Topology of spaces of smooth functions and gradient-like flows with prescribed singularities on surfaces

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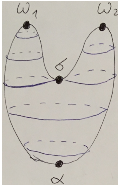
Space of smooth functions with prescribed types of local singularities

Let M be a smooth orientable closed 2D surface, and $f_0 \in C^\infty(M)$ have only critical points of types A_i, D_j, E_k (e.g. a Morse function).



Recall that a point $P \in M$ is *critical* for $f \in C^\infty$ if $\mathrm{d} f(P) = 0$.
A function f is called *Morse* if all its critical points are *non-degenerate* (of type A_1), i.e. $\mathrm{d}^2 f(P)$ is non-degenerate. By the Morse lemma, locally $f = \pm x^2 \pm y^2 + f(P)$ in suitable coordinates near each critical point P.

Let $\mathcal{F} = \mathcal{F}(f_0)$ be the set of functions $f \in C^\infty(M)$ having the same types of critical points as f_0.

Let $\mathcal{D}^0(M)$ be the identity path component of $\mathcal{D}(M) = \mathrm{Diff}^+(M)$ endowed with C^∞-topology. The group $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ acts on $\mathcal{F}$ by "left-right changes of coordinates".

Problem 1. Describe the topology of the space $\mathcal{F}$, endowed with C^∞-topology, and its decomposition into $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$- and $\mathcal{D}^0(M)$-orbits.

Suppose $\Omega \in \Lambda^n(M)$ is a volume form on a n-manifold $M = M^n$.
Let $\mathcal{P} \subset M$ be a finite set. For any vector field ξ on $M' := M \setminus \mathcal{P}$, we assign the $(n-1)$-form $\beta = i_\xi \Omega \in \Lambda^{n-1}(M')$. Clearly, this assignement is 1-to-1, and $\xi \in \mathrm{Ker}\,\beta$. The flow of ξ is volume-preserving $\iff$ $\mathrm{d}\beta = 0$.
Indeed: $L_\xi \Omega = (i_\xi \mathrm{d} + \mathrm{d} i_\xi)\Omega = \mathrm{d} i_\xi \Omega = \mathrm{d}\beta$, so $L_\xi \Omega = 0 \iff \mathrm{d}\beta = 0$.
By abusing language, a closed $(n-1)$-form β will be called a *flow*.

Suppose now that $n = \dim M = 2$. Denote $\mathcal{Z}_\beta := \{P \in M' \mid \beta(P) = 0\}$ equilibrium points.

Space of Morse flows with prescribed types of local singularities

A closed 1-form β on $M' = M \setminus \mathcal{P}$ will be called a *Morse flow* on M if, in a neighbourhood of every point $P \in \mathcal{P} \cup \mathcal{Z}_\beta$, $\exists$ local coordinates x, y s.t. $x(P) = y(P) = 0$ and either $\beta = \mathrm{d}(2xy) = \mathrm{d}(Im(z^2))$, $P \in \mathcal{Z}_\beta$, or $\beta = \pm\frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2} = \pm\mathrm{d}(Im(\ln z))$, $P \in \mathcal{P}$, $z := x + iy$.
Geometrically, $\mathcal{P}_\beta := \mathcal{P} = \{$*sources* and *sinks* of $\beta\}$, $\mathcal{Z}_\beta = \{$*saddle points* of the flow $\beta\}$.

A closed 1-form β on $M' = M \setminus \mathcal{P}$ will be called a *gradient-like flow* on M if $\exists$ a Morse function $f \in C^\infty(M)$, called an *energy function* of β, such that

(i) $\mathcal{P} = \{$local extremum points of $f\}$, $\quad \mathrm{d}f \wedge \beta|_{M \setminus \mathcal{C}_f} > 0$,

(ii) in a neighbourhood of each point $P \in \mathcal{C}_f$, $\exists$ local coordinates x, y s.t. either $f = f(P) + x^2 - y^2$, $\beta = \mathrm{d}(2xy) = \mathrm{d}(Im(z^2))$ and $P \in \mathcal{Z} = \mathcal{C}_f \setminus \mathcal{P}$, or $f = f(P) \pm (x^2 + y^2)$, $\beta = \pm\frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2} = \pm\mathrm{d}(Im(\ln z))$ and $P \in \mathcal{P}$.

