Unipotent group structures on quintic del Pezzo varieties
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Seminar on Algebraic Transformation Groups
February 2023

§1. Motivation

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Problem 27 on **Hirzebruch**'s (1954) problem list: Fix $n \in \mathbf{N}^{\geq 1}$, and

Classify all the pairs (X, Δ) such that:

- X: complex projective manifold of $\dim_{\mathbf{C}}(X) = n$.
- $\Delta \subseteq X$ effective reduced (boundary) divisor such that $X \smallsetminus \Delta \cong \mathbf{A}^n$.
- $b_2(X) = 1$.

Kodaira (1974): In that case,

- X is a *Fano* manifold (i.e., $\det(T_X) = \mathcal{O}_X(-K_X)$ is *ample*),
- $-K_X = m\Delta$, $m \in \mathbf{N}^{\geq 1}$.

Example: $(X, \Delta) \cong (\mathbf{P}^n, \mathbf{P}^{n-1})$.

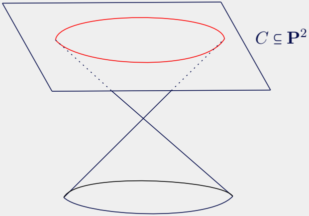
Recall (Kobayashi-Ochiai): The Fano index of X is the maximum $\iota_X \in \mathbf{N}$ such that $-K_X = \iota_X A$ for some A ample divisor. Moreover, $1 \leq \iota_X \leq n+1$, and $\iota_X = n+1$ (resp. $\iota_X = n$) iff $X \cong \mathbf{P}^n$ (resp. $X \cong \mathbf{Q}^n \subseteq \mathbf{P}^{n+1}$).

Known cases:

- $n = 1$: $(X, \Delta) \cong (\mathbf{P}^1, \{\text{pt}\})$
- $n = 2$: $(X, \Delta) \cong (\mathbf{P}^2, \{\text{line}\})$
- $n = 3$: Several authors (1978–1993).

Furushima (1993):

$$(X, \Delta) \cong \begin{cases} (\mathbf{P}^3, \{\text{plane}\}) & (\iota_X = 4) \\ (\mathbf{Q}^3, \mathbf{Q}_0^2) & (\iota_X = 3) \\ (V_5, S_i) \quad i = 1, 2 & (\iota_X = 2) \\ (V_{22}, S_i) \quad i = 1, 2 & (\iota_X = 1) \end{cases}$$



$\Delta = \mathbf{Q}_0^2 = \mathrm{Cone}(C) \subseteq \mathbf{P}^3$

Kuznetsov–Prokhorov–Shramov (2018)

These are the only Fano 3-folds with $b_2(X) = 1$ and infinite $\mathrm{Aut}(X)$.

§1. Motivation

Recall (Iskovskikh '70s): The 3-fold V_5 is obtained by considering successive smooth hyperplane sections

$$\mathrm{Bl}_{p_1,p_2,p_3,p_4}(\mathbf{P}^2) \simeq S_5 \subseteq V_5 \subseteq W_5 \subseteq Z_5 \subseteq G \simeq \mathrm{Gr}(2,5) \hookrightarrow \mathbf{P}^9$$

The case $n \geq 4$ remains open. However, there are some remarkable results:

- **Prokhorov** (1994): If (X,Δ) is such that $\iota_X = 3$, then we have that $(X,\Delta) \cong (W_5, \Delta_i)$ with $i = 1,2,3,4$.
- **Prokhorov-Zaidenberg** (2018): Each member of the 1-dimensional family of Fano fourfolds $W_{18} \subseteq \mathbf{P}^{18}$ with $\iota_{W_{18}} = 2$ admit at least two different compactifications $\mathbf{A}^4 \hookrightarrow W_{18}$.

In all these examples, the automorphism group is **infinite**.

§2. Additive structures

§2. $\mathbf{G}_a^n$-STRUCTURES

We will impose some addional geometric restrictions by considering

- G, a connected linear algebraic group.
- X, an irreducible normal projective variety.

Definition

A G-**structure** on X is a regular action $G \times X \longrightarrow X$ such that for a general point $x_0 \in X$ we have that:

1. The stabilizer $\mathrm{Stab}(x_0)$ is trivial.
2. The orbit $G \cdot x_0 \cong G$ is dense.

In particular, $G \hookrightarrow X$ is an equivariant compactification.

