Dynamics and Multivalued Groups

Mikhail Kornev

Steklov Mathematical Institute of RAS

SIMC Youth Race
March 13–17, 2023

Introduction

- In different areas of research, multivalued products on spaces appear
- The literature on multivalued groups and their applications is large and includes articles since XIX century mostly in the context of hypergroups
- In 1971, S. P. Novikov and V. M. Buchstaber gave the construction, predicted by characteristic classes. This construction describes a multiplication, with a product of any pair of elements being a non-ordered multiset of n points

Introduction

- It led to the notion of n-valued groups which was given axiomatically and developed by V. M. Buchstaber
- At present, a number of authors are developing n-valued (finite, discrete, topological or algebra geometric) group theory together with applications in various areas of Mathematics and Mathematical Physics

Introduction

- Since 1996, V. M. Buchstaber and A. P. Veselov and became develop some applications of n-valued group theory to discrete dynamical systems
- In 2010, V. Dragović showed the associativity equation for 2-valued group explains the Kovalevskaya top integrability mechanism

Introduction

We will talk about

- Symbolic Dynamics
- Tiling theory
- Multivalued Group theory
- their connections and some author's results

Combinatorics on Words Preliminaries

- *Alphabet* A is a finite set, consisting of letters
- A^* stands for the *monoid of finite words* in an alphabet A
- A^ω stands for the set of *right infinite words*
- A word $w \in A^\omega$ is *periodic* if it is of the form $w = uvvv\ldots$ for some $u, v \in A^*$
- A word $w \in A^\omega$ is *aperiodic* (or, *quasi-periodic*) if it is not periodic
- *Factor* is a finite continuous subword u in $w = \ldots u \ldots$
- Denote by $|w|$ the length of a word $w \in A^*$

Combinatorics on Words Preliminaries

- Let A and B be alhabets. A *morphism* is a map $\mathcal{F} : A^* \to B^*$ satisfying
$$\mathcal{F}(xy) = \mathcal{F}(x)\,\mathcal{F}(y)$$
for all words $x, y \in A^*$, i. e., $\mathcal{F}$ is a homomorphism of monoids
- A morphism is defined by the images $\mathcal{F}(a)$ of the letters $a \in A$

Combinatorics on Words Preliminaries

- In some cases, one can define a limit

$$a \to \mathcal{F}(a) \to \mathcal{F}(\mathcal{F}(a)) \to \ldots \to \mathcal{F}^{\infty}(a)$$

- It is easy to see that the word $w = \mathcal{F}^{\infty}(a)$ will be *a fixed point*, i. e., $\mathcal{F}(w) = w$

Examples of Morphisms

Example (Fibonacci Morphism)

$$\mathcal{F} : \{0,1\}^* \to \{0,1\}^*,\ 0 \mapsto 01,\ 1 \mapsto 0$$

The *infinite Fibonacci word* $\Phi := \mathcal{F}^\infty(0)$ is

$$\Phi = 0100101001001010010100100101001\ldots$$

Examples of Morphisms

Example (Thue-Morse Morphism)

$$\mathcal{F} : \{0,1\}^* \to \{0,1\}^*,\ 0 \mapsto 01,\ 1 \mapsto 10$$

The *Thue-Morse sequence* $\mathcal{F}^\infty(0)$ is

$$T = 01101001100101101001011001101001\ldots$$

Examples of Morphisms

Example (Tribonacci Morphism)

$\mathcal{F} : \{a, b, c\}^* \to \{a, b, c\}^*$

$$\mathcal{F} : \begin{cases} a \mapsto abc, \\ b \mapsto ac, \\ c \mapsto b \end{cases}$$

The *infinite tribonacci word* $\mathcal{F}^\infty(a)$ is

$$abcacbabcbacabcacbacabcb\ldots$$

The Factor Complexity

- The *factor complexity* of an infinity word w is the function $f_w(n)$ defined as the number of its factors of length n
- One can show that for an infinite word w there exists $C \in \mathbb{N}$ such that
$$f_w(n) \leqslant C$$
for evey $n \in \mathbb{N}$

The Factor Complexity

Theorem (M. Morse and G. Hedlund, 1940)

Let w be an aperiodic infinite word. Then for any $n \in \mathbb{N}$

$$f_w(n) \geqslant n + 1$$

Definition

In the case of equality $f_w(n) = n + 1$, a word w is called *Sturmian*

Some easy properties:

- $f_w(n) \leqslant |A|^n$ where A is an alphabet
- $f_w(n)$ is non-decreasing function