Geometrically, $\mathcal{P}_\beta := \mathcal{P} = \{$*sourses* and *sinks* of $\beta\} = \{$local extremum points of $f\}$, $\mathcal{Z}_\beta := \mathcal{Z} = \mathcal{C}_f \setminus \mathcal{P} = \{$*saddle points* of $\beta\} = \{$saddle critical points of $f\}$.

Let β_0 be a Morse flow on M. Let $\mathcal{B} = \mathcal{B}(\beta_0)$ be the set of all gradient-like flows β having the same types of local singularities as β_0 (in particular, $|\mathcal{Z}_\beta| = |\mathcal{Z}_{\beta_0}|$ and $|\mathcal{P}_\beta| = |\mathcal{P}_{\beta_0}|$).

Problem 2. Describe the topology of the space $\mathcal{B} = \mathcal{B}(\beta_0)$, endowed with C^∞-topology, and its decomposition into $\mathcal{D}^0(M)$-orbits and into classes of (orbital) topological equivalence.

Problem 3. Characterize gradient-like flows among all Morse flows.

Characterization of 2D gradient-like flows

Theorem 1 (E.K. 2021 [14], generalizing 2D-case of a result by S.Smale [2])

(a) The space $\mathcal{B}(\beta_0)$ of gradient-like flows is non-empty $\iff$ β_0 has at least one sink and at least one source.

(b) Morse flow β is gradient-like $\iff$ (i) β has at least one sink and at least one source, (ii) every separatrix of β has both endpoints at equilibria $\mathcal{Z}_\beta \cup \mathcal{P}_\beta$, (iii) $\nexists$ an oriented cycle $P_1 P_2 \dots P_{k-1} P_k$ $(k \ge 2)$ formed by oriented separatrices of β, $P_k = P_1$.

(c) The space $\mathcal{B}(\beta_0)$ of gradient-like flows is open in the space of Morse flows.

Type of singularity			Normal form	Restrictions	Gradient-like flow
min	$A^{+,+}_{2i-1}$		$u^{2i} + v^2$	$i \ge 1,$	$\kappa \frac{u\,\mathrm{d}\,v - v\,\mathrm{d}\,u}{u^2+v^2}$
max	$A^{+,-}_{2i-1}$		$-(u^{2i} + v^2)$	$\kappa \in \mathbb{R}_{>0}$	$-\kappa \frac{u\,\mathrm{d}\,v - v\,\mathrm{d}\,u}{u^2+v^2}$
saddle	A^-_1		$u^2 - v^2$		$\mathrm{d}(2uv)$
	$A^{-,\eta}_{2i+1}$		$\eta(u^{2i+2} - v^2)$	$i \ge 1, \eta = \pm$	$\eta\,\mathrm{d}(2uv)$
	D^{η}_{2i+3}		$\eta u^{2i+2} + uv^2$	$i \ge 1, \eta = \pm$	$\mathrm{d}(\eta u^{2i+1} v + c_{2i} v^3)$
	E^{η}_7		$\eta(u^3 + uv^3)$	$\eta = \pm 1$	$\eta\,\mathrm{d}(u^2 v + c v^4)$
triv	A^{η}_{2i}		$\eta(u^{2i+1} + v^2)$	$i \ge 1, \eta = \pm$	$\eta\,\mathrm{d}(v - uv)$
	D^{+}_{2i+2}		$u^{2i+1} + uv^2$	$i \ge 1$	$\mathrm{d}\,v$
	E^{η}_6		$\eta(u^3 + v^4)$	$\eta = \pm 1$	$\eta\,\mathrm{d}(v - uv)$
	E^{η}_8		$u^3 + \eta v^5$	$\eta = \pm$	$\mathrm{d}(v - \eta u)$
mult	D^-_{2i+2}		$u^{2i+1} - uv^2$	$i \ge 1$	$\mathrm{d}(u^{2i} v - c_{2i-1} v^3)$

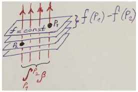
Here $c_j = (2j+1)/(3j+6)$ and $c = 5/12$.

