Higgs Bundles with Nilpotent Structures

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Cayley-Hamilton Theorem

Let V be a vector space over a field k of dimension r, and $A \in \mathrm{End}_k(V)$. Let us recall the following classical theorem:

Cayley-Hamilton Theorem

We denote the characteristic polynomial of A by:

$$\mathrm{char}(A) := \det(\lambda - A) = \lambda^r + a_1\lambda^{r-1} + \cdots + a_r.$$

Then we have:

$$A^r + a_1 A^{r-1} + \cdots + a_r = 0.$$

Remark

Equivalently speaking, V is a module over $k[\lambda]/(\mathrm{char}(A))$ with λ acts on V as A.

Cayley-Hamilton Theorem

We know that the above theorem holds for endomorphism of free modules of finite ranks over a commutative Noetherian ring:

Other settings

- What if the ring consists of functions on algebraic varieties, e.g., algebraic curves?
- Suppose we work on an algebraic curve, what if the endomorphism is nilpotent at some fixed points?
- What if the endomorphism is symplectic or special orthogonal?

Higgs Bundles and Hitchin Maps

Let X be a smooth projective curve over a field k of genus $g(X) \geq 2$.

Definition

A rank r Higgs bundle on X is a pair (E, θ) where E is vector bundle of rank r and $\theta : E \to E \otimes \omega_X$ is $\mathcal{O}_X$-linear.

- The moduli space of semi-stable Higgs bundles of rank r, degree d, $\mathbf{Higgs_{r,d}}$ is a normal quasi-projective variety and it is smooth if $(r, \deg d) = 1$.
- In this setting, Hitchin map takes the form:

$$h_{r,d} : \ \mathbf{Higgs_{r,d}} \to \mathbf{A} := \oplus_{\mathbf{i=1}}^{\mathbf{r}} \mathbf{\Gamma}(\mathbf{X}, \omega_{\mathbf{X}}^{\otimes \mathbf{i}}), \ (\mathbf{E}, \theta) \mapsto \text{coeff. of char}(\theta)$$

which is proper and flat.

Spectral Curves

What are fibers of $h_{r,d}$? To study fibers of the Hitchin map, we introduce the concept of a spectral curve.

Spectral Curves

Closed points in $\mathbf{A}$ corresponds to coefficients of characteristic polynomials of Higgs bundles. Thus we can use these polynomials to define curves in $\mathrm{Tot}(\omega_X)$. We denote the one, corresponding to a closed point $a \in \mathbf{A}$, by X_a .

Generic Fibers of Hitchin Maps

By Cayley-Hamilton theorem, we can easily see that a Higgs bundle (E, θ) lie in the fiber $h_{r,d}^{-1}(a)$ can be seen as a sheaf on X_a.

Theorem Beauville-Narasimhan-Ramanan [BNR89]

If a is generic such that X_a is integral , then $h_{r,d}^{-1}(a)$ is isomorphic to moduli of torsion free sheaves of rank one on X_a of degree $d + n(n-1)(g(X)-1)$, i.e., compactified Jacobian.

This holds for higher dimensional varieties. But to calculate generic fibers, for example, the number of connected components, we need to know Neron-Severi groups of corresponding spectral varieties.

Theorem Su-W [SW21]

Let k be algebraically closed and X smooth projective of dimension ≥ 2 over k. Then under some conditions, we have an isomorphism of Picard varieties: $\mathrm{Pic}(X_a) \cong \mathrm{Pic}(X)$ for generic smooth spectral varieties X_a.

(Quasi-)Parabolic Vector Bundles

Let k, r be as before, and X now is a smooth projective curve. We choose a finite set D of $X(k)$.

Definition

A parabolic vector bundle is a rank r vector bundle E on X which for every $x \in D$ is endowed with a filtration

$$E|_x = F^0(x) \supset F^1(x) \supset \cdots \supset F^{\sigma_x}(x) = 0.$$

And we denote $\dim F^{j-1}(x)/F^j(x) = m^j(x)$ and call $(D, P) := (D, \{m^j(x)\}_x)$ the parabolic type.

