,k=1}^{d} g_{jk}(x) dx_j dx_k.$$

Magnetic potential is a real-valued one-form

$$\mathbf{A} = \sum_{j=1}^{d} A_j(x) dx_j.$$

Electric potential is a real-valued function $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$.

The magnetic Schrödinger operator

$$H_\hbar = \sum_{j,k=1}^{d} \frac{1}{\sqrt{\det g(x)}} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - A_j(x) \right) \times \\ \times \left[g^{jk}(x) \sqrt{\det g(x)} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_k} - A_k(x) \right) \right] + \hbar V(x), \quad i = \sqrt{-1},$$

$(g^{jk}(x))$ is the inverse matrix of $g(x) = (g_{jk}(x))$, $\hbar > 0$ is the semiclassical parameter (Planck constant).

- It describes the motion of a quantum charged particle in an external electromagnetic field.
- One can consider the case of a smooth manifold (either compact or noncompact, of bounded geometry) as well as the case of non-exact magnetic field (then $\hbar$ takes a discrete set of values).

Assumptions

$$H_\hbar = \sum_{j,k=1}^{d} \frac{1}{\sqrt{\det g(x)}} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - A_j(x)\right) \times$$
$$\times \left[g^{jk}(x)\sqrt{\det g(x)}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_k} - A_k(x)\right)\right] + \hbar V(x), \quad \hbar > 0.$$

For any $j, k = 1, \dots, d$, we have $g_{jk} \in C_b^\infty(\mathbb{R}^d)$, that is, for any $\alpha \in \mathbb{Z}_+^d$,

$$\sup_{x\in\mathbb{R}^d} |\partial^\alpha g_{jk}(x)| < \infty.$$

$g(x) = (g_{jk}(x))$ is positive definite uniformly on $x \in \mathbb{R}^d$:

$$\inf_{x\in\mathbb{R}^d} g(x) \geqslant \varepsilon_0 > 0.$$

Assumptions

$$H_\hbar = \sum_{j,k=1}^{d} \frac{1}{\sqrt{\det g(x)}} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - A_j(x)\right) \times$$
$$\times \left[g^{jk}(x)\sqrt{\det g(x)} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - A_k(x)\right)\right] + \hbar V(x), \quad \hbar > 0,$$

For the magnetic field $\mathbf{B}$ defined by

$$\mathbf{B} = d\mathbf{A} = \sum_{j<k} B_{jk}(x) dx_j \wedge dx_k, \quad B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}.$$

we have $B_{jk} \in C_b^\infty(\mathbb{R}^d)$.
Finally, $V \in C_b^\infty(\mathbb{R}^d)$.

Setting of the problem

- The magnetic Schrödinger operator

$$H_\hbar = \sum_{j,k=1}^{d} \frac{1}{\sqrt{\det g(x)}} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - A_j(x)\right) \times$$
$$\times \left[g^{jk}(x)\sqrt{\det g(x)} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_k} - A_k(x)\right)\right] + \hbar V(x), \quad \hbar > 0.$$

  is a uniformly elliptic second order differential operator.
- It is essentially self-adjoint in the Hilbert space $L^2(\mathbb{R}^d, dv_g)$ with initial domain $C_c^\infty(\mathbb{R}^d)$, where $dv_g = \sqrt{\det g(x)}dx$ is the Riemannian volume form.
- We study spectral properties of $H_\hbar$ in the semiclassical limit $\hbar \to 0$.
- First, in the case, when **B** is of maximal rank ($d = 2n$), we give a rough asymptotic description of the spectrum of $H_\hbar$ as $\hbar \to 0$.

Model operator

For any $x_0 \in \mathbb{R}^d$. the model operator $\mathcal{H}^{(x_0)}$ is the magnetic Schrödinger operator on $C^\infty(T_{x_0}\mathbb{R}^d) \cong C^\infty(\mathbb{R}^d)$

$$\mathcal{H}^{(x_0)} = -\sum_{j,k=1}^{d} g^{jk}(x_0)\left(\frac{\partial}{\partial v_j} - iA_{j,x_0}(v)\right) \times \\ \times \left(\frac{\partial}{\partial v_k} - iA_{k,x_0}(v)\right) + V(x_0), \quad v \in \mathbb{R}^d \cong T_{x_0}\mathbb{R}^d,$$

with the magnetic potential

$$\mathbf{A}_{x_0} = \sum_{j=1}^{d} A_{j,x_0}(v)dv_j = \frac{1}{2}\sum_{j<k} B_{jk}(x_0)(v_j dv_k - v_k dv_j),$$

and constant magnetic field

$$d\mathbf{A}_{x_0} = \sum_{j<k} B_{jk}(x_0) dv_j \wedge dv_k = \mathbf{B}_{x_0}.$$