Problem 4. Describe the topology of the space $\mathbb{F} = \mathbb{F}(f_0) = \{(f, \beta) \mid f \in \mathcal{F}(f_0)$ is an energy function of the flow $\beta\}$, endowed with C^∞-topology, and its decomposition into $\mathcal{D}^0(M)$-orbits and into classes of topological equivalence of f or β.

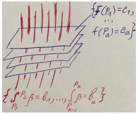
Solution to problem 4. Denote $s := \max\{0, \chi(M) + 1\}$. Let $\mathcal{D}^0_s(M)$ be the identity path component of $\mathcal{D}_s(M) = \{\phi \in \mathcal{D}(M) \mid N_s \subseteq \mathrm{Fix}(\phi)\}$, $N_s \subset M$, $|N_s| = s$.

Theorem 2 (E.K. 2012, 2016, 2021 [10, 11, 13, 14])

For any function $f_0 \in C^\infty(M)$, whose all critical points have A-D-E types (e.g. Morse), the projection $p : \mathbb{F} = \mathbb{F}(f_0) \to \mathbb{F}/\mathcal{D}^0_s(M) =: \mathcal{M} = \mathcal{M}(f_0)$ is a homotopy equivalence, where $\mathcal{M}$ is a manifold of $\dim \mathcal{M} = 2s + |\mathcal{C}_{f_0}| + |\mathcal{C}^{extr*}_{f_0}| + |\mathcal{C}^{triv}_{f_0}| + 2|\mathcal{C}^{saddle}_{f_0}| + 3|\mathcal{C}^{mult}_{f_0}|$.
Moreover:



(a) There exist two transversal fibrations on $\mathcal{M}$: $|\mathcal{C}_{f_0}|$-codimensional and $2s + |\mathcal{C}_{f_0}|$-dimensional (and induced stratifications) s.t. $\forall (f, \beta), (f_1, \beta_1) \in \mathbb{F}$
f, f_1 belong to the same $\mathcal{D}^0(M)$–orbit (resp. $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$-orbit) $\iff$ $p(f, \beta), p(f_1, \beta_1)$ belong to the same $|\mathcal{C}_{f_0}|$-codimensional fiber (resp.stratum)



β, β_1 belong to the same $\mathcal{D}^0(M)$–orbit (resp. orbital topol. equivalent) $\iff$ $p(f, \beta), p(f_1, \beta_1)$ belong to the same $2s + |\mathcal{C}_{f_0}|$-dimensional fiber (resp.stratu

(b) The projection $p : \mathbb{F} \to \mathcal{M}$ induces a homotopy equivalence between every $\mathcal{D}^0(M)$-invariant subset $I \subseteq \mathbb{F}$ and its image $p(I)$ in $\mathcal{M}$.

Proof: $\mathcal{D}^0_s(M)$ is contractible [1, 4, 5], acts freely on $\mathbb{F}$. $\Rightarrow$ $\mathbb{F} \approx \mathcal{M} \times \mathcal{D}^0_s(M)$.
Local coord.on $\mathcal{M}$: $p(f, \beta) \mapsto \left(\{f(P)\}_{P \in \mathcal{C}_f}, \left\{\int_{P_1}^{P_2} \beta\right\}_{P_i \in \mathcal{C}^{extr*}_f \cup \mathcal{C}^{triv}_f \cup \mathcal{C}^{saddle}_f \cup \mathcal{C}^{mult}_f}, \{x_j\}_{j=1}^{2s}\right)$. □

Space of functions with prescribed types of local singularities

Solution to problem 2:

Lemma 1 (E.K. and D.Permyakov 2010 [8] for Morse case)

The forgetful map $\mathrm{Forg} : \mathbb{F} \to \mathcal{F}$, $(f, \beta) \mapsto f$, *is surjective. It admits a homotopy inverse mapping* $i : \mathcal{F} \to \mathbb{F}$ *and corresponding homotopies respecting the projections* $q : \mathcal{F} \to \mathcal{F}/\mathcal{D}^0(M)$ *and* $q \circ \mathrm{Forg}$.