(Quasi-)parabolic Higgs bundles

Definition

Let E be a parabolic vector bundle. A **parabolic Higgs field** on E is a $\mathcal{O}_X$-homomorphism $\theta : E \to E \otimes_{\mathcal{O}_X} \omega_X(D)$ with the property that at each x, it takes each $F^j(x)$ to $F^{j+1}(x)$. A **parabolic Higgs bundle** is a pair (E, θ) with E a parabolic vector bundle and θ is a parabolic Higgs field on E.

In particular, θ is nilpotent at each marked point. In fact, locally around marked points θ is a topologically nilpotent element in the sense of Kazhdan-Lusztig [KL88]

Parabolic Hitchin Maps

Definition

We can also define (semi-)stable parabolic Higgs bundles, and construct coarse moduli spaces $\mathbf{Higgs}_{P,\mathrm{GL}_r}$ and the corresponding parabolic Hitchin map via characteristic polynomials:

$$h_{P,\mathrm{GL}_r} : \mathbf{Higgs}_{P,\mathrm{GL}_r} \to \mathbf{A}_P \subset \prod_{i=1}^{r} \mathrm{H}^0(X, (\omega_X(D))^{\otimes i})$$

$$(E, \theta) \to \text{coeff of char}(\theta).$$

In the following, we shall talk about generic fibers of the map.

Spectral Curves

Given a Higgs bundle (E, θ) with $h_{P,\mathrm{GL}_r}((E, \theta)) = a \in \mathbf{A}_P$, E can be viewed as a torsion-free sheaf of generic rank one on X_a. However:

Geometry of Spectral Curves

Unless all flags are complete flags, we have:

- X_a will be singular;
- $p_a(X_a) > \dim h_P^{-1}(a)$ which means that $h_P^{-1}(a)$ is a proper subvariety of the compactified Jacobian of X_a.

The strategy is to resolve the singularieties of X_a.

Singularities of Spectral Curves

By the Newton polygon of the characteristic polynomial $f(\lambda)$:

Kazhdan-Lusztig Map for Type A

Assuming k is algebraically closed. Locally around the marked point, we have a decomposition:

$$f_a(\lambda) = \prod f_{a,i}(\lambda), f_{a,i}(\lambda) \in k[[t]][\lambda]$$

where each $f_{a,i}$ is an Eisenstein polynomial and $\{\deg f_{a,i}\}$ forms a conjugate partition of $\{m^j(x)\}$.

Remark

This is a way to describe image of the Kazhdan-Lusztig map of a nilpotent orbit whose partition is $\{\deg f_{a,i}\}$.

Generic Fibers

Under certain generic condition on the characteristic polynomial $f_a(\lambda)$ and its irreducible factors $\{f_{a,i}\}$:

Theorem Su-W-Wen [SWW22a]

For generic $a \in \mathbf{A}_P$, there is a one to one correspondence between:

$$\left\{\text{Parabolic Higgs bundle } (E,\theta) \in h^{-1}_{P,\mathrm{GL}_r}(a)\right\} \leftrightarrow$$

$$\{\text{ line bundles } \mathcal{M} \text{ over } \overline{X_a}\}$$

where $\overline{X_a}$ is the normalization of X_a. This holds over an arbitrary field.

Remark

We can also prove similar results for parabolic $\mathrm{SL}_r\,/\mathbf{PGL}_r$-Higgs bundles. It turns out that generic fibers are (roughly) dual Abelian varieties.