Once Again: The Fibonacci Word

- Consider the Fibonacci word constructed above

$$\Phi = 0100101001001010010100100101001001010100\ldots$$

- There is another way to construct Φ
- Consider the following recursive sequence $\{\Phi_k\}$ of *finite Fibonacci words*

$$\Phi_{k+1} = \Phi_k\Phi_{k-1}, \text{ where } \Phi_0 = 0, \; \Phi_1 = 01$$

- $\{|\Phi_k|\}$ is the *Fibonacci sequence*:

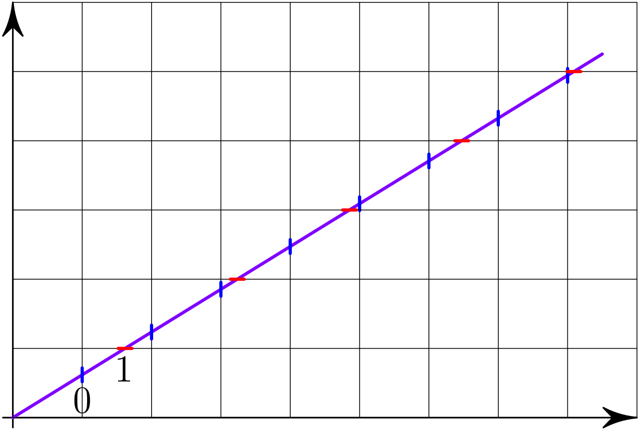
$$|\Phi_k| = F_{k+2}, \; F_{k+2} = F_{k+1} + F_k, \; F_0 = 0, \; F_1 = 1$$

- In this setting $\Phi = \lim_n \Phi_n$

$$\Phi_2 = 010$$
$$\Phi_3 = 01001$$
$$\Phi_4 = 01001010$$

The Fibonacci Word is Sturmian

- It turns out that the Fibonacci word is Sturmian
- It follows from the geometric interpretation of Sturmian words



$$y(x) = \psi x, \;\; \psi = 1/\varphi, \, \varphi = (1+\sqrt{5})/2$$
$$\Phi_5 = 0100101001001$$

Some Properties of the Fibonacci Word

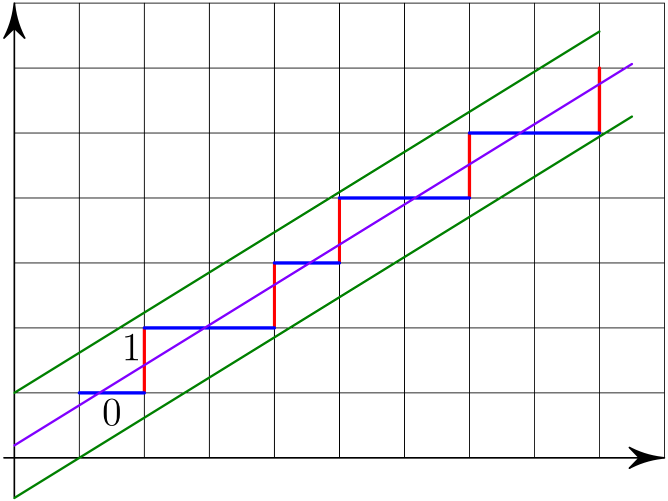
- The factors 11 and 000 are absent in Φ
- The last two letters of a Fibonacci word are alternately 01 and 10
- The nth digit of Φ is

$$2 + \lfloor n\varphi \rfloor - \lfloor (n+1)\,\varphi \rfloor\,,$$

where $\varphi = (1+\sqrt{5})/2$ is the golden rartio

The Fibonacci Word and Quasi-Quasicrystals

Cut-and-projection method gives



$$y(x) = \psi x + \frac{1-\psi}{2},\ \ \psi = 1/\varphi,\ \varphi = (1+\sqrt{5})/2$$

Balanced Words

Definition

An infinity word w in the alphabet $\{a, b\}$ is called *balanced* if for any two factors u and v of the same length n

$$||u|_a - |v|_b| = 1$$

where $|-|_a$ denotes the number of letters a (the Hamming weight).