Main examples:

- $G = \mathbf{G}_m^n = ((\mathbf{C}^\times)^n, \cdot) \rightsquigarrow X$ **toric variety** $\rightsquigarrow$ combinatorics
- $G = \mathbf{G}_a^n = (\mathbf{C}^n, +) \rightsquigarrow X$ **variety with $\mathbf{G}_a^n$-structure**

Hassett–Tschinkel (1999): There is a correspondence

$$\left\{ \begin{array}{l} \mathbf{G}_a^n\text{-structures on the} \\ \text{projective space } \mathbf{P}^n \end{array} \right\}_{/\sim} \leftrightarrow \left\{ \begin{array}{l} \text{commutative local Artin} \\ \mathbf{C}\text{-algebras } A \text{ of } \dim_{\mathbf{C}}(A) = n+1 \end{array} \right\}_{/\sim}$$

Example ($n = 2$): The algebras $A_i = \mathbf{C}[X,Y]/\mathcal{I}_i \cong_{\mathbf{C}\text{-v.s.}} \mathbf{C}^3$ with

$$\mathcal{I}_1 = \langle X^2, XY, Y^2 \rangle \quad \text{and} \quad \mathcal{I}_2 = \langle XY, Y - X^2 \rangle$$

define additive structures on $\mathbf{P}^2 = \mathbf{P}(A_i)$ via $\exp(\mathfrak{m}_{A_i})$:

Let $(a_1, a_2) \in \mathbf{G}_a^2$, then $\exp([a_1 X + a_2 Y]) \curvearrowright A_i$ induces

$$\rho_1 : [x_0, x_1, x_2] \mapsto [x_0 + a_2 x_2, x_1 + a_1 x_2, x_2]$$

$$\rho_2 : [x_0, x_1, x_2] \mapsto [x_0 + a_1 x_1 + (a_2 + \frac{1}{2}a_1^2) x_2, x_1 + a_1 x_2, x_2]$$

where ρ_1 (resp. ρ_2) have infinitely many (resp. 3) orbits.

Suprunenko (1966): $a(n) := \#\{\mathbf{G}_a^n\text{-structures on } \mathbf{P}^n\}/\sim$

n	1	2	3	4	5	≥ 6
$a(n)$	1	2	4	9	25	$+\infty$

Theorem (Hassett-Tschinkel, 1999)

Let X be a smooth projective 3-fold with $b_2(X) = 1$.

If X admits a $\mathbf{G}_a^3$-structure, then $X \cong \mathbf{P}^3$ or $\mathbf{Q}^3 \subseteq \mathbf{P}^4$.

Their proof uses the following ingredients:

Let X be smooth proj. with a $\mathbf{G}_a^n$-structure such that $b_2(X) = r$. Then

- $\Delta = X \smallsetminus \mathbf{A}^n = \Delta_1 \cup \ldots \cup \Delta_r$, with Δ_i irreducible divisor.
- $-K_X = \sum_{i=1}^r a_i \Delta_i$, with $a_i \geq 2$. In particular, if $b_2(X) = 1$ then $\iota_X \geq 2$.
- $(n = 3)$ By Furushima, they are reduced to exclude the case $X \cong V_5$, i.e., a 3-dim. linear section of $\mathrm{Gr}(2,5) \hookrightarrow \mathbf{P}(\Lambda^2 \mathbf{C}^5) \cong \mathbf{P}^9$.
- The contradiction[1] comes from a $\mathbf{G}_a^3$-equivariant Sarkisov link $V_5 \dashrightarrow \mathbf{Q}^3$ studied by Furushima–Nakayama (1989). $\square$

[1]Alternatively, we can use the fact that $\mathrm{Aut}(V_5) \cong \mathrm{PGL}_2(\mathbf{C}) \not\ltimes \mathbf{G}_a^3$.

§3. Results

Fujita (1980s): Classification of Fano n-folds X such that $\iota_X = n-1$, i.e., *del Pezzo varieties*. They are classified according to their degree $d \in \{1, \ldots, 8\}$.