Spectrum of the model operator

- Let B_{x_0} be a skew-symmetric operator in $T_{x_0}\mathbb{R}^d \cong \mathbb{R}^d$ such that

$$\mathbf{B}_{x_0}(u,v) = g_{x_0}(B_{x_0}u, v), \quad u, v \in T_{x_0}\mathbb{R}^d.$$

Since B_{x_0} is skew-symmetric of rank $2n$, zero is an eigenvalue of multiplicity $d-2n$ and its non-zero eigenvalues have the form

$$\pm i a_j(x_0), j = 1, \ldots, n, \quad a_j(x_0) > 0,$$

- Denote

$$\Lambda_{\mathbf{k}}(x_0) = \sum_{j=1}^{n} (2k_j+1) a_j(x_0) + V(x_0), \quad \mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}_+^n.$$

Spectrum of the model operator

- In the maximal rank case $d = 2n$, the spectrum of $\mathcal{H}^{(x_0)}$ consists of eigenvalues of infinite multiplicity (Landau levels)

$$\sigma(\mathcal{H}^{(x_0)}) = \left\{\Lambda_{\mathbf{k}}(x_0) \,:\, \mathbf{k} \in \mathbb{Z}_+^n\right\}.$$

In particular, the lowest eigenvalue of $\mathcal{H}^{(x_0)}$ is

$$\Lambda_0(x_0) = \sum_{j=1}^{n} a_j(x_0) + V(x_0).$$

- In the case $d > 2n$, the spectrum is the half-line

$$\sigma(\mathcal{H}^{(x_0)}) = [\Lambda_0(x_0), +\infty),$$

Description of the spectrum

Theorem (Yu.K. 2020)

Assume that $\mathbf{B}$ *is of maximal rank* ($d = 2n$). *Then, for any* $K > 0$, *there exists* $c > 0$ *such that for any* $\hbar > 0$ *the spectrum of* $H_\hbar$ *in* $[0, K\hbar]$ *is contained in the* $c\hbar^{5/4}$*-neighborhood of* $\hbar\Sigma$, *where*

$$\Sigma = \bigcup_{x_0 \in \mathbb{R}^d} \Sigma_{x_0} = \left\{ \Lambda_{\mathbf{k}}(x_0) \,:\, \mathbf{k} \in \mathbb{Z}_+^n, x_0 \in \mathbb{R}^d \right\}.$$

L. Charles (2021): for a compact manifold, better estimate $c\hbar^{3/2}$ instead of $c\hbar^{5/4}$.

Lower bound

$$\inf \sigma(H_\hbar) \geqslant \hbar\Lambda_0 - c\hbar^2, \quad \hbar > 0,$$

where

$$\Lambda_0 = \inf \left\{ \Lambda_0(x_0) \,:\, x_0 \in \mathbb{R}^d \right\}$$

Band structure and gaps

- Σ is a closed subset of $\mathbb{R}$:

$$\Sigma = \bigcup_{\mathbf{k}\in\mathbb{Z}_+^n} [\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}],$$

  where, for any $\mathbf{k} \in \mathbb{Z}_+^n$, we have the band

$$[\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}] = \{\Lambda_{\mathbf{k}}(x_0) : x_0 \in \mathbb{R}^d\}.$$

- In some cases, Σ has gaps: $[\Lambda_0, +\infty) \setminus \Sigma \neq \emptyset$ and Theorem implies the existence of gaps in the spectrum of $H_\hbar$.
  For instance, if $V(x) \equiv 0$ and a_j can be chosen to be constants:

$$a_j(x) \equiv a_j, \quad x \in \mathbb{R}^d, \quad j = 1, \ldots, n,$$

  then Σ is a countable discrete set.
  The set Σ may also have gaps if the functions a_j are not constants, but vary slow enough.