- The Fibonacci word is an example of balanced word

$$\Phi = 0100101001001010010100100101001001010100\ldots$$

- For the Thue-Morse word, however, it is not the case: see, e. g., 00 and 11

$$T = 0110100110010110100101100110100 1\ldots$$

Geometric Words

Definition

An infinite word in two-letter alphabet is called *geometric* if it encodes intersections of a fixed line $y = \alpha x + \rho$ with vertical and horizontal lines of integer lattice

- If α is rational the dynamics is periodic
- If α is irrational the one is qusi-periodic

Sturmian Words are Geometric

Corollary

For an infinite word in 2-letter alphabet the following conditions are equivalent

- $f_w(n) = n + 1$
- w *is aperiodic and balanced*

Markov's Result

Theorem (A. A. Markov, 1882)

Let $\alpha = [0; a_1, a_2, \ldots]$ *be the continued fraction expansion,* $\alpha \in (0, 1)$. *Then the word* $S(\alpha)$ *encoded by a line* $y = \alpha x$ *can be written as follows*

$$S(\alpha) = \lim_k S_k(\alpha)$$

where

$$S_k = S_{k-1}^{a_k} S_{k-2}$$

with the initial conditions $S_{-1} = b$ *и* $S_0 = a$. *The letters* a *and* b *correspond to vertical and horizontal intersections respectively*

For the word length sequence $\{|S_k|\}$ we have $|S_{-1}| = 1$, $|S_0| = 1$ and

$$|S_k| = a_k |S_{k-1}| + |S_{k-2}|$$

Markov's Result

Example

- Consider the line $y = \psi x$ where $\psi = 1/\varphi$, $\varphi = (1 + \sqrt{5})/2$

$$\psi = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

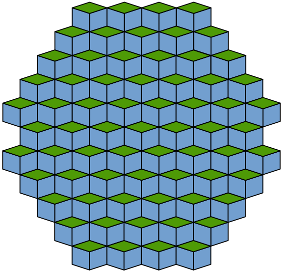
- In this case, $S_n = S_{n-1}S_{n-2}$ — the Fibonacci word

Tilings

Definition

A simple tiling of $\mathbb{R}^d$:

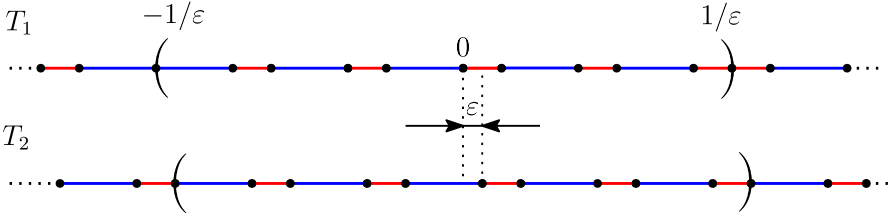
- There are only a finite number of tile types, up to translation
- Each tile is a polytope
- Tiles meet full-facet to full-facet



The ε-closeness

Definition

We say that tilings T_1 and T_2 are ε-close if they are agree on a ball of radius $1/\varepsilon$ around the origin, up to translation of size ε or less



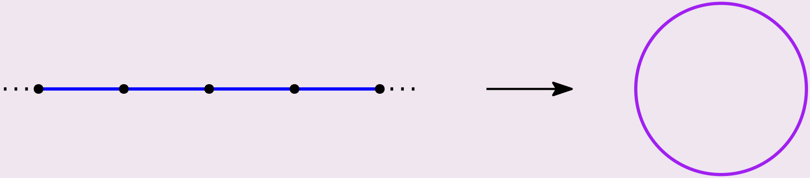
Tiling Spaces

Definition

- The *orbit* of a tiling T is the set $\mathcal{O}(T) := \{T - x \mid x \in \mathbb{R}^d\}$ of translates of T
- A *tiling space* Ω is a set that is closed under translation, and complete in the tiling metric
- The *hull* Ω_T of a tiling T is the closure of $\mathcal{O}(T)$ with respect to the ε-closure property

Example

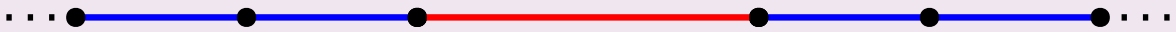
- Consider a simple 1-dimensional tilling T_0 with just one kind of tile. Suppose its length is 1 and its color is blue
- Obviously, $T_0 = T_0 - 1$. So, Ω_{T_0} is a circle