Topological Mirror Symmetry

Theorem Su-W-Wen[SWW22b]

There is a numerical invariant Δ_P depends on the parabolic type (D, P) such that the equality of (stringy) Hodge numbers

$$h^{p,q}(\mathbf{Higgs}^{\mathbf{L}}_{\mathbf{P},\mathrm{SL}_{\mathbf{r}}}) = \mathbf{h}^{\mathbf{p,q}}_{\mathbf{st}}(\mathbf{Higgs}^{\mathbf{e}}_{\mathbf{P},\mathbf{PGL}_{\mathbf{r}}}; \alpha_{\mathbf{PGL}_{\mathbf{r}}})$$

holds for all p, q provided $d \equiv \lambda e(\mod \Delta_P)$ for some $\lambda \in \mathbb{Z}_+$. In particular, if there is a point $x \in D$ such that the filtration at x is Borel then $\Delta_P = 1$ and:

$$h^{p,q}(\mathbf{Higgs}^{\mathbf{L}}_{\mathbf{P},\mathrm{SL}_{\mathbf{r}}}) = \mathbf{h}^{\mathbf{p,q}}_{\mathbf{st}}(\mathbf{Higgs}^{\mathbf{e}}_{\mathbf{P},\mathbf{PGL}_{\mathbf{r}}})$$

Topological Mirror Symmetry

Previous Work

- Topological mirror symmetry was proposed by Hausel-Thaddeus [HT03] and proved in rank 2 cased by working on global nilpotent cones.
- The parabolic cases in rank 2, 3 for full flags was proved by Gothen and Oliveira [GO19].
- The non-parabolic case was proved in general by Groechenig-Wyss-Ziegler [GWZ20] via p-adic integration.

Hitchin Systems for SO and Sp: Nilpotent Structures

In a recent work by Fu-Ruan-Wen , they conjectured and proved for Richardson orbits:

Conjecture Fu-Ruan-Wen[FRW22]

Given two special nilpotent orbits $\mathbb{O} \subset \mathfrak{sp}_{2r}$ (resp. ${}^S\mathbb{O} \subset \mathfrak{so}_{2r+1}$), related via Springer correspondence, can we find equivariant finite covers $X \to \overline{\mathbb{O}}$ (resp. ${}^SX \to \overline{{}^S\mathbb{O}}$), such that $(X, \mathbb{O})$ and $({}^SX, {}^S\mathbb{O})$ are mirrors in some sense, e.g., share the same stringy E-polynomials.

Remark

Each nilpotent orbits in Sp_{2r} (resp. SO_{2r+1}) corresponds to a partition, and we say a nilpotent orbit is special if and only if its conjugate is still a partition of a nilpotent orbits in Sp_{2r} (resp. SO_{2r+1}).

All the following results and conjectures are work in progress with Xiaoyu Su, Xueqing Wen, Yaoxiong Wen and Weiqiang He.

Construction of Moduli Spaces

Goals: Globalization

We can make conjectures on topological mirror symmetry for certain SO_{2r+1}/Sp_{2r} Hitchin systems over curves which can be treated as global versions of the above one.

What kind of moduli spaces we should expect? Let (X, D) be as before.

A General Principle

Since we want to explore mirror symmetry phenomena for nilpotent orbits $(\mathbb{O}, {}^S\mathbb{O})$ which are Springer dual to each other, we want to consider Sp_{2r}(resp. SO_{2r+1}) Higgs bundles with nilpotent structures at marked points. It is reasonable to require Higgs fields lie in the nilpotent closures $(\overline{\mathbb{O}}, \overline{{}^S\mathbb{O}})$ at marked points.

Construction of Moduli Spaces

Candidates

Given a Sp_{2r}/SO_{2r+1}-Higgs bundles $(E, (\cdot,\cdot), \theta)$,we have two different choices of extra structures at marked points:

- A decreasing filtration $\{F^{\cdot}E\}$ given by Jacobson-Morozov theorem, $\theta : F^iE \to F^{i+1}E$ and $\theta(x) \in \overline{\mathbb{O}_x}$ (resp. $\overline{{}^S\mathbb{O}_x}$) $\rightsquigarrow$ ${}^{JM}\mathbf{Higgs}_{Sp_{2r},\overline{\mathbb{O}}}$ (resp. ${}^{JM}\mathbf{Higgs}_{SO_{2r+1},\overline{{}^S\mathbb{O}}}$).