Proof: follows from "uniform" Morse lemma (E.K. 2009 [7] for Morse case) or from "uniform" reduction of a function near critical points to normal forms (A.Orevkova [15] for E_6, E_8 and A_μ cases), by the same arguments as in Morse case [8, Theorem 2.5 (B)]. □

Theorem 3 (E.K. 2012, 2016, 2021 [10, 11, 13, 14])

For any function $f_0 \in C^\infty(M)$, *whose all critical points have A-D-E types (e.g. a Morse function), the space* $\mathcal{F} = \mathcal{F}(f_0)$ *has the homotopy type of a manifold* $\mathcal{M} = \mathcal{M}(f_0)$ *having dimension* $\dim \mathcal{M} = 2s + |\mathcal{C}_{f_0}| + |\mathcal{C}_{f_0}^{extr*}| + |\mathcal{C}_{f_0}^{triv}| + 2|\mathcal{C}_{f_0}^{saddle}| + 3|\mathcal{C}_{f_0}^{mult}|$. *Moreover:*

(a) *There exists a surjective submersion* $\kappa = p \circ i : \mathcal{F} \to \mathcal{M}$ *and a stratification (resp.* $|\mathcal{C}_{f_0}|$*-codimensional fibration) on* $\mathcal{M}$ *such that every* $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$*-orbit (resp.* $\mathcal{D}^0(M)$*–orbit) in* $\mathcal{F}$ *is the* κ*-preimage of a stratum (resp. fiber) in* $\mathcal{M}$.

(b) *The map* κ *induces a homotopy equivalence between every* $\mathcal{D}^0(M)$*-invariant subset* $I \subseteq \mathcal{F}$ *and its image* $\kappa(I) \subseteq \mathcal{M}$. *In particular, it induces a homotopy equivalence between every* $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$*–orbit (resp.* $\mathcal{D}^0(M)$*–orbit) from* $\mathcal{F}$ *and the corresponding stratum (resp. fiber) in* $\mathcal{M}$.

In particular, $\pi_k(\mathcal{F}) \cong \pi_k(\mathcal{M})$, $H_k(\mathcal{F}) \cong H_k(\mathcal{M})$. *Thus* $H_k(\mathcal{F}) = 0$ *for all* $k > \dim \mathcal{M}$.

Solution to problem 3. Denote $\mathcal{F}^1 = \{f \in \mathcal{F} \mid f(\mathcal{C}_f^{extr}) = \pm 1, \sum_{P \in \mathcal{C}_f^{saddle} \cup \mathcal{C}_f^{triv} \cup \mathcal{C}_f^{mult}} f(P) = 0\}$.

Lemma 2 (E.K. 2021 [14])

The forgetful map $\mathrm{Forg} : \mathbb{F}^1 \to \mathcal{B}$, $(f, \beta) \mapsto \beta$, *is surjective. It admits a homotopy inverse mapping* $i : \mathcal{B} \to \mathbb{F}^1$ *and corresponding homotopies respecting the projections* $q : \mathcal{B} \to \mathcal{B}/\mathcal{D}^0(M)$ *and* $q \circ \mathrm{Forg}$. *Here* $\mathbb{F}^1 = \mathbb{F}^1(f_0) = \{(f, \beta) \in \mathbb{F} \mid f \in \mathcal{F}^1\}$.

