On the scalar approach to the weak asymptotics problem for $\mathcal{GN}$-systems

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Hermite-Padé approximations

$$\vec{f} = (f_1, \ldots, f_p), \quad f_j = \sum_{k=0}^{\infty} \frac{c_{jk}}{z^{k+1}}, \quad \vec{n} = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p$$

Problem

Find a vector of polynomials $(Q_{\vec{n},1}, \ldots, Q_{\vec{n},p}) \not\equiv 0$ *and a polynomial* $Q_{\vec{n},0}$, $Q_{\vec{n},j} \in \mathbb{C}[z]$, *such that* $\deg Q_{\vec{n},j} \leq n_j$ *and*

$$(Q_{\vec{n},0} + Q_{\vec{n},1} f_1 + \cdots + Q_{\vec{n},p} f_p)(z) = O\left(\frac{1}{z^{|\vec{n}|+p}}\right), \quad z \to \infty.$$

Problem

Find a polynomial $P_{\vec{n}} \not\equiv 0$ *and a vector of polynomials* $(P_{\vec{n},1}, \ldots, P_{\vec{n},p})$, $P_{\vec{n},j}, P_{\vec{n}} \in \mathbb{C}[z]$, *such that* $\deg P_{\vec{n}} \leq |\vec{n}|$ *and*

$$(P_{\vec{n},j} + P_{\vec{n}} f_j)(z) = O\left(\frac{1}{z^{n_j+1}}\right), \quad z \to \infty, \quad j = 1, \ldots, p.$$

Asymptotic problems of Hermite-Padé polynomials

- $\Omega\colon \mathbb{N} \to \mathbb{N}^p, \Omega(n) = \vec{n}, |\vec{n}| \to \infty$ while $n \to \infty$

Weak asymptotic problem of H-P polynomials

To describe the weak limit of the sequence of counting measures of zeros of the corresponding polynomials.

Strong asymptotic problem of H-P polynomials

To describe the limit of the sequence of the corresponding polynomials as the limit of the sequence of functions in some domain $D \subset \mathbb{C}$.

Example: weak asymptotics

- $f_1(z) = \widehat{\sigma}_1(z), f_2(z) = \widehat{\sigma}_2(z)$
- $\Omega(n) = \vec{n} = (n, n)$
- $P_{\vec{n}} \not\equiv 0, P_{\vec{n},1}, P_{\vec{n},2}:$ $\deg P_{\vec{n}} \le 2n$ and

$$(P_{\vec{n},j} + P_{\vec{n}} f_j)(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty, \quad j = 1, 2.$$

- $\sigma_1, \sigma_2:$ $\operatorname{supp} \sigma_j \subset E_j \subset \mathbb{R}$, $\mathring{E}_1 \cap \mathring{E}_2 = \varnothing$, $\frac{d\sigma_j}{dx} > 0$ a.e. on E_j

Example: weak asymptotics

Vector potential and energy

For $\nu = (\nu_1, \nu_2)$, $\operatorname{supp} \nu_j \subset E_j$:

- $W^\nu(z) = (W_1^\nu(z), W_2^\nu(z))$
  - $W_1^\nu(z) = 2V^{\nu_1}(z) + V^{\nu_2}(z)$
  - $W_2^\nu(z) = V^{\nu_1}(z) + 2V^{\nu_2}(z)$
- $I(\nu) = 2(I(\nu_1, \nu_2) + I(\nu_1, \nu_1) + I(\nu_2, \nu_2))$

Energy minimization problem

$$\begin{cases} J(\nu) \to \min \\ |\nu_1| = \frac{1}{2}, |\nu_2| = \frac{1}{2} \end{cases}$$

has the unique solution $\lambda = (\lambda_1, \lambda_2)$; it is completely described via the equilibrium property

$$\begin{cases} W_j^\lambda(x) = x_j, & x \in \operatorname{supp} \lambda_j \\ W_j^\lambda(x) \geqslant x_j, & x \notin \operatorname{supp} \lambda_j \end{cases}$$

Moreover, $\frac{1}{|\vec{n}|} P_{\vec{n}} \xrightarrow{*} \lambda$ as $n \to \infty$.