Fu–M. (2019): Let X be a smooth del Pezzo variety of dimension n admitting a $\mathbf{G}_a^n$-structure. Then,

$$X \cong \begin{cases} \mathrm{Gr}(2,5) \cap L \subseteq \mathbf{P}^9 \text{ linear section} & (b_2(X) = 1,\ d = 5,\ 4 \leq n \leq 6) \\ \mathbf{P}^2 \times \mathbf{P}^2,\ \mathrm{Bl}_p(\mathbf{P}^3),\ \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 & (b_2(X) \geq 2) \end{cases}$$

Several remaining issues in the case $b_2(X) = 1$

(A) What is the boundary divisor $\Delta = X \smallsetminus \mathbf{A}^n$? (cf. Hirzebruch's problem)

(B) What about **singular** varieties? (cf. Equivariant MMP)

(C) What if the ground field $k \neq \overline{k}$? (cf. $k = \mathbf{C}(Y)$ function field)

Main Theorem (Dubouloz–Kishimoto–M., 2022)

Let $X_L := \mathrm{Gr}(2,5) \cap L \subseteq L \subseteq \mathbf{P}^9$ be a n-dimensional linear section. Then,

1. If X_L is smooth, there is a unique $\mathbf{G}_a^n$-structure on X_L as long as $4 \leq n \leq 6$. Moreover, we can describe $\Delta = X_L \setminus \mathbf{A}^n$.
2. If X_L is a terminal 3-fold, there exists (a unique) $\mathbf{G}_a^3$-structure on X_L if and only if $\mathrm{Sing}(X_L) = \{3 \text{ nodes}\}$.
3. If X_L is a surface with canonical singularities, there exists a $\mathbf{G}_a^2$-structure on X_L if and only if
   - $(\rho_{X_L} = 1)$ $\mathrm{Sing}(X_L) = 1\,\mathrm{A}_4$. Here, there are two $\mathbf{G}_a^2$-structures.
   - $(\rho_{X_L} = 2)$ $\mathrm{Sing}(X_L) = 1\,\mathrm{A}_3$. Here, the $\mathbf{G}_a^2$-structure is unique.

Actually, more is true

Let k be a field of characteristic zero, and let Y be a k-form of X_L. Then, along the proof, is possible to take into account the action of $\mathrm{Gal}(\overline{k}/k)$ in order to analyze the existence of $\mathbf{G}_{a,k}^n$-structures on Y.

§4. Some ingredients

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We analyze equivariant Sarkisov links that are obtained as linear projections from "maximal" linear subspaces.

Recall (cf. Todd, 1930): Let $G := \mathbf{G}(1,4) \cong \mathrm{Gr}(2,5) \hookrightarrow \mathbf{P}(\Lambda^2\mathbf{C}^5) \cong \mathbf{P}^9$.

To every the projective linear flag in $\mathbf{P}^4$

$$\Lambda_\bullet : \quad \Lambda_0 := \{p\} \subseteq \Lambda_1 := \ell_0 \subseteq \Lambda_2 := \Pi \subseteq \Lambda_3 := \Lambda \subseteq \Lambda_4 = \mathbf{P}^4$$

we can associate a Schubert variety of type (a,b) as follows:

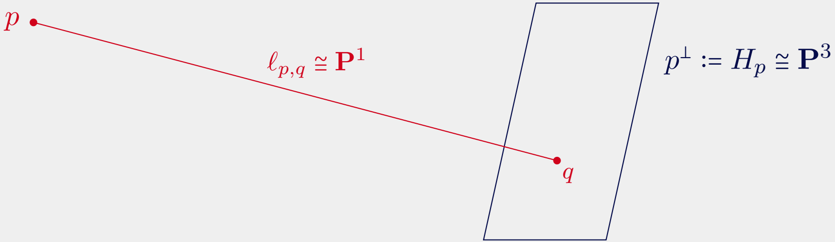
$$\sigma_{a,b}(\Lambda_\bullet) := \{\ell \subseteq \mathbf{P}^4 \text{ such that } \ell \cap \Lambda_{3-a} \neq \varnothing \text{ and } \ell \subseteq \Lambda_{4-b}\} \subseteq \mathbf{G}(1,4)$$