Theorem 4 (E.K. 2021 [14])

For any gradient-like flow β_0 *on* M, *the space* $\mathcal{B} = \mathcal{B}(\beta_0)$ *has the homotopy type of the manifold* $\mathcal{M}^1 = \mathcal{M}^1(f_0) := \mathbb{F}^1/\mathcal{D}_s^0(M)$, *where* f_0 *is an energy function of* β_0. *Moreover:*

(a) *There exists a surjective submersion* $\lambda = p \circ i : \mathcal{B} \to \mathcal{M}^1$, *a stratification and a* $(|\mathcal{Z}_{\beta_0}| + 2s - 1)$*-dimensional fibration on* $\mathcal{M}^1$ *s.t. every class of orbital topological equivalence (resp.* $\mathcal{D}^0(M)$*-orbit) in* $\mathcal{B}$ *is the* λ*-preimage of a stratum (resp. fibre) in* $\mathcal{M}^1$.

(b) *The map* λ *induces a homotopy equivalence between each* $\mathcal{D}^0(M)$*-invariant set* $I \subseteq \mathcal{B}$ *and its image* $\lambda(I) \subseteq \mathcal{M}^1$. *In particular, between every class of orbital topol. equivalence (resp.* $\mathcal{D}^0(M)$*–orbit) in* $\mathcal{B}$ *and the corresponding stratum (resp. fibre) in* $\mathcal{M}^1$.

(c) *All fibres and strata in* $\mathcal{M}^1$ *(and, thus, all* $\mathcal{D}^0(M)$*–orbits and all classes of orbital topol. equivalence in* $\mathcal{B}$*) are homotopy equivalent either to a point, or to* T^2, $SO(3)/G$, *or* S^2, *in dependence on whether* $\chi(M) < 0$, $\chi(M) = 0$, $\chi(M) \cdot |\mathcal{Z}_{\beta_0}| > 0$, *or* $\chi(M) > 0 = |\mathcal{Z}_{\beta_0}|$, *respectively, where* $G \subset SO(3)$ *is a finite subgroup.*

In particuar, $\pi_k(\mathcal{B}) \cong \pi_k(\mathcal{M}^1)$, $H_k(\mathcal{B}) \cong H_k(\mathcal{M}^1)$. *Thus* $H_k(\mathcal{B}) = 0$ *for all* $k > \dim \mathcal{M}^1$.

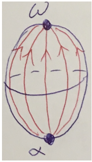
Let M be S^2 or T^2, f_0 a Morse function. Endow all functional spaces with C^∞-topology.
Denote $\mathcal{F}^1 := \{f \in \mathcal{F} \mid f(\mathcal{C}_f^{extr}) = \pm 1, \sum_{P \in \mathcal{C}_f^{saddle}} f(P) = 0\}$,
$\mathbb{F}^1 := \{(f,\beta) \mid f \in \mathcal{F}^1$ is an energy function of the flow $\beta\}$, $\qquad \mathcal{M}^1 := \mathbb{F}^1/\mathcal{D}_s^0(M)$.

Observation (E.K. 2016 [13]): $\forall (f,\beta) \in \mathbb{F}^1$, consider the natural complex structure J on M (i.e. an operator field $J: T_PM \to T_PM$, $P \in M$, such that $J^2 = -id$) defined by the conditions $J^*\beta = -\mathrm{d}\, f$ and $J^*(\mathrm{d}\, f) = \beta$.
Then $\beta^{\mathbb{C}} := \beta + i\mathrm{d}\, f$ is a meromorphic 1-form on (M,J), with simple zeros and poles, and real periods. In particular, $\mathrm{Res}_P\, \beta^{\mathbb{C}} \in i\mathbb{R}$ at each pole $P \in \mathcal{P}_\beta$. Thus,

$\mathcal{M}^1 \approx \{$real-normalized meromorphic 1-forms $(J,\beta^{\mathbb{C}})$ on M with simple poles and zeros$\}$.

Here J is a standard complex structure on $M = S^2 = \overline{\mathbb{C}}$, or $J = J_\lambda$ on $M = T^2_\lambda = \mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z})$, where $\lambda \in \mathbb{C}$, $Im\lambda > 0$.

