Orders on Free Metabelian Groups

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Orderable Groups

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In this talk, all groups are countable groups.

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3. Every free group is bi-orderable.
4. The Klein bottle group $K = \langle a, b \mid a^{-1}ba = b^{-1} \rangle$ is left-orderable but not bi-orderable.
5. There exist torsion-free groups that are not left-orderable. One of them is

$$\langle a, b \mid a^2ba^2 = b, b^2ab^2 = a \rangle.$$

One can check for all $\varepsilon, \delta \in \{-1, 1\}$, the following equation holds:

$$(a^\varepsilon b^\delta)^2 (b^\delta a^\varepsilon)^2 = 1.$$

Orders on $\mathbb{Z}^2$

Every order on $\mathbb{Z}^2$ corresponds to a line passing through the origin in the plane, where the line separates positive lattice points and negative lattice points.

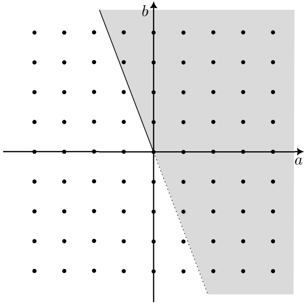

Figure: A biorder on $\mathbb{Z}^2$

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An order given by such construction is called a *lexicographical order leading by the quotient*.

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Let G be an extension of A by Q and suppose A, Q are bi-orderable. And P_A and P_Q are positive cones of some bi-orders on A, Q respectively. In addition if we assume P_A is invariant under the action of Q, then $P := P_A \cup \pi^{-1}(P_Q)$ defines a positive cone associated to a bi-order on G.

Convex subgroups relative to an order

A subgroup H of an orderable group G is called *convex* with respect to an order $\leqslant$ if for any pair of elements $f_1 \leqslant f_2$ in H, the condition $f_1 \leqslant g \leqslant f_2$ implies $g \in H$.

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Proposition

Let G be a finitely generated orderable group that is an extension of A by Q. If A is convex with respect to order $\leqslant$, then $\leqslant$ is a lexicographical order leading by the quotient there the order on Q is induced by $\leqslant$.

Convex hull of the derived subgroups

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Theorem (Wang, 2022)

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Corollary

Any bi-invariant order on M_2 is a lexicographical order leading by the quotient with respect to the extension of the derived subgroup by the abelianization.

Convex hull of the derived subgroups

Let $\mathcal{LO}(G)$ be the set of all left-orders on G. It carries a natural topology whose sub-basis is the family of sets of the form $V_g = \{P_\leqslant \mid 1 \leqslant g\}$ for $g \in G$. The space $\mathcal{LO}(G)$ is a closed subset of the Cantor set and is metrizable. And the space of all bi-orders $\mathcal{O}(G)$ is a closed subspace of $\mathcal{LO}(G)$.

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Theorem (Rivas-Tessera, 2016)

The space of left-orders of a virtually solvable group is either finite or a Cantor set.

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Let $a_1^{t_1} a_2^{t_2} \dots a_n^{t_n}$ be a non-trivial element in M_n for $n \geqslant 3$. Then there exists a bi-invariant order such that $a_1^{t_1} a_2^{t_2} \dots a_n^{t_n} \in \overline{M}'_n$.

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Theorem (Wang, 2022)

Let $\leqslant$ be a bi-invariant order on M_n, $n \geqslant 3$, then $M_n/\overline{M}_n'$ is not trivial. Equivalently, it is a free abelian group of rank at least 1.

Computable Groups and Computable Orders

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A left-order (bi-order) on $G = \langle X \rangle$ is computable with respect to the generating set $X = \{x_1, x_2, \dots\}$ if the set $\{(u, v) \in (X \cup X^{-1})^* \times (X \cup X^{-1})^* \mid u \preceq v\}$ is recursive.