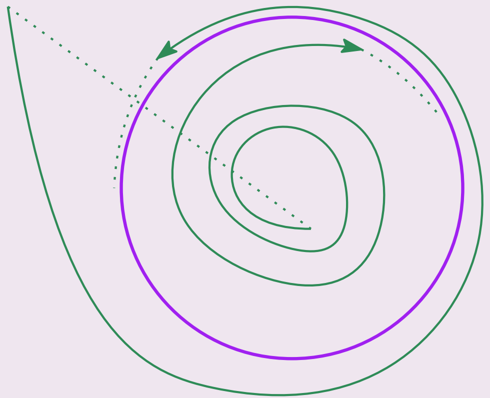
Tiling Spaces

Example

- Consider an 1-dimensional tilling T_1 with one red tile of length 2 and other blue tiles of length 1
- Any tiling with one red tile is in $\mathcal{O}(T_1)$, and hence in Ω_{T_1}
- Tilings with no red tiles are also in Ω_{T_1} by simple reasons
- So, Ω_{T_1} looks like the circle Ω_{T_0} and the line $\mathcal{O}(T_1)$ with both ends of the line asymptotically approaching the circle



Tiling Spaces



Tiling Spaces

Theorem

If T is a simple tiling then Ω_T is compact

- For a tiling T one can approximate the space Ω_T via CW complexes Γ_n from the *Gähler's construction*
- There is a sequence of forgetful maps $f_n : \Gamma_{n+1} \to \Gamma_n$. The space Γ_n knows about surrounding n layers in some sence
- Hence, one can form an inverse limit and it will homeomorphic to Ω_T

$$\Omega_T = \varprojlim \Gamma_n$$

- In the case of substitution tilings, it is more convenient to use the *Anderson-Putnam construction* of $\Gamma_n' s$

Topological Invariants of Tiling Spaces

- Ω_T has one connected component, but uncountably many path-component
- Each path component in a tiling space is an orbit under $\mathbb{R}^d$. Such an orbit of an aperiodic tiling is contractible, so $\pi_n(\Omega_T) = 0$ and $H_n(\Omega_n; A) = 0$ for $n > 0$, A is abelian
- Čech cohomology does better

$$\check{H}^* \left(\varprojlim \Gamma_n \right) \cong \varinjlim \check{H}^*(\Gamma_n) \cong \varinjlim H^*(\Gamma_n)$$

Example

$\check{H}^1$ of the Fibonacci tiling space is $\mathbb{Z} \oplus \varphi\mathbb{Z}$, $\varphi = (1 + \sqrt{5})/2$

Prodefinition of n-valued Groups

Prodefinition

A *hypergroup* is a promonoidal category structure on a discrete poset X, whose promultiplication $X \times X \to \mathcal{G}(X)$ takes values in the 2-category of non-empty groupoids, with some additional groupal properties

Recall, a promonoidal category is a category $\mathcal{C}$ together with

- A profunctor (promultiplication) $\mathcal{C} \times \mathcal{C} \twoheadrightarrow \mathcal{C}$
- A profunctor (prounit) $J : 1 \twoheadrightarrow \mathcal{C}$
- Associativity $P \circ (P \times 1) \stackrel{\sim}{=} P \circ (1 \times P)$
- Unit isomorphisms $P \circ (J \times 1) \stackrel{\sim}{=} 1$, and $P \circ (1 \times J) \stackrel{\sim}{=} 1$

A fancy arrow $A \twoheadrightarrow B$ means a functor $B^{\mathrm{op}} \times A \to \mathrm{Set}$. The composition of $F : A \twoheadrightarrow B$ and $G : B \twoheadrightarrow C$ is defined to be

$$(G \circ F)(c, a) = \int^{a \in A} F(b, a) \otimes G(c, b)$$

Symmetric Powers of a Space

- For a topological space X, let $(X)^n$ denote its n-fold symmetric power, i. e., $(X)^n = X^n/\Sigma_n$ where the symmetric group Σ_n acts by permuting the coordinates
- An element of $(X)^n$ is called an n-subset of X or just an n-set. It is a subset with multiplicities of total cardinality n

Example

The spaces $(\mathbb{C})^n = \mathbb{C}^n/\Sigma_n$ and $\mathbb{C}^n$ are identified using the map $\mathcal{S} : \mathbb{C}^n \to \mathbb{C}^n$ whose components are given by

$$(z_1, z_2, \dots, z_n) \to \sigma_r(z_1, z_2, \dots, z_n),\ 1 \leqslant r \leqslant n,$$

where σ_r is the r-th elementary symmetric polynomial

n-valued Group Structure

An *n-valued multiplication* on X is a map

$$\mu : X \times X \to (X)^n \; : \; \mu(x, y) = x * y = [z_1, z_2, \dots, z_n], \; z_k = (x * y)_k$$