Thm (S.Grushevsky, I.Krichever 2010 [9]): the map $\beta^{\mathbb{C}} \mapsto (\mathcal{P}_\beta, \{\mathrm{Res}_P\, \beta^{\mathbb{C}}\}_{P \in \mathcal{P}_\beta})$ is injective.

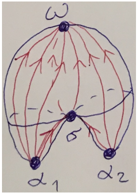


Example 1. $M = S^2$, f_0 has 2 critical points.

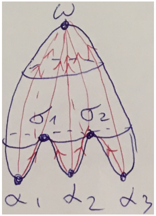
Then $\mathbb{F}^1 \sim \mathcal{M}^1 \approx \{\beta^{\mathbb{C}}_{\alpha,\omega} = \frac{i\,\mathrm{d}\,z}{z-\alpha} - \frac{i\,\mathrm{d}\,z}{z-\omega}\} \sim \{\alpha \in \overline{\mathbb{C}}\} \approx S^2$.

No zeros at all. $\quad\Rightarrow\quad$ No multiple zeros.

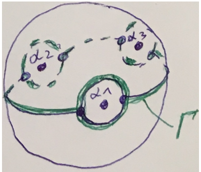
Examples 2, 3 and 4 (on a sphere)



Example 2. $M = S^2$, f_0 has 2 minima, 1 saddle, 1 maximum.
Then $\mathbb{F}^1 \sim \mathcal{M}^1 \approx \{\beta^{\mathbb{C}}_{\alpha_1,\alpha_2,\omega} = \frac{i\,dz}{z-\alpha_1} + \frac{i\,dz}{z-\alpha_2} - \frac{2i\,dz}{z-\omega}\} \approx$
$\approx \{(\alpha_1,\alpha_2,\omega) \in Q_3(S^2)\}/((\alpha_1,\alpha_2,\omega) \sim (\alpha_2,\alpha_1,\omega)) \sim$
$\sim SO(3)/\langle \mathrm{diag}(1,-1,-1)\rangle \approx L(4,1)$.
Here $L(m,n) := \{(z,w) \in \mathbb{C}\times\mathbb{C} \mid |z|^2+|w|^2 = 1\}/((z,w) \sim (e^{2\pi i/m}z, e^{2\pi in/m}w))$ is a lens space.
No multiple zeros, since $\beta^{\mathbb{C}}_{0,1,\infty} = \frac{i\,dz}{z} + \frac{i\,dz}{z-1} = i\,\frac{2z-1}{z(z-1)}\,dz$ has a simple zero at $z = \frac{1}{2}$.



Example 3. $M = S^2$, f_0 has 3 minima, 2 saddles, 1 maximum. Then
$\mathbb{F}^1 \sim \mathcal{M}^1 \approx \{\beta^{\mathbb{C}}_{\alpha_1,\alpha_2,\alpha_3,\omega} = \frac{i\,dz}{z-\alpha_1} + \frac{i\,dz}{z-\alpha_2} + \frac{i\,dz}{z-\alpha_3} - \frac{3i\,dz}{z-\omega} \mid \text{no multiple zeros}\} \sim$
$\sim \{(\alpha_1,\alpha_2,\alpha_3,\omega) \in Q_4(S^2) \mid \omega \in \Gamma_{\alpha_1,\alpha_2,\alpha_3}\}/((\alpha_1,\alpha_2,\alpha_3,\omega) \sim (\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\omega))$
$\sim \bigcup_{\phi\in SO(3)} (\phi\{\alpha_1^\circ,\alpha_2^\circ,\alpha_3^\circ\}) \times (\phi(\Gamma_{\alpha_1^\circ,\alpha_2^\circ,\alpha_3^\circ})) \subset Q_3(S^2)\times S^2$.
Here $\beta^{\mathbb{C}}_{0,1,\infty,\omega} = \frac{i\,dz}{z} + \frac{i\,dz}{z-1} - \frac{3i\,dz}{z-\omega} = i\,\frac{-z^2+2(1-\omega)z+\omega}{z(z-1)(z-\omega)}\,dz$ has a multiple zero $\iff$ $(\omega-1)^2+\omega = 0$, i.e. $\omega = \frac{1\pm i\sqrt{3}}{2}$. No multiple zeros $\iff$ $\omega \in \overline{\mathbb{C}} \setminus \{0,1,\infty,\frac{1\pm i\sqrt{3}}{2}\}$.