Main examples:

1. $\sigma_{2,2}(\Pi) = \{\ell \subseteq \mathbf{P}^4 \text{ such that } \ell \subseteq \Pi\} =: P_\Pi \cong \mathbf{P}^2$.
2. $\sigma_{3,1}(p \in \Lambda) = \{\ell \subseteq \mathbf{P}^4 \text{ such that } p \in \ell \subseteq \Lambda\} =: P_{p,\Lambda} \cong \mathbf{P}^2$.
3. $\sigma_{3,0}(p) = \{\ell \subseteq \mathbf{P}^4 \text{ such that } p \in \ell\} =: V_p \cong \mathbf{P}^3$.

The Schubert variety $\sigma_{3,0}(p) \subseteq G$ can be visualized as follows:

$$\sigma_{3,0}(p) = \left\{\ell \subseteq \mathbf{P}^4 \text{ such that } p \in \ell\right\} \simeq \mathbf{P}^3 \subseteq \mathbf{G}(1,4)$$



In particular, the Hilbert scheme of such volumes is $\Sigma_{3,0}(G) \simeq (\mathbf{P}^4)^\vee \simeq \mathbf{P}^4$.

§4. Some ingredients

Following **Piontkowski–Van de Ven** (1999), we compute $\mathrm{Aut}(X)$ and the induced action on the Hilbert scheme $\Sigma_{a,b}(X)$. For example:

1. For $G = \mathrm{Gr}(2,5)$, we have that

$$\Sigma_{2,2}(G) \cong G, \quad \Sigma_{3,1}(G) \cong \mathbf{P}(T_{\mathbf{P}^4}), \quad \Sigma_{3,0}(G) \cong \mathbf{P}^4.$$

2. For $Z := G \cap H \subseteq H \cong \mathbf{P}^8$ smooth hyperplane section, we have that

$$\Sigma_{2,2}(Z) \cong \mathbf{Q}^3 \subseteq \mathbf{P}^4, \quad \Sigma_{3,1}(Z) \cong \mathrm{Bl}_p(\mathbf{P}^4), \quad \Sigma_{3,0}(Z) \cong \{\mathrm{pt}\}.$$

3. For $W := G \cap H \cap H' \subseteq H \cap H' \cong \mathbf{P}^7$ smooth linear section of codimension 2, we have that

$$\Sigma_{2,2}(W) \cong \{\mathrm{pt}\}, \quad \Sigma_{3,1}(W) \cong C \subseteq \mathbf{P}^2, \quad \Sigma_{3,0}(W) = \varnothing.$$

§4. Some ingredients

We will examine linear projections from linear subspaces (cf. Fujita 1981). To do it equivariantly, we need:

Blanchard's Lemma

Let G be a connected algebraic group, and let $f : X \to Y$ a proper morphism between algebraic varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

If G acts regularly on X, then there is a unique regular action of G on Y such that f is G-equivariant.

If we assume additionally that $G = \mathbf{G}_a^n$, we prove:

Corollary (DKM, 2022)

Let $n = \dim(X)$ and $m = \dim(Y)$. Then,

- Any $\mathbf{G}_a^n$-structure on X induces a unique $\mathbf{G}_a^m$-structure on Y.
- The generic fiber X_η of $f : X \to Y$ admits a $\mathbf{G}_{a,k(Y)}^{n-m}$-structure.

§5. Sketch of Proof

Start with $G := \mathrm{Gr}(2,5) \subseteq \mathbf{P}^9$.

Existence of a $\mathbf{G}_a^6$-structure (from birational geometry):

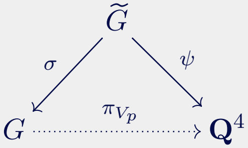
Let $\mathbf{P}^3 \cong V_p \subseteq G \hookrightarrow \mathbf{P}^9$ be a $\sigma_{3,0}$-volume, and consider the linear projection

$$\pi_{V_p} : G \dashrightarrow \mathbf{P}^5,$$

whose image is the smooth quartic $\mathrm{Gr}(2,4) \cong \mathbf{Q}^4 \subseteq \mathbf{P}^5$.