- *Associativity.* The n^2-sets

$$[x * (y * z)_1, x * (y * z)_2, \dots, x * (y * z)_n],$$
$$[(x * y)_1 * z, (x * y)_2 * z, \dots, (x * y)_n * z]$$

  are equal for all $x, y, z \in X$
- *Unit.* $e \in X$ such that $e * x = x * e = [x, x, \dots, x]$ for all $x \in X$
- *Inverse.* A map inv: $X \to X$ such that

$$e \in \mathrm{inv}(x) * x \text{ and } e \in x * \mathrm{inv}(x) \text{ for all } x \in X$$

The map μ defines an n-valued group structure on X if it is associative, has a unit and an inverse

Example: 2-valued Group Structure on $\mathbb{Z}_+$

- Consider the semigroup of nonnegative integers $\mathbb{Z}_+$
- Define the multiplication $\mu\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2$ by the formula $x * y = [x + y, |x - y|]$
- *The unit:* $e = 0$
- *The inverse:* $\mathrm{inv}(x) = x$.
- *The associativity:* one has to verify that the 4-subsets of $\mathbb{Z}_+$

$$[x + y + z, |x - y - z|, x + |y - z|, |x - |y - z||]$$

and

$$[x + y + z, |x + y - z|, |x - y| + z, ||x - y| - z|]$$

are equal for all nonnegative integers x, y, z

Example: n-valued Group Structure on $\mathbb{C}$

- Define the multiplication $\mu\colon \mathbb{C} \times \mathbb{C} \to (\mathbb{C})^n$ by the formula

$$x * y = [(\sqrt[n]{x} + \varepsilon^r \sqrt[n]{y})^n, \quad 1 \leqslant r \leqslant n],$$

  where $\varepsilon \in \mathbb{Z}_n$ is a primitive nth root of unity
- *The unit:* $e = 0$
- *The inverse:* $\mathrm{inv}(x) = (-1)^n x$
- The multiplication is described by the polynomial equations

$$p_n(x, y, z) = \prod_{k=1}^{n} \big(z - (x * y)_k\big) = 0$$

  For instance,

$$p_1 = z - x - y, \quad p_2 = (z + x + y)^2 - 4(xy + yz + zx),$$
$$p_3 = (z - x - y)^3 - 27xyz$$

Homomorphisms of n-valued Groups

Definition

A map $f : X \to Y$ is called *a homomorphism of n-valued groups* if

- $f(e_X) = e_Y$
- $f(\mathrm{inv}_X(x)) = \mathrm{inv}_Y(f(x))$ for all $x \in X$
- $\mu_Y(f(x), f(y)) = (f)^n \mu_X(x, y)$ for all $x, y \in X$

So, the class of all n-valued groups forms a category **MultValGrp**

Reducible n-valued Groups

- For each $m \in \mathbb{N}$, an n-valued group on X, with some multiplication μ, can be regarded as an mn-valued group by using as the multiplication the composition

$$X \times X \xrightarrow{\mu} (X)^n \xrightarrow{(D)^m} (X)^{mn}, \quad \text{where } D \text{ is diagonal}$$

Definition

An n-valued group on X is called *reducible* if there is an isomorphism $f : X \to Y$ where Y is an n-valued group with a multiplication $\mu_n = \mu_k^m$, $n = mk$

Kernels and Images

Lemma

Let $f : X \to Y$ be a homomorphism of n-valued groups. Then

- $\ker(f) = \{x \in X \mid f(x) = e_Y\}$ *is an n-valued group*
- $f(x_1) = f(x_2) \Leftrightarrow (f)^n(zx_1) = (f)^n(zx_2)$ *for all* $z \in \ker(f)$
- *Suppose that the map* $\mathrm{inv} : X \to X$ *is uniquely defined. Then* $\ker(f) = \{e\}$ *if and only if any equality* $f(x_1) = f(x_2)$ *implies* $x_1 = x_2$
- $\mathrm{Im}(f) = \{y \in Y \mid y = f(x), x \in X\}$ *is an n-valued group*

Coset Groups

- Let G be a (1-valued) group with the multiplication μ_0, the unit e_G, and $\mathrm{inv}_G(u) = u^{-1}$
- Let $A \hookrightarrow \mathrm{Aut}G$ be a finite group of order n
- Denote by X the quotient space G/A of G, and denote by $\pi : G \to X$ the quotient map
- Define the n-valued multiplication $\mu : X \times X \to (X)^n$ by the formula

$$\mu(x, y) = [\pi(\mu_0(u, v^a)) \mid a \in A]$$

  where $u \in \pi^{-1}(x),\ v \in \pi^{-1}(y)$ and v^a is the image of the action of $a \in A$ on $v \in G$