Here $\Gamma_{\alpha_1,\alpha_2,\alpha_3} \subset S^2$ is a chord diagram (see Fig.) corresponding to a 3-point configuration $\{\alpha_1,\alpha_2,\alpha_3\} \in Q_3(S^2)$, $\{\alpha_1^\circ,\alpha_2^\circ,\alpha_3^\circ\} = \{0,1,\infty\}$.
The camera $\mathrm{Forg}^{-1}[f] \sim SO(3)\times[0,1]/((\phi,0) \sim (\phi\,\mathrm{diag}(1,-1,-1),1))$,
the wall $\mathrm{Forg}^{-1}[g] \sim SO(3)/\langle\mathrm{diag}(1,-1,-1)\rangle \approx L(4,1)$.

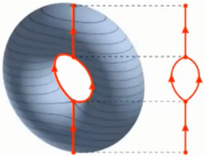
Example 4. $M = S^2$, f_0 has 3 minima, 1 multi-saddle D_4^-, 1 maximum.
$\mathbb{F}^1 \sim \mathcal{M}^1 \approx \{\beta^{\mathbb{C}}_{\alpha_1,\alpha_2,\alpha_3,\omega} \mid \text{multiple zero}\} \approx SO(3)/S_3 \approx L(6,1)$.

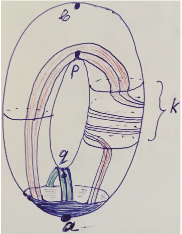
Example 5 (on a torus)

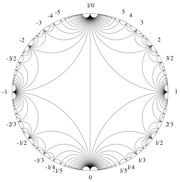
Example 5. $M = T^2$, f_0 has 4 critical points.

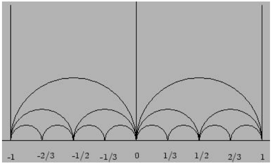
If $f \in \mathcal{F}^1$ is a strong Morse function (has 4 critical values), then the preimage of its $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$-orbit $[f]$ under the projection $\mathrm{Forg} : \mathbb{F}^1 \to \mathcal{F}^1$ ("blue camera" $\mathrm{Forg}^{-1}[f] \sim T^2 \times S^1$) is open in $\mathbb{F}^1$.

If $g \in \mathcal{F}^1$ is a non-strong Morse function (has 3 critical values), then the preimage of its $\mathcal{D}^0(\mathbb{R}) \times \mathcal{D}^0(M)$-orbit $[g]$ under the projection $\mathrm{Forg} : \mathbb{F}^1 \to \mathcal{F}^1$ (a "blue wall" $\mathrm{Forg}^{-1}[g] \sim T^2$) has codimension 1 in $\mathbb{F}^1$.









Assign to each camera $\mathrm{Forg}^{-1}[f] \mapsto \pm[e_q^1]$, a primitive cycle in $H_1(T^2)$. It is a bijection.

Then $\mathbb{F}^1 \sim T^2 \times F^\circ$ with decomposition into cameras and walls. Here cameras $\mathrm{Forg}^{-1}[f]$ correspond to vertices of the Farey graph F, walls $\mathrm{Forg}^{-1}[g]$ correspond to edges of F. Here F° is a graph obtained from the Farey graph F by attaching a loop to each vertex.