Example: strong asymptotics

- $d\sigma(x) = (1+x)^{\alpha}(1-x)^{\beta}x^{\gamma}\,dx, \quad \alpha,\beta,\gamma > -1$
- $E_1 = [-1,0],\ E_2 = [0,1]$
- $\sigma_1 = \sigma|_{E_1},\ \sigma_2 = \sigma|_{E_2}$

- $P_{\vec{n}}(z) = P^1_{\vec{n}}(z) \cdot P^2_{\vec{n}}(z)$
- $P^j_{\vec{n}}(z) = \alpha_j^n \Phi_j^n(z)(F_j(z) + O(1)), \quad z \in \mathbb{C} \setminus E_i$
- $F_j(z) \in H(\mathbb{C} \setminus E_j)$,

$$\Phi_j(z): \quad \Phi^3 + 3\Phi^2 + \left(3 - \frac{27}{4}z^2\right)\Phi + 1 = 0$$

Favorite applications of H-P

Hexagon tilings

- K. Johansson, *Non-intersecting paths, random tilings and random matrices*, Probab. Theory Related Fields 123 (2002), pp. 225–280
- M. Duits and A. B. J. Kuijlaars, *The two-periodic Aztec diamond and matrix valued orthogonal polynomials*, J. Eur. Math. Soc. 23.**4** (2021), pp. 1075–1131

Constructive analytic continuation

A. V. Komlov, *The polynomial Hermite-Padé m-system for meromorphic functions on a compact Riemann surface*, Sbornik: Mathematics 212.**12** (2021), pp. 1694–1729

Main class of functions

Class $A^{\circ}(\Sigma), \#\Sigma < +\infty$ of analytic functions which

- are holomorphic at each point $z_0 \in \overline{\mathbb{C}} \setminus \Sigma$;
- can be analytically continued over any path $\gamma \subset \overline{\mathbb{C}} \setminus \Sigma$;
- have at lest one point of Σ as a branch point.

H. Stahl's theory for Padé polynomials

- $f \in A^\circ(\Sigma)$, $\mathrm{cap}(\Sigma) = 0$
- $P_n, Q_n \in \mathbb{C}[z]$: $Q_n \not\equiv 0$, $\deg Q_n \le n$ and

$$(P_n + Q_n f)(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty.$$

- H. Stahl, *Structure of extremal domains associated with an analytic function*, Complex Variables Theory Appl. 4.**4** (1985), pp. 339–354
- H. Stahl, *Extremal domains associated with an analytic function. I, II*, Complex Variables Theory Appl. 4.**4** (1985), pp. 311–324, 325–338
- H. Stahl, *Orthogonal polynomials with complex valued weight function. I, II*, Constr. Approx. 2.**3** (1986), pp. 225–240, 241–251
- H. Stahl, *The Convergence of Padé Approximants to Functions with Branch Points*, Journal of Approximation Theory 91.**2** (1997), pp. 139–204, ISSN: 0021-9045

H. Stahl's theory for Padé polynomials

H. Stahl's theory: geometric part

There exists a unique compact set $S = S(f)$ which has minimal capacity among elements of $\{\Gamma \subset \mathbb{C} \mid f \in H(\mathbb{C} \setminus \Gamma)\}$ and possesses the following properties:

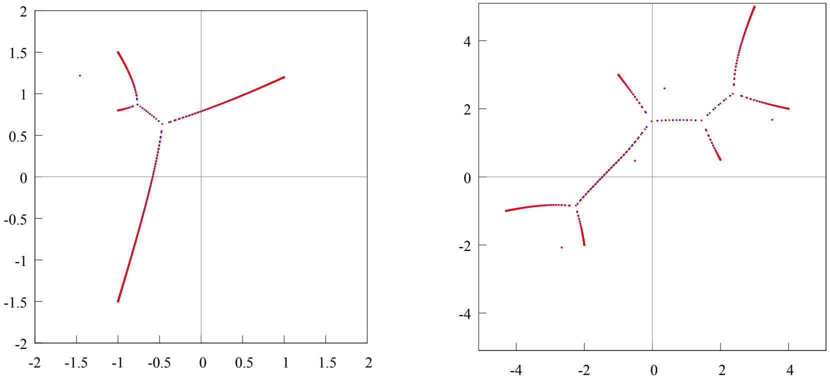
- $D := \overline{\mathbb{C}} \setminus S$ is a domain;
- $S = \bigcup_{j=1}^{m} S_j$, where S_j are analytic arcs;
- $\frac{\partial g_D(z,\infty)}{\partial n^+} = \frac{\partial g_D(z,\infty)}{\partial n^-}, \quad z \in \mathring{S}_j$.