Theorem

The multiplication μ defines some n-valued coset group structure (G, A) with the unit $e_X = \pi(e_G)$ and the non-ambiguity defined map $\mathrm{inv}(u) = \pi(u^{-1})$ where $\pi(u) = x$

Example

- Consider $G = \{a, b \mid a^2 = b^2 = e\}$
- The interchange of a and b is an element of order 2 of Aut$G$
- Then we have on the set $X = G/A = \{u_{2n}, u_{2n+1}\}$, $n \geqslant 0$ where

$$u_{2n} = [(ab)^n, (ba)^n], \;\; u_{2n+1} = [a(ba)^n, b(ab)^n]$$

- The multiplication:

$$u_k * u_\ell = [u_{k+\ell}, u_{|k-\ell|}]$$

- Thus, X is isomorphic to the 2-valued group on $\mathbb{Z}_+$ constructed above

n-valued Dynamics

Definition

An *n-valued dynamics* T on a space Y is a map $T : Y \to (Y)^n$

- If Y is a state space then the n-valued dynamics T defines possible states $T(y) = [y_1, \dots, y_n]$ at the moment $(t+1)$ as a state function of y at the moment t

Example

1. Consider
   $F(x, y) = b_0(x)y^n + b_1(x)y^{n-1} + \cdots + b_n(x),\ x, y \in \mathbb{C}.$
2. The equation $F(x, y) = 0$ defines an n-valued dynamics

$$T \colon \mathbb{C} \to (\mathbb{C})^n \ : \ T(x) = [y_1, \dots, y_n]$$

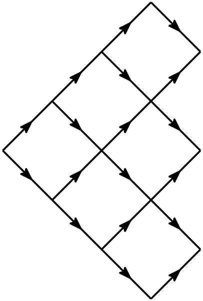
where $[y_1, \dots, y_n]$ — n-set of roots of $F(x, y) = 0$

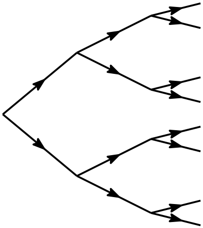
n-valued Growth Function

- Let $T\colon Y \to (Y)^n$ be an n-valued dynamics. For any $y \in Y$ define the n-valued growth function $\xi_y\colon \mathbb{N} \to \mathbb{N}$ where $\xi_y(k)$ — the number of different points in the set $T^k(y)$

Problem

Characterize such n-valued dynamics T that functions $\xi_y(k)$ have polynomial growth for any $y \in Y$


polynomial growth


exponential growth

n-valued Actions

An action of n-valued group X on a space Y is defined by the map

$$\varphi\colon X \times Y \to (Y)^n \ : \ \varphi(x, y) = x \cdot y = [y_1, \dots, y_n]$$

such that

- for any $x_1, x_2 \in X$ and $y \in Y$ the following n^2-sets coincide:

  $x_1 \cdot (x_2 \cdot y) = [x_1 \cdot y_1, \dots, x_1 \cdot y_n]$ and $(x_1 x_2) \cdot y = [z_1 \cdot y, \dots, z_n \cdot y]$

  where $x_2 \cdot y = [y_1, \dots, y_n]$ и $x_1 x_2 = [z_1, \dots, z_n]$
- $e \cdot y = [y, \dots, y]$ for any $y \in Y$

n-valued Cyclic Dynamics

Definition

An n-valued group $X := \langle x \rangle$ is called *cyclic* if it is generated by the only element $x \in X$

Definition

Consider n-valued dynamics $T \colon Y \to (Y)^n$ with $X = \langle a \rangle$. The generator a is called the *generator of the cyclic dynamics T*

Theorem (A. A. Gaifullin, P. V. Yagodovskii, 2007)

An n-valued dynamics T has a generator $a \in X$ if and only if there exists such a dynamics $T^{-1} \colon Y \to (Y)^n$ that for any y_1, $y_2 \in Y$ the multiplicity of y_2 in $T(y_1)$ equals the multiplicity of y_1 in $T^{-1}(y_2)$

n-valued Cyclic Group Growth Problem

- Let $X = \langle a \rangle$ be a cyclic n-valued group
- Then there is the left action of X on itself

$$T : X \to (X)^n, \;\; T(x) = a \cdot x$$

- Recall $\xi_a(k)$ is a number of different elements in $T^k(a)$

Notation

Denote by $\mathbb{G}_\varphi(G)$ the n-valued group obtained from the construction above for some ordinary group G and some automorphism group element φ