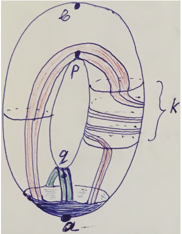
Example 5 (addition)

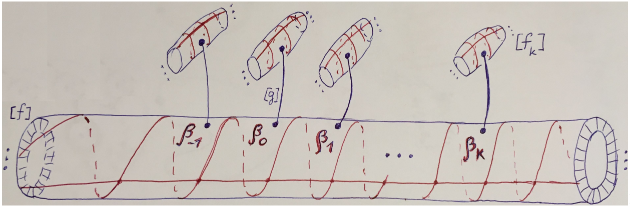
Recall that each blue camera $\mathrm{Forg}^{-1}[f]$ is fibred (resp. stratified) by red fibres (resp. red strata), that surve as equivalence classes of flows in the blue camera.

- How is a blue camera $\mathrm{Forg}^{-1}[f]$ stratified by red strata (i.e. by orbital topological equivalence classes of flows)?

  Answer: the blue camera $\mathrm{Forg}^{-1}[f] \approx (0,1) \times (\mathbb{R}^2/(1,-1)\mathbb{Z}) \times T^2$ is divided by **red walls** $(0,1) \times (\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}) \times T^2$ (represented by β having a separatrix connecting p to q) into **red cameras** $(0,1) \times (0,1) \times (k, k+1) \times T^2$ marked by β_k.
- How neighbour blue cameras are attached to each other along a blue wall?

  Answer: blue camera $\mathrm{Forg}^{-1}[f]$ is attached along its **blue wall** $\mathrm{Forg}^{-1}[g_k] \approx \{0\} \times (0,1) \times (k, k+1) \times T^2$ to $\mathrm{Forg}^{-1}[f_k]$.



$\mathcal{M}^1 =$ 

$\times T^2$

To see a whole red camera = open red stratum (representing a whole class of orbital topological equivalence of Morse-Smale flows, i.e. structurally stable flows), we should take a red camera ignoring blue walls. So, we should take two red strata from two neighbour blue cameras and glue them together along their common blue wall.

Корректность определения градиентоподобного потока в неморсовском случае

Проверим корректность определения градиентоподобного потока, т.е. совместность условий (i)–(iii). Для этого нужно проверить, что если f и β имеют вид, указанный в таблице, то выполнены свойства (i) и (ii). Свойство (i) очевидно. Для проверки свойства (ii) вычислим 2-форму $d f \wedge \beta$ для основных случаев таблицы:

$$d(u^{2i}+v^2)\wedge\frac{u\,d v - v\,d u}{u^2+v^2} = 2\frac{iu^{2i}+v^2}{u^2+v^2}\omega,$$

$$d(u^{2i+2}-v^2)\wedge d(uv) = 2((i+1)u^{2i+2}+v^2)\omega,$$

$$d(u^{i+1}+\eta uv^j)\wedge d(u^i v+\eta c_{i,j}v^{j+1}) = (((i+1)u^i+\eta v^j)(u^i+\eta(j+1)c_{i,j}v^j)-\eta iju^i v^j)\omega =$$
$$= ((i+1)u^{2i}+(j+1)c_{i,j}v^{2j})\omega,$$

$$d(u^{2i+1}+v^2)\wedge d(v-uv) = ((2i+1)u^{2i}(1-u)+2v^2)\omega,$$

$$d(u^{2i+1}+uv^2)\wedge d v = ((2i+1)u^{2i}+v^2)\omega,$$

$$d(u^3+v^4)\wedge d(v-uv) = (3u^2(1-u)+4v^4)\omega,$$

$$d(u^3+\eta v^5)\wedge d(v-\eta u) = (3u^2+5v^4)\omega,$$

где обозначено $\omega = d u\wedge d v$, $c_{i,j} = \frac{ij-1}{(i+1)(j+1)}$. Тогда $c_{j+1,2} = \frac{2j+1}{3j+6} = c_j$ и $c_{2,3} = \frac{5}{12} = c$.

Мы получили, что в проколотой окрестности начала координат множитель при ω положителен. Значит, 2-форма $d f\wedge\beta$ задает положительную ориентацию на M, что и доказывает (ii). Таким образом, условия (i)–(iii) совместны. □

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