Functions of the operator

- For any $\varphi \in \mathcal{S}(\mathbb{R})$, the operator $\varphi(H_\hbar/\hbar)$ is defined by the spectral theorem. It is a smoothing operator in $L^2(\mathbb{R}^d, dv_g)$ with smooth Schwartz kernel $K_{\varphi(H_\hbar/\hbar)} \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$:

$$\varphi(H_\hbar/\hbar)u(x) = \int_{\mathbb{R}^d} K_{\varphi(H_\hbar/\hbar)}(x,x')u(x')dv_g(x'), \quad u \in L^2(\mathbb{R}^d, dv_g).$$

- If the spectrum is discrete (a compact manifold M): $H_\hbar u_{\hbar,j} = \nu_{\hbar,j} u_{\hbar,j}$ with a complete orthonormal system $\{u_{\hbar,j} \in C^\infty(M), j = 0, 1, 2, \ldots\}$, then

$$K_{\varphi(H_\hbar/\hbar)}(x,x') = \sum_{j=0}^{\infty} \varphi(\nu_{\hbar,j}/\hbar) u_{\hbar,j}(x) \overline{u_{\hbar,j}(x')}.$$

- Theorem: For any $\varepsilon > 0$ and $k \in \mathbb{Z}_+$,

$$\left| K_{\varphi(H_\hbar/\hbar)}(x,x') \right|_{C^k} = \mathcal{O}(\hbar^\infty), \quad |x - x'| > \varepsilon.$$

- For the proof, we use the finite propagation speed property of solutions of hyperbolic equations.

Full off-diagonal asymptotics

Theorem (Yu.K. 2022)

The following asymptotic expansion holds true as $\hbar \to 0$:

$$K_{\varphi(H_\hbar/\hbar)}(x_0+v, x_0+v')$$
$$\cong \hbar^{-\frac{d}{2}} \sum_{r=0}^{\infty} F_{r,x_0}(\hbar^{-1/2}v, \hbar^{-1/2}v')\kappa_{x_0}^{-\frac{1}{2}}(v)\kappa_{x_0}^{-\frac{1}{2}}(v')\hbar^{\frac{r}{2}}, x_0 \in \mathbb{R}^d, v, v' \in \mathbb{R}^d.$$

- κ_{x_0} is a smooth function on $T_{x_0}\mathbb{R}^d \cong \mathbb{R}^d$ defined by

$$\kappa_{x_0}(v) = \sqrt{\det g(x_0+v)}, \quad v \in \mathbb{R}^d.$$

- $F_{r,x_0}(v, v')$ are some smooth functions.
- Full off-diagonal expansions for the (generalized) Bergman kernels Dai-Liu-Ma04,Ma-Marinescu07,Yu.K.18, ..., goes back to Bismut-Lebeau localization technique in index theory.

Full off-diagonal asymptotics

Theorem (Yu.K. 2022)

For any $j, m, m' \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that, for any $N \in \mathbb{N}$, there exists $C > 0$ such that for any $p \geqslant 1$, $x_0 \in \mathbb{R}^d$ and $v, v' \in T_{x_0}\mathbb{R}^d$,

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial v^{\alpha} \partial v'^{\alpha'}} \left(K_{\varphi(H_\hbar/\hbar)}(x_0+v, x_0+v') \right.\right.$$
$$\left.\left. - \hbar^{-\frac{d}{2}} \sum_{r=0}^{j} F_{r,x_0}(\hbar^{-\frac{1}{2}} v, \hbar^{-\frac{1}{2}} v') \kappa_{x_0}^{-\frac{1}{2}}(v) \kappa_{x_0}^{-\frac{1}{2}}(v') \hbar^{\frac{r}{2}} \right) \right|_{C^{m'}}$$
$$\leqslant C\hbar^{-\frac{j-m+1}{2}} (1 + \hbar^{-\frac{1}{2}}|v| + \hbar^{-\frac{1}{2}}|v'|)^M (1 + \hbar^{-\frac{1}{2}}|v - v'|)^{-N}.$$

Here $C^{m'}$ is the $C^{m'}$-norm for the parameter $x_0 \in \mathbb{R}^d$.