The Case of $\mathbb{Z}/3 * \mathbb{Z}/3$ with $\mathbb{Z}/2 < \mathrm{Aut}$

Proposition

For the group $\mathbb{Z}/3 * \mathbb{Z}/3 = \langle a, b \mid a^3 = b^3 = 1 \rangle$ *and the automorphism* $\varphi : a \mapsto b$ *the corresponding 2-valued group* $\mathbb{G}_\varphi(\mathbb{Z}/3 * \mathbb{Z}/3)$ *has the growth function*

$$\xi_{[a,b]}(k) = F_{k+3} - 1 = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{k+3} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+3} \right) - 1.$$

In particular, the growth is exponential:

$$\xi_{[a,b]}(k) \sim \frac{\varphi^{k+3}}{\sqrt{5}}$$

where $k \to \infty$ *and* $\varphi = (1+\sqrt{5})/2$.

n-bonacci Sequence

Definition

The n-bonacci sequence $\{F_k^{(n)}\}$ is defined recursively as follows:

$$F_k^{(n)} = F_{k-1}^{(n)} + \ldots + F_{k-n}^{(n)},$$

initial conditions are $F_0 = \ldots = F_{n-2} = 0$ и $F_{n-1} = 1$.

Example

Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Tribonacci sequence:

$$0, 0, 1, 1, 2, 4, 7, 13, 24, \ldots$$

The Case of $\mathbb{Z}/m * \mathbb{Z}/m$ with $\mathbb{Z}/2 < \mathrm{Aut}$

Proposition

The number S_k of new words, appearing on the step k, equals

$$S_k = F^{(m-1)}_{k+m-2}$$

when $k \geqslant -(m-2)$.

The Case of $\mathbb{Z}/m * \mathbb{Z}/m$ with $\mathbb{Z}/2 < \mathrm{Aut}$

Proposition (M. K.)

For the group $\mathbb{Z}/m * \mathbb{Z}/m = \langle a, b \mid a^m = b^m = 1 \rangle$, $m \geqslant 3$ *with the automorphim* $\varphi : a \mapsto b$ *we have*

$$\xi_{[a,b]}(k) \sim \frac{r^{k+1}}{mr - 2(m-1)}$$

where $k \to \infty$ *and* r *is the positive root of the polinomial* $\chi(\lambda) = \lambda^n - \lambda^{n-1} - \ldots - 1$. *In particular,* $\mathbb{G}_\varphi(\mathbb{Z}/m * \mathbb{Z}/m)$ *has the polinomial growth if and only if* $m = 2$

The Case of $(\mathbb{Z}/2)^{*s}$ with $\mathbb{Z}/s < \mathrm{Aut}$

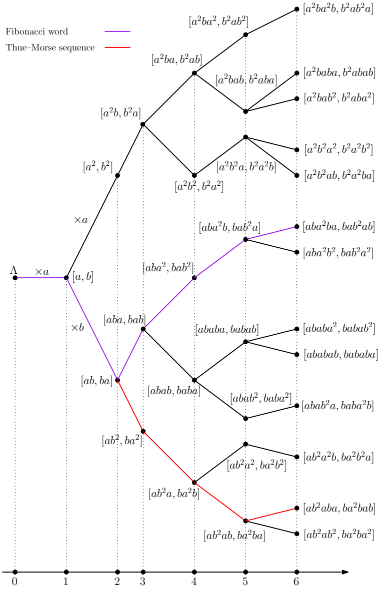
Proposition

For the group $(\mathbb{Z}/2)^{*s} = \langle a_1, \dots, a_s \mid a_1^2 = \dots = a_s^2 = 1 \rangle$ *with the automorphism* $a_i \mapsto a_{i+1}$ *(indices move modulo* s*) we have the* s*-valued group with the growth*

$$\xi_{[a_1,\dots,a_s]}(k) = \begin{cases} \dfrac{(s-1)^k - 1}{s-2} + 1, & s \geqslant 3 \\ k+1, & s = 2 \end{cases}$$

