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Means in quantum physics and somewhere else

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Arguments of the talk

- Quantum Covariance is defined using means
- Quantum Fisher Information is defined using means
- The relation between QFI and QC requires a mean formula
- Uncertainty Principle(s) is a mean inequality
- The Jensen inequality has a mean version
- Is Stam inequality a mean inequality?

Pascal (and the Greeks)

Car enfin qu'est-ce que l'homme dans la nature? Un néant a l'égard de l'infini, un tout a l'égard du néant, un milieu entre rien et tout.

Pensées

□ □ □

The ancient Greeks defined a list of ten distinct "means", including most of the well-known means that we still use today.

The basic inequality for the harmonic, geometric and arithmetic means.

$$\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \sqrt{xy} \leq \frac{x+y}{2}$$

Axioms for numerical means

Let $\mathbb{R}^+ = (0, +\infty)$.
A *mean* for pair of positive numbers is a function $m(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that
i) $m(x,x) = x$;
ii) $m(x,y) = m(y,x)$;
iii) $x \leq x' \quad y \leq y' \quad \Longrightarrow \quad m(x,y) \leq m(x',y')$;
iv) for $t > 0$ one has $m(tx,ty) = t \cdot m(x,y)$;
v) $m(\cdot,\cdot)$ is continuous.

$$\mathcal{M}_{nu} := \{m(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ | m \text{ is a mean } \}$$

Representing functions for numerical means

$\mathcal{F}_{nu}$ is the class of functions $f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that
iii) $f(1) = 1$;
iv) $tf(t^{-1}) = f(t)$;
iii) $x \leq x' \quad \Longrightarrow \quad f(x) \leq f(x')$;
iv) f is continuous.

Proposition

There is bijection betwen $\mathcal{M}_{nu}$ and $\mathcal{F}_{nu}$ given by the formula

$$m_f(x, y) := yf(xy^{-1})$$

Examples of representing functions - I

Table:

Name	function	mean
arithmetic	$\frac{1+x}{2}$	$\frac{x+y}{2}$
geometric	$\sqrt{x}$	$\sqrt{xy}$
harmonic	$\frac{2x}{x+1}$	$\frac{2}{x^{-1}+y^{-1}}$
logarithmic	$\frac{x-1}{\log x}$	$\frac{x-y}{\log x - \log y}$

Examples of representing functions - II

Table:

Name	function	mean
Heinz $\beta \in \left[0, \frac{1}{2}\right]$	$\frac{x^\beta + x^{1-\beta}}{2}$	$\frac{x^\beta y^{1-\beta} + x^{1-\beta} y^\beta}{2}$
Wigner − Yanase	$\left(\frac{1+\sqrt{x}}{2}\right)^2$	$\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2$
WYD $\beta \in \left(0, \frac{1}{2}\right]$	$\frac{\beta(1-\beta)(x-1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)}$	$\frac{\beta(1-\beta)(x-y)^2}{(x^\beta - y^\beta)(x^{1-\beta} - y^{1-\beta})}$

Regular and non-regular means: the function $\tilde{f}$.

$$\mathcal{F}_{nu}^{r} := \{f \in \mathcal{F}_{nu} | f(0) := \lim_{t \to 0} f(t) > 0\}$$

$$\mathcal{F}_{nu}^{n} := \{f \in \mathcal{F}_{nu} | f(0) = 0\}$$

$$\mathcal{F}_{nu} = \mathcal{F}_{nu}^{r} \dot{\cup} \mathcal{F}_{nu}^{n}$$

$$\tilde{f}(x) := \frac{1}{2}\left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right]$$

Proposition

The function $\mathcal{F}_{nu}^{r} \ni f \to \tilde{f} \in \mathcal{F}_{nu}^{n}$ is a bijection.

Operator means

Kubo-Ando 1980

Let $D_n := \{A \in M_{n,sa} | A > 0\}$.

A *mean* is a function $m : \mathcal{D}_n \times \mathcal{D}_n \to \mathcal{D}_n$ such that

(i) $m(A, A) = A$,

(ii) $m(A, B) = m(B, A)$,

(iii) $A < A', B < B' \implies m(A, B) < m(A', B')$,

(iv) m is continuous,

(v) $Cm(A, B)C^* \leq m(CAC^*, CBC^*)$, for every $C \in M_n$.

Property (vi) is the *transformer inequality*.

Operator monotone functions

M_n = complex matrices

Definition

$f : (0, +\infty) \to R$ is operator monotone iff $\forall A, B \in M_{n,sa}$ and $\forall n = 1, 2, ...$

$$0 \leq A \leq B \quad \Longrightarrow \quad 0 \leq f(A) \leq f(B).$$

Definition

φ is a Pick function if it is analytic in the upper half plane and map the latter into itself.