All of these moduli spaces admit natural Hitchin maps.

Generic Spectral Curves

We have the following interesting fact:

Theorem He-Su-W-Wen-Wen

Given two moduli spaces ${}^{JM}\mathbf{Higgs}_{Sp_{2r},\overline{\mathbb{O}_1}}, {}^{JM}\mathbf{Higgs}_{SO_{2r+1},\overline{\mathbb{O}_2}}$, they have same generic spectral curves if and only if the two special nilpotent orbits are related via Springer correspondence and have same dimension.

The singularities of spectral curves are determined by Kazhdan-Lusztig maps.

Generic Fibers

Theorem He-Su-W-Wen-Wen

Generic fibers for the Hitchin map associated with ${}^{JM}\mathbf{Higgs}_{Sp_{2r},\overline{\mathbb{O}}}$ are Prym varieties of normalized spectral curves.

However we can not identify generic fibers of ${}^{JM}\mathbf{Higgs}_{SO_{2r+1},\overline{{}^{S}\mathbb{O}}}$ simply with line bundles on normalized spectral curves, due to the "collapse"-phenomena in Springer correspondence. Moreover,

Duality Fails

Generic fibers of ${}^{JM}\mathbf{Higgs}_{Sp_{2r},\overline{\mathbb{O}}}$ and ${}^{JM}\mathbf{Higgs}_{SO_{2r+1},\overline{{}^{S}\mathbb{O}}}$ are not dual Abelian varieties.

Topological Mirror Symmetry

We assume $\mathbb{O}$ and ${}^S\mathbb{O}$ are both Richarson orbits. For $\mathbb{O}$ (resp. ${}^S\mathbb{O}$), we have a set of parabolic subgroups $\{P_i\}_{i=1}^{\ell}$ (resp. $\{{}^LP_i\}_{i=1}^{\ell}$) such that

$$T^*Sp_{2r}/P_i \to \overline{\mathbb{O}},\ T^*SO_{2r+1}/{}^LP_i \to \overline{{}^S\mathbb{O}}$$

are generically finite maps.

Theorem Fu-Ruan-Wen [FRW22]

We denote the Lusztig canonical quotient by $\overline{A(\mathbb{O})}$ (resp. $\overline{A({}^S\mathbb{O})}$) , then

$$\overline{A(\mathbb{O})} = \overline{A({}^S\mathbb{O})}$$

If we denote the mapping degree by ν_{P_i} (resp. $\nu_{{}^LP_i}$), then:

$$\nu_{P_i} \cdot \nu_{{}^LP_i} = \#\overline{A(\mathbb{O})},\ \forall i = 1, \ldots, \ell.$$

This motivates the following construction:

Better Candidates

For each pair $(P_i, {}^L P_i)$, we choose the extra structure:

- A decreasing filtration $\{F^{\cdot}E\}$ given by P_i(resp. ${}^L P_i$), $\theta : F^i E \to F^{i+1}E$ and $\theta(x) \in \overline{\mathbb{O}_x}$ (resp. $\overline{{}^S\mathbb{O}_x}$) $\rightsquigarrow$ ${}^{P_i}\mathbf{Higgs}_{Sp_{2r}, \overline{\mathbb{O}}}$ (resp. ${}^{{}^L P_i}\mathbf{Higgs}_{SO_{2r+1}, \overline{{}^S\mathbb{O}}}$).

We have the following pre-theorem:

Pre-Theorem [He-Su-W-Wen-Wen]

Topological mirror symmetry of $({}^{P_i}\mathbf{Higgs}_{Sp_{2r}, \overline{\mathbb{O}}}, {}^{{}^L P_i}\mathbf{Higgs}_{SO_{2r+1}, \overline{{}^S\mathbb{O}}})$ holds.

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