In particular, the growth is polynomial if and only if $s = 2$

$\mathbb{Z}/3 * \mathbb{Z}/3$ and Symbolic Dynamics



$\mathbb{Z}/3 * \mathbb{Z}/3$ and Symbolic Dynamics

An algorithm construction of a directed tree Γ, as vertices having the elements of 2-valued group $\mathbb{G}$:

0. We start with the vertex, corresponding to the empty set Λ — the root of our tree
1. Add the vertex $[a, b]$ adjacent to the root
2. Add two edges to the last vertex: each of them corresponds to an addition a letter (a or b) on the right hand side. Now we have two words of length 2: $[a^2, b^2]$ and $[ab, ba]$

$\mathbb{Z}/3 * \mathbb{Z}/3$ and Symbolic Dynamics

Definition

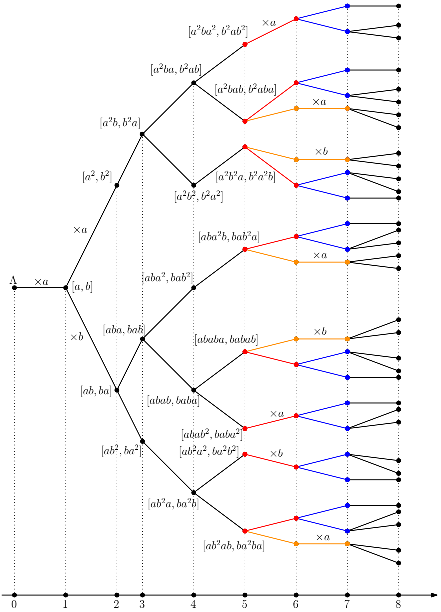
We say that a word is *cube-free* (it doesn't agree with the common use) if any word in the (natural) normal form of the group $\mathbb{Z}/3 * \mathbb{Z}/3 = \langle a, b \mid a^3 = b^3 = 1 \rangle$

4. On the step k we start with all cube-free words of length $k - 1$ and add for each vertex 1 or 2 edges according to the principle:
   - If a word ends with the first power of a letter then we will add 2 edges, corresponding to the multiplications with a and b
   - If a word ends with the square of a letter then we will add exactly one edge, corresponding to the remaining letter
   - The edge, corresponding to the multiplication with a, lies higher than the other one

Properties of Γ

- On the level k of the tree Γ top down, all cube-free words of length k place in lexicographic ascending order and their number is F_{k+1}. Using the binary notation $a \leftrightarrow 0$, $b \leftrightarrow 1$, this order coincides with the natural order on the binary numbers
- If one picts, down to top, the vertex having the number F_k on each k-level of Γ then the resulting vertex sequence will form the route $ab(aab)$ in Γ

Properties of Γ



Properties of Γ

The latter can be formulated more generally in the following

Proposition (M. K.)

For an infinite cube-free word Ψ*, consider the factor sequence* $\{\Theta_k\}$ *of the form*

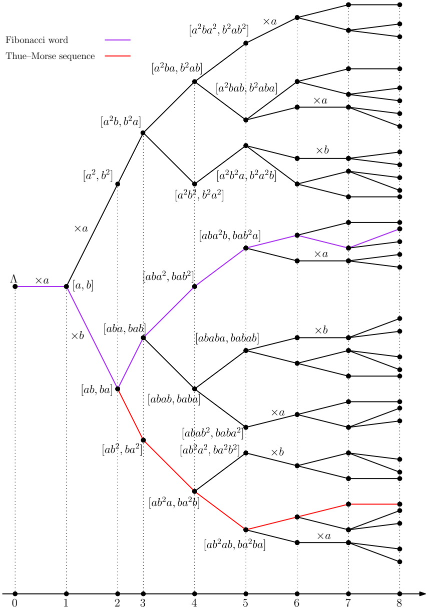
$$\Psi aabaabaab\ldots = \Psi(aab)$$

$$\Theta_1 = \Psi,\ \Theta_2 = \Psi a,\ \Theta_3 = \Psi aa,\ \Theta_4 = \Psi aab,\ \Theta_5 = \Psi aaba,\ \ldots$$

where the last letter of pre-period word Ψ *differs from* a*. Then the number* Q_k *of cube-free words satisfies the recursive equality, with words being grater or equal* Θ_k *lexicographically:*

$$Q_k = Q_{k-1} + Q_{k-2}.$$

Properties of Γ



Conclusion

- This construction of the tree might give some fruitful intuition about quasi-periodic words
- At present, there are gaps in the n-valued-group growth study
- The items above will be the subjects of further study

Thank you!