The blow-up of of the indeterminacy locus $V_p \cong \mathbf{P}^3$ induces the following:

§5. Sketch of Proof ($X_L = \mathrm{Gr}(2,5) \cap L \hookrightarrow \mathbf{P}^9$)



Here:

1. **Sharoyko** (2009): $\mathrm{Gr}(2,4) \cong \mathbf{Q}^4 \subseteq \mathbf{P}^5$ admits a *unique* $\mathbf{G}_a^4$-structure.
2. $\widetilde{G} := \mathrm{Bl}_{V_p}(G) \cong$ Locally trivial $\mathbf{P}^2$-bundle $\psi : \widetilde{G} \longrightarrow \mathbf{Q}^4$.
3. $\widetilde{G} \cong \mathbf{P}(E)$, where E is a canonically defined[2] rank 3 v.b. on $\mathbf{Q}^4$. In particular, it carries a canonical $\mathbf{G}_a^4$-linearization.
4. We use the properties of E to extend the $\mathbf{G}_a^4$-structure on $\mathbf{Q}^4$ to a unique $\mathbf{G}_a^6$-structure on $\widetilde{G}$. We conclude by Blanchard's lemma. □

[2] More precisely, is the extension of a spinor bundle and the trivial line bundle.

§5. Sketch of Proof ($X_L = \mathrm{Gr}(2,5) \cap L \hookrightarrow \mathbf{P}^9$)

Uniqueness of a $\mathbf{G}_a^6$-structure (from birational geometry):

Recall that $\Sigma_{3,0}(G) \cong \mathbf{P}^4$, and $\mathrm{Aut}(G) \cong \mathrm{PGL}_5(\mathbf{C}) \curvearrowright \Sigma_{3,0}(G)$ is transitive.

Let $\mathbf{G}_a^6 \times \mathrm{Gr}(2,5) \longrightarrow \mathrm{Gr}(2,5)$ be any $\mathbf{G}_a^6$-structure. Then:

1. Borel fixed-point theorem: There is $\mathbf{G}_a^6$-stable $\sigma_{3,0}$-volume $V_p \subseteq G$.
2. (Corollary of) Blanchard's lemma: there is a $\mathbf{G}_a^4$-structure on $\mathbf{Q}^4$, which is known to be unique (Sharoyko, 2009).
3. Together with the uniqueness of the $\mathbf{G}_a^6$-linearization of E (where $\widetilde{G} \cong \mathbf{P}(E)$), we get a unique $\mathbf{G}_a^6$-structure on G preserving V_p.
4. Since $\mathrm{Aut}(G)$ acts transitively on $\Sigma_{3,0}(G) \simeq (\mathbf{P}^4)^\vee$, we are done. $\square$

Lie theory (Arzhantsev 2011, Devyatov 2015)

Using the fact that $\mathrm{Gr}(2,5) \cong \mathrm{GL}_5 / P$ is a rational homogeneous space, we can exhibit the desired $\mathbf{G}_a^6$-structure explicitly in Plücker coordinates.

The boundary divisor $\Delta = G \setminus \mathbf{A}^6$:

Let $\mathbf{P}^2 \cong P_\Pi \subseteq G \hookrightarrow \mathbf{P}^9$ be a $\sigma_{2,2}$-plane, and the (surjective) linear projection

$$\pi_{P_\Pi} : G \dashrightarrow \mathbf{P}^6.$$

We check that there is a unique divisor $R \in |\mathcal{O}_G(1) \otimes \mathcal{I}^2_{P_\Pi}|$ such that

$$\begin{array}{ccccccccc}
E, \ \widetilde{R} & \hookrightarrow & & & \mathrm{Bl}_{P_\Pi}(G) & & \hookleftarrow & \widetilde{R}, & E \\
\downarrow \quad \downarrow & & & \swarrow & & \searrow & & \downarrow & \downarrow \cong \\
P_\Pi \subseteq R & \hookrightarrow & G & & \dashrightarrow & & \mathbf{P}^6 \hookleftarrow & T \cong \mathbf{P}^1 \times \mathbf{P}^2 & \subseteq \langle T \rangle \cong \mathbf{P}^5
\end{array}$$

is equivariant. □

Moreover, among the infinite $\mathbf{G}_a^6$-structures on $\mathbf{P}^6$, the induced action on $\mathbf{P}^6$ is the naive (i.e. toric) one.