Löwner Theorem

Löwner 1932

Theorem

f is operator monotone iff it is the restriction of a Pick function.

$\mathcal{F}_{op}$

Usually one consider o.m. functions that are:
i) normalized i. e. $f(1) = 1$;
ii) symmetric i.e. $tf(t^{-1}) = f(t)$.

$\mathcal{F}_{op}$:= family of normalized symmetric o. m. functions.

Examples

$$\frac{1+x}{2}, \quad \sqrt{x}, \quad \frac{2x}{1+x}.$$

Kubo–Ando theorem

$\mathcal{M}_{op}$:= family of matrix means.

Kubo and Ando (1980) proved the following, fundamental result.

Theorem

There exists a bijection between $\mathcal{M}_{op}$ and $\mathcal{F}_{op}$ given by the formula

$$m_f(A,B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

$$[A,B]=0 \quad \Longrightarrow \quad m_f(A,B) := Af(BA^{-1}).$$

Kubo–Ando inequality

Examples of operator means

$$\frac{A+B}{2}$$

$$A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$$

$$2(A^{-1}+B^{-1})^{-1}$$

Fundamental inequality

$$2(A^{-1}+B^{-1})^{-1} \leq m_f(A,B) \leq \frac{A+B}{2} \qquad \forall f \in \mathcal{F}_{op}$$

Chentsov Theorem

(Using the classical **Chentsov theorem** it is possible to understand what should be called *Quantum Fisher Information*.) .

Chentsov Theorem.
On the simplex $\mathcal{P}_n^1$ the Fisher information is the only Riemannian metric contracting under an arbitrary coarse graining T, namely for any tangent vector X at the point ρ we have

$$g^m_{T(\rho)}(TX, TX) \leq g^n_\rho(X, X)$$

Remark
Coarse graining = stochastic map = linear, positive, trace preserving.

Monotone metrics (or QFI) according Chentsov-Morozova

$$D_n^1 := \{\rho \in M_{n,sa} | \mathrm{Tr}(\rho) = 1 \quad \rho > 0\} = \text{faithful states}$$

Definition

A quantum Fisher information is a Riemaniann metric on D_n^1 contracting under an arbitrary coarse graining T, namely

$$g^m_{T(\rho)}(TA, TA) \leq g^n_\rho(A, A).$$

(quantum) coarse graining = linear, (completely) positive, trace preserving map.

Petz theorem

$$L_\rho(A) := \rho A \qquad R_\rho(A) := A\rho$$

Petz theorem

There is bijection among quantum Fisher information(s) and operator monotone functions (and/or operator means) given by the formula

$$\langle A, B\rangle_{\rho,f} := \mathrm{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

g-Covariance (according to Petz)

To each operator monotone $g \in \mathcal{F}_{op}$ one associate the means $m_g(\cdot, \cdot)$.

Define the g-covariance as

$$\mathrm{Cov}_\rho^g(A, B) := \mathrm{Tr}(m_g(L_\rho, R_\rho)(A_0)B_0)$$

If

$$g(x) = \frac{1+x}{2}$$

then
$m_g =$ arithmetic mean
and $\mathrm{Cov}_\rho^g(A, B)$ is the standard covariance introduced by Schrödinger and Roberston.

Fundamental formula: Relation between QFI and covariance

Theorem (Gibilisco-Imparato-Isola)

If f is regular then

$$\frac{f(0)}{2}\langle i[\rho, A], i[\rho, B]\rangle_{\rho,f} = \mathrm{Cov}_\rho(A, B) - \mathrm{Cov}_\rho^{\tilde{f}}(A, B).$$

Introducing the *metric adjusted skew information*

$$I_\rho^f(A) := \frac{f(0)}{2}||i[\rho, A]||^2_{\rho,f} = \frac{f(0)}{2}||\dot{\rho}_A(0)||^2_{\rho,f}$$

Proposition

$$I_\rho^f(A) = \mathrm{Var}_\rho(A) - \mathrm{Var}_\rho^{\tilde{f}}(A).$$

$$\mathrm{Var}_\rho(A) \geq \frac{f(0)}{2}||i[\rho, A]||^2_{\rho,f}$$

The dynamical UP (g-version)

Theorem

Let $A_1 \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det\left\{\mathrm{Cov}_\rho^g(A_h, A_j)\right\} \geq \det\left\{g(0)f(0)\langle i[\rho, A_h], i[\rho, A_j]\rangle_{\rho,f}\right\}$$

for $h, j = 1, \ldots, N$,
for all $g, f \in \mathcal{F}_{op}$.