On-diagonal asymptotics

Corollary (Yu.K. 2022)

For any $x_0 \in \mathbb{R}^d$, there exists a sequence of distributions $f_r(x_0) \in \mathcal{S}'(\mathbb{R}), r \geqslant 0$, such that the following asymptotic expansion holds true as $\hbar \to 0$ uniformly on x_0:

$$K_{\varphi(H_\hbar/\hbar)}(x_0, x_0) \sim \hbar^{-\frac{d}{2}} \sum_{r=0}^{\infty} \langle f_r(x_0), \varphi \rangle \hbar^{\frac{r}{2}}, \quad \langle f_r(x_0), \varphi \rangle = F_{r,x_0}(0,0).$$

Corollary: semiclassical trace formula

In the compact case

$$\operatorname{tr} \varphi(H_\hbar/\hbar) \sim \sum_{r=0}^{\infty} \langle f_r, \varphi \rangle \hbar^{\frac{d-r}{2}}, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

The Gutzwiller trace formula (for the zero energy level).

Leading coefficient

- Recall that, for any $x_0 \in \mathbb{R}^d$. the model operator $\mathcal{H}^{(x_0)}$ is the magnetic Schrödinger operator on $C^\infty(T_{x_0}\mathbb{R}^d) \cong C^\infty(\mathbb{R}^d)$ with constant magnetic field $\mathbf{B}_{x_0}$.
- For any $\varphi \in \mathcal{S}(\mathbb{R})$, $K_{\varphi(\mathcal{H}^{(x_0)})}(v, v') \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ the smooth integral kernel of the operator $\varphi(\mathcal{H}^{(x_0)})$:

$$\varphi(\mathcal{H}^{(x_0)})u(v) = \int_{\mathbb{R}^d} K_{\varphi(\mathcal{H}^{(x_0)})}(v, v')u(v')dv', \quad u \in C_c^\infty(\mathbb{R}^d).$$

- Then

$$\langle f_0(x_0), \varphi \rangle = K_{\varphi(\mathcal{H}^{(x_0)})}(0, 0), \quad x_0 \in \mathbb{R}^d.$$

Leading coefficient

In the maximal rank case $d = 2n$

$$\langle f_0(x_0), \varphi \rangle = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} a_j(x_0) \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \varphi(\Lambda_{\mathbf{k}}(x_0)),$$

where $\Lambda_{\mathbf{k}}(x_0) = \sum_{j=1}^n (2k_j + 1) a_j(x_0) + V(x_0), \mathbf{k} \in \mathbb{Z}_+^n$.
In the case $d > 2n$,

$$\begin{aligned}\langle f_0(x_0), \varphi \rangle =& \frac{1}{(2\pi)^{d-n}} \prod_{j=1}^{n} a_j(x_0) \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \int_{\mathbb{R}^{d-2n}} \varphi(\Lambda_{\mathbf{k}}(x_0) + |\xi|^2) d\xi \\ =& \frac{|S^{d-2n-1}|}{2(2\pi)^{d-n}} \prod_{j=1}^{n} a_j(x_0) \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \int_0^{+\infty} \varphi(\tau)(\tau - \Lambda_{\mathbf{k}}(x_0))_+^{d/2-n-1} d\tau.\end{aligned}$$

Higher order coefficients

In the maximal rank case $d = 2n$

$$\langle f_r(x_0), \varphi \rangle = \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \sum_{j=1}^{N} f_{r,\mathbf{k},j}(x_0)\varphi^{(j-1)}(\Lambda_{\mathbf{k}}(x_0)).$$

In the general case $d > 2n$

$$\langle f_r(x_0), \varphi \rangle = \sum_{\mathbf{k}\in\mathbb{Z}_+^n} \sum_{j=1}^{N} f_{r,\mathbf{k},j}(x_0) \int_0^{+\infty} \varphi^{(j-1)}(\tau)(\tau - \Lambda_{\mathbf{k}}(x_0))_+^{d/2-n-1} d\tau$$

Asymptotic localization of Schwartz kernels

Theorem (Yu.K. 2022)

Assume that, for some $x_0 \in \mathbb{R}^d$, the rank of $\mathbf{B}_{x_0}$ equals d and an interval (α, β) does not contain any $\Lambda_{\mathbf{k}}(x_0)$ with $\mathbf{k} \in \mathbb{Z}_+^n$.
For any $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} \varphi \subset (\alpha, \beta)$,

$$\left|K_{\varphi(H_\hbar/\hbar)}(x_0, x_0)\right|_{C^k} = \mathcal{O}(\hbar^\infty), \quad k = 0, 1, \ldots, \quad \hbar \to 0.$$