Regular and Context-free Languages

Let X be a generating set of G. A language $\mathcal{L}$ over X is a subset of X^*, the free monoid (including the empty word) generated by X and X^{-1}. A language is *regular* if it is accepted by a finite state automaton and is *context-free* if it is accepted by a pushdown machine.

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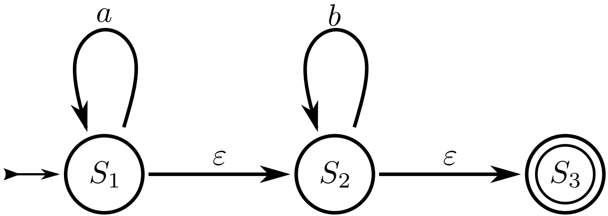

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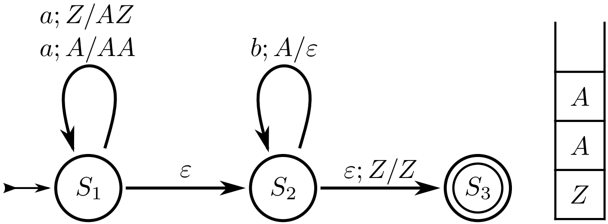

Figure: A Pushdown Machine accepting $\{a^n b^n \mid n \geqslant 0\}$.

Complexity of Orders

Let $G = \langle X \rangle$ be a finitely generated orderable group. A left-order $\leqslant$ on G is called *regular (context-free)* if the positive cone can be evaluated as a regular (context-free) language, i.e., there exists a regular (context-free) language $\mathcal{L}$ over X such that $\pi(\mathcal{L}) = P_{\leqslant}$.

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Theorem (Rourke-Wiest, 2000)

Mapping class groups of compact surfaces with a finite number of punctures and non-empty boundary admit a regular left-order.

Complexity of Orders

Theorem (Hermiller-Šunić, 2017)

Let A and B be two nontrivial, finitely generated, left-orderable groups. There exists no left-order on $G = A * B$ such that its positive cone is represented by a regular language.

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Theorem (Antolín-Rivas-Su, 2021)

Suppose that A and B are groups admitting regular left-orders. Then $(A * B) \times \mathbb{Z}$ admits a regular left-order.

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Recall that by Magnus embedding, a free metabelian group of rank n embeds into the wreath product of two free abelian groups of rank n. It naturally inherits a computable left-order (bi-orders) from the wreath product.

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Let M_n be the free metabelain group of rank n and A_n, T_n free abelian groups of rank n. The generating sets of M_n, A_n, T_n are respectively $X = \{x_1, x_2, \ldots, x_n\}$, $A = \{a_1, a_2, \ldots, a_n\}$ and $T = \{t_1, t_2, \ldots, t_n\}$. The Magnus embedding is given by the homomorphsim $\varphi(x_i) = a_i t_i$.

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Thus every free metabelian group of finite rank is computably left-orderable (bi-orderable).

Complexity of Orders

Theorem (Antolín-Rivas-Su, 2021)

The metabelian Baumslag-Solitar group $BS(1, n)$ admits a regular bi-order if and only if $n = 0, 1$ and admits a regular left-order if and only if $n \geqslant -1$.

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Theorem (Wang, 2022)

Let M_n be the free metabelian group of rank n. Then every M_n is computably bi-orderable. Moreover, M_n admits a regular bi-order if and only if $n = 1$.

Further Discussions

Theorem (Downey-Krutz, 1986)

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Theorem (Solomon, 2002)

Let G *be a computable torsion-free abelian group then* G *admits a presentation such that* G *has a computable order over that presentation.*

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Theorem (Darbinyan, 2020)

There exists a computable bi-orderable group G, which does not have a presentation with computable bi-order. G can be chosen to be two-generated solvable group of derived length 3.

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Proposition

A non-abelian finitely generated metabelian group is bi-orderable if and only if it is an extension of Q-orderable $\mathbb{Z}Q$-module A and a free abelian group Q.

Thank you for your attention!