Conclusion - Dynamical UP case

For the dynamical UP quantum g-covariances coming from regular g (constant $g(0) \neq 0$) do have uncertainty relations.

Quantum g-covariances coming from nonregular g (constant $g(0) = 0$) do NOT have (non-trivial) uncertainty relations.

g-version of Robertson UP

Theorem

Let $A_1 \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det \left\{ \mathrm{Cov}_\rho^g(A_h, A_j) \right\} \geq \det \left\{ -i \cdot g(0) \cdot \mathrm{Tr}(\rho [A_h, A_j]) \right\},$$

for $h, j = 1, \ldots, N$,
for all $g \in \mathcal{F}_{op}$.

Remark: $g(0)$ is the best constant in the above inequalities.

The end

Quantum g-covariances coming from regular g (constant $g(0) \neq 0$) do have uncertainty relations.

Quantum g-covariances coming from nonregular g (constant $g(0) = 0$) do NOT have uncertainty relations.

The usual quantum covariance has the most demanding one (since $g(0) = \frac{1}{2}$ only for the arithmetic mean).

After all Schrödinger and Robertson were right ...

Rao inequality for the harmonic mean

Suppose $X, Y\colon \Omega \to (0, +\infty)$ are positive random variables.
Working on a result by Fisher on ancillary statistics Rao obtained the following proposition by an application of Hölder's inequality together with the harmonic-geometric mean inequality.

$$\mathrm{E}(m_h(X, Y)) \leq m_h(\mathrm{E}(X), \mathrm{E}(Y)).$$

C.R. Rao. *R.A. Fisher: the founder of modern statistics*. Statistical Science, **7** (1992), 34–48.

C.R. Rao. *Seven inequalities in statistical estimation theory*. Student, **1** (1996), no. 3, 149–158.

How general is the Rao inequality?

Linearity of the expectation operator trivially implies

$$\mathrm{E}(m_a(X,Y)) = m_a(\mathrm{E}(X),\mathrm{E}(Y)).$$

On the other hand the Cauchy-Schwartz inequality implies

$$\mathrm{E}(m_g(X,Y)) \leq m_g(\mathrm{E}(X),\mathrm{E}(Y)).$$

It is natural to ask about the generality of this result. For example, does it hold also for the logarithmic mean? To properly answer this question it is better to choose one of the many axiomatic approaches to the notion of a mean.

Concavity and convexity

The functions in the tables above are all concave (even operator concave!)

However, there exist non-concave functions in $\mathcal{F}_{num}$.
Consider for example the function

$$g(x) = \frac{1}{4}\begin{cases} x+3 & 0 \le x \le 1, \\ 3x+1 & x \ge 1. \end{cases}$$

This piece-wise affine function is convex and belongs to $\mathcal{F}_{num}$.

Jensen inequality for mean: commutative case

Theorem
Take a function $f \in \mathcal{F}_{num}$.
The inequality

$$\mathrm{E}(m_f(X,Y)) \leq m_f(\mathrm{E}(X), \mathrm{E}(Y))$$

holds for arbitrary positive random variables X and Y if and only if f is concave.
Proof: Jensen inequality ...

P. Gibilisco, F. Hansen. An inequality for expectation of means of positive random variable. *Annals of Functional Analysis*, 8(1), pp.142 – 151, 2017.

Question

Question: what about a noncommutative version?
Yes for the harmonic mean, due to ...
B.L.S. Prakasa Rao. An inequality for the expectation of harmonic mean of random matrices. *Technical Report*, Indian Statistical Institute, Delhi, (1998).
C.R. Rao. Statistical proofs of some matrix inequalities. *Lin. Alg. Appl.*, **321** (2000), 307 – 320.

What happens for other noncommutative means?

Jensen inequality for means: noncommutative case

If ρ is a density matrix and A is self-adjoint then the expectation of A in the state ρ is defined by setting $\mathrm{E}_\rho(A) = \mathrm{Tr}(\rho A)$.

Theorem

Take **any** $f \in \mathcal{F}_{op}$. Then

$$\mathrm{E}_\rho(m_f(A, B)) \leq m_f(\mathrm{E}_\rho(A), \mathrm{E}_\rho(B)),$$

Remark. Why is it so? All the function in $\mathcal{F}_{op}$ are (operator) concave making the operator case quite different from the numerical one.

P. Gibilisco, F. Hansen. An inequality for expectation of means of positive random variable. *Annals of Functional Analysis*, 8(1), pp.142 – 151, 2017.