Moreover, if an interval $[\alpha, \beta]$ does not contain any $\Lambda_{\mathbf{k}}(x_0)$ with $\mathbf{k} \in \mathbb{Z}_+^n$, then the Schwartz kernel of the spectral projection $E_{[\hbar\alpha, \hbar\beta]}$ of the operator $H_\hbar$ associated with $[\hbar\alpha, \hbar\beta]$ satisfies

$$\left|E_{[\hbar\alpha, \hbar\beta]}(x_0, x_0)\right| = \mathcal{O}(\hbar^\infty), \quad \hbar \to 0.$$

Magnetic walls and Iwatsuka model

$\mathbb{R}^2$ with Euclidean metric $g = dx^2 + dy^2$ and magnetic field

$$\mathbf{B} = B(x,y)dx \wedge dy.$$

- B depends only on the first coordinate, i.e. $B(x,y) = B(x)$;
- B is a monotone function of x;
- There exist $B_-, B_+ \in \mathbb{R} \setminus \{0\}$, $B_- B_+ > 0$ such that

$$\lim_{x\to\pm\infty} B(x) = B_\pm.$$

- The particle is subject to a strong magnetic field on the right half plane, and to a weaker one on the left half plane.
- $B_- B_+ < 0$ corresponds to a magnetic wave guide.

Magnetic walls and Iwatsuka model

- Magnetic potential

$$A_1 = 0, \quad A_2 = \beta(x) := \int_0^x B(s)ds, \quad x \in \mathbb{R} \left(B = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right).$$

- Landau Hamiltonian

$$H_B = -\frac{\partial^2}{\partial x^2} + \left(\frac{\partial}{\partial y} - i\beta(x) \right)^2.$$

Theorem (Iwatsuka, 1985)

If $B_- \neq B_+$, then H_B has absolutely continuous spectrum.

Iwatsuka, A.: Examples of absolutely continuous Schrödinger operators in magnetic fields. Publ. Res. Inst. Math. Sci. 21, 385–401 (1985)

Asymptotic localization for Iwatsuka model

Corollary (Yu.K. 2022)

Assume that $I = [a, b]$ *doesn't contain any Landau level* $(2k+1)B_-$ *and* $(2k+1)B_+$ *with* $k \in \mathbb{Z}_+$.
Then $B^{-1}(I) = [B^{-1}(a), B^{-1}(b)]$ *is a compact interval and the Schwartz kernel of the spectral projection* $E_{\hbar I}$ *of the operator*

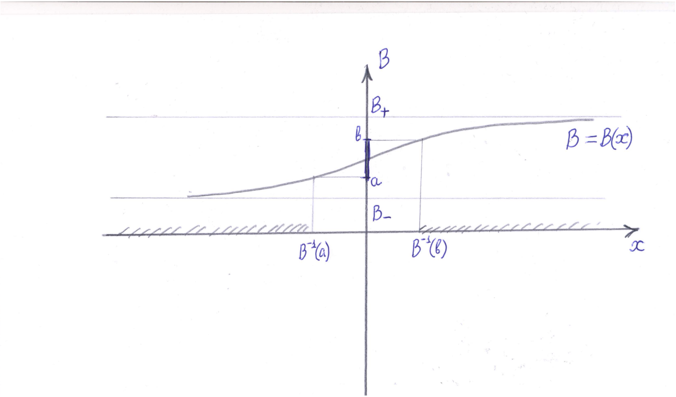
$$H_\hbar = -\hbar^2 \frac{\partial^2}{\partial x^2} + \left(\hbar \frac{\partial}{\partial y} - i\beta(x)\right)^2$$

associated with $\hbar I = [\hbar a, \hbar b]$ *satisfies*

$$|E_{\hbar I}(x, y, x, y)| = \mathcal{O}(\hbar^\infty), \quad \hbar \to 0$$

for any (x, y) *outside the strip* $B^{-1}(I) \times \mathbb{R} \subset \mathbb{R}^2$.

Asymptotic localization for Iwatsuka model



For any (x, y) outside the strip $B^{-1}(I) \times \mathbb{R} \subset \mathbb{R}^2$.

$$|E_{\hbar I}(x, y, x, y)| = \mathcal{O}(\hbar^\infty), \quad \hbar \to 0$$