Smooth linear sections $W \subseteq Z \subseteq G$:

Let H, H' be general hyperplanes in $\mathbf{P}^9 = \mathbf{P}(\Lambda^2\mathbf{C}^5)$, and consider

1. $Z := G \cap H \subseteq \mathbf{P}^8$ smooth quintic del Pezzo fivefold.
2. $W := G \cap H \cap H' \subseteq \mathbf{P}^7$ smooth quintic del Pezzo fourfold.

For a $\sigma_{2,2}$-plane $\mathbf{P}^2 \cong P_\Pi \subseteq W \subseteq Z$, the diagrams

$$\begin{array}{ccccccccc}
E, & \widetilde{R} & \hookrightarrow & \mathrm{Bl}_{P_\Pi}(Z) & \hookleftarrow & & \widetilde{R}, & E & \\
\downarrow & \downarrow & \swarrow & & \searrow & & \downarrow & \downarrow \cong & \\
P_\Pi \subseteq & R & \hookrightarrow Z & \dashrightarrow & \mathbf{P}^5 & \hookleftarrow & S \cong \mathbf{F}_1 & \subseteq \langle S \rangle \cong \mathbf{P}^4 &
\end{array}$$

$$\begin{array}{ccccccccc}
E, & \widetilde{R} & \hookrightarrow & \mathrm{Bl}_{P_\Pi}(W) & \hookleftarrow & & \widetilde{R}, & E & \\
\downarrow & \downarrow & \swarrow & & \searrow & & \downarrow & \downarrow \cong & \\
P_\Pi \subseteq & R & \hookrightarrow W & \dashrightarrow & \mathbf{P}^4 & \hookleftarrow & C \cong \mathbf{P}^1 & \subseteq \langle C \rangle \cong \mathbf{P}^3 &
\end{array}$$

are equivariant, and induce the naive $\mathbf{G}_a^n$-structure on $\mathbf{P}^n$. □

§6. The singular case

§6. Sketch of Proof (singular case)

The case of terminal threefolds:

1. By Prokhorov's classification of G-del Pezzo threefolds (2013), we are left to analyze threefolds X with $s(X) \in \{1, 2, 3\}$ nodes.
2. In the case of $s(X) = 1$ or 2 nodes, a $\mathbf{Q}$-factorialization is isomorphic to a projectivization $\mathbf{P}(E)$ that does not admit a $\mathbf{G}_a^3$-structure.

Proposition (DKM 2022)

Let E be a simple vector bundle on a normal projective variety X (i.e., $\mathrm{End}(E) \simeq \mathbf{C}$). Then, $\mathbf{P}(E)$ does not admit an additive structure.

3. If $s(X) = 3$ there is a birational map $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbf{P}^3) \to X$, where p_1, p_2, p_3 are points in general position. The result follows.

§6. Sketch of Proof (singular case)

The case of surfaces with canonical singularites:

1. By the classification of canonical del Pezzo surfaces, together with the work of Derenthal–Loughran (2010), we are left to study uniqueness of $\mathbf{G}_a^2$-structures on $X_L \subseteq \mathbf{P}^5$ with $\mathrm{Sing}(X_L) \in \{1\,\mathrm{A}_3, 1\,\mathrm{A}_4\}$.
2. The explicit projective models obtained by Cheltsov–Prokhorov (2021) allow us to conclude. □

Application

Application

Let k be a field of characteristic zero. It follows from the proof:

Corollary (DKM 2022)

Let X be a k-form of a smooth del Pezzo variety of degree 5, and assume that $\dim(X) \in \{4, 5\}$. Then X is the **trivial** k-form, i.e.,

$$X \simeq \mathrm{Gr}(2, k^5) \cap L.$$

It is worth noting that the above result is **not true** for $\mathrm{Gr}(2, 5)$ nor V_5: there are non-trivial forms for these varieties.

Remark: It also follows from the proof that among the 4 compactifications $\mathbf{A}^4 \hookrightarrow (W_5, \Delta)$ studied by Prokhorov (1994), only 1 of them is **additive**.

Thanks for your attention!