H. Stahl's theory: analytic part

- $\frac{1}{n}\chi(Q_n) \xrightarrow{*} \lambda$, where λ is the equilibrium measure of S;
- $|(f - P_n/Q_n)(z)|^{1/n} \xrightarrow{\text{cap}} e^{-2g_D(z,\infty)}, \quad z \in D$.

Examples of Stahl compacts

$$f_1(z) = \sqrt[4]{\frac{1.0-(-1+0.8i)z}{1-(1+1.2i)z} \cdot \frac{1-(-1+1.5i)z}{1-(-1-1.5i)z}}$$
$$f_2(z) = \frac{z}{\sqrt[6]{(1+(4.3+1i)z)(1-(2+5i)z)(1+(2+2i)z)(1+(1-3i)z)(1-(4+2i)z)(1-(3+5i)z)}}$$



Numerical estimation of $S(f_1)$ Numerical estimation of $S(f_2)$

Generalized Nikishin system

Let's start from a vector of measures $\sigma = (\sigma_1, \dots, \sigma_p)$; we will construct the vector of measures $(\mu_1, \dots, \mu_p)$ using the following inductive process:

$$
\begin{aligned}
d\mu_1(x) &= d\sigma_1(x),\\
d\mu_2(x) &:= \langle \sigma_1, \sigma_2 \rangle := \widehat{\sigma}_2(x) d\sigma_1(x),\\
d\mu_3(x) &:= \langle \sigma_1, \sigma_2, \sigma_3 \rangle := \langle \sigma_1, \langle \sigma_2, \sigma_3 \rangle \rangle,\\
&\vdots\\
d\mu_p(x) &:= \langle \sigma_1, \dots, \sigma_p \rangle = \langle \sigma_1, \langle \sigma_2, \dots, \sigma_p \rangle \rangle
\end{aligned}
$$

Measure composed via Nikishin's rule

We will say that the measure μ_p is composed from the vector of measures σ via the Nikishin's rule; we will denote it as $\mu(\sigma)$.

Generalized Nikishin system

A. I. Aptekarev and V. G. Lysov, *Systems of Markov functions generated by graphs and the asymptotics of their Hermite-Padé approximants*, Sbornik: Mathematics 201.**2** (2010), pp. 183–234

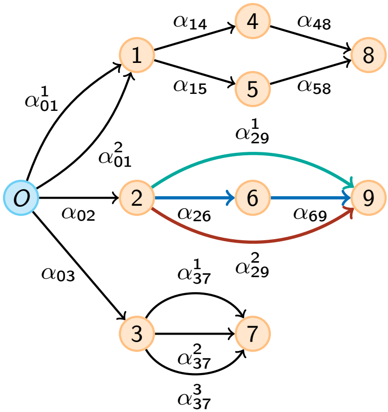
Generating graph

- Oriented graph $\Gamma := \Gamma(\mathcal{V}, \mathcal{E}, O)$
- Set of vertices $\#\mathcal{V} < +\infty$, $\mathcal{V} = \{A, B, C, \dots\}$; set of edges $\#\mathcal{E} < +\infty$, $\mathcal{E} = \{\alpha, \beta, \gamma, \dots\}$
- Unique vertex O such that for any vertex $A \in \mathring{\mathcal{V}} := \mathcal{V} \setminus O$ exists an oriented path from O to A.

$$\forall \alpha \in \mathcal{E} \quad \longrightarrow \quad E_\alpha := [a_\alpha, b_\alpha] \subset \mathbb{R},\ \sigma_\alpha : \frac{d\sigma_\alpha}{dx} > 0 \text{ a.e. on } E_\alpha;$$

we assume that if edges α and β have common vertex, corresponding segments E_α and E_β do not intersect.

Generalized Nikishin system



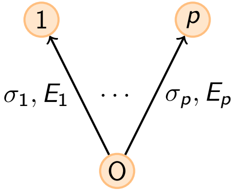
- $\mathcal{T}_9 = \{t_9^1, t_9^2, t_9^3\}$
- $t_9^1 = (\alpha_{02}, \alpha_{29}^1)$
- $t_9^2 = (\alpha_{02}, \alpha_{26}, \alpha_{69})$
- $t_9^3 = (\alpha_{02}, \alpha_{29}^2)$

- $\mu_{t_9^j} = \mu(t_9^j)$
- $\widehat{\mu}_9(z) := (\widehat{\mu}_{t_9^1} + \widehat{\mu}_{t_9^2} + \widehat{\mu}_{t_9^3})(z)$

$\mathcal{GN}$ system

- $\mathcal{GN}(\Gamma) = \{\widehat{\mu}_A(z) \mid A \in \mathring{\mathcal{V}}\}$

Classical systems as special cases of $\mathcal{GN}$-system

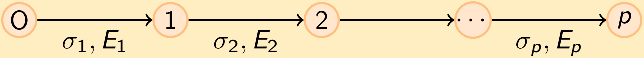


Angelesco system, 1919

$$\mathcal{A}:\ \{\widehat{\sigma}_j(z)\}_{j=1}^{p},\ S(\sigma_j)\subset E_j\subset\mathbb{R}:\ E_j\cap E_k=\varnothing$$

Nikishin system, 1986

$$\mathcal{N}:\ \mu(\sigma_1),\ldots,\mu(\sigma_1,\ldots,\sigma_p),\ S(\sigma_j)\subset E_j\subset\mathbb{R}:\ E_j\cap E_{j+1}=\varnothing$$



Main results for classical systems

Angelesco system

- A. A. Gonchar and E. A. Rakhmanov, *On the convergence of simultaneous Padé approximants for systems of functions of Markov type*, Proc. Steklov Inst. Math., 157 (1981), pp. 31–50
- A. I. Aptekarev, *Asymptotics of simultaneously orthogonal polynomials in the Angelesco case*, Sbornik: Mathematics 64.**1** (1989), pp. 57–84

Nikishin system

- E. M. Nikishin, *Asymptotic behavior of linear forms for simultaneous Padé approximants*, Russian Math. (Iz. VUZ) **2** (1986), pp. 33–41
- A. I. Aptekarev, *Strong asymptotics of multiply orthogonal polynomials for Nikishin systems*, Sbornik: Mathematics 190.**5** (1999), pp. 631–669

Vector potential equilibrium problem

Initial data

- $\vec{E} = (E_1, \dots, E_m) \Subset \mathbb{C}^m$
- $\mathcal{A} = (a_{kj})_{k,j=1}^{m} \in \mathbb{R}^{m\times m} \quad \mathcal{A} \geq 0$
- $\mathcal{C} = (c_{kj})_{k,j=1}^{r,m} \in \mathbb{R}^{r\times m}, \quad \operatorname{rank} \mathcal{C} = r$
- $b = (b_1, \dots, b_p) \in \mathbb{R}^r, \quad b \neq 0$

- $a_{jj} > 0$
- $a_{kj} = 0$ if $k \neq j$ and $E_k \cap E_j \neq \varnothing,\ k, j = 1, \dots, m$
- $\left\{x \in \mathbb{R}^m : \sum_{j=1}^{m} c_{kj} x_j = b_k,\ k = 1, \dots, r;\ x_j \geqslant 0,\ j = 1, \dots, m\right\}$ is bounded and nonempty.

Vector potential equilibrium problem

- $M^+(\vec{E}) := M^+(E_1) \times \cdots \times M^+(E_m)$

Vector potential and interaction energy

- $W^\mu = (W_1^\mu, \ldots, W_m^\mu)\,, \quad W_k^\mu(z) = \sum_{j=1}^m a_{kj} V^{\mu_j}(z)$
- $J(\mu) = \sum_{k,j=1}^m a_{kj} I\,(\mu_k, \mu_j)$

$$\mathrm{M}^+_{\mathcal{C},b}(\vec{E}) := \left\{ \mu \in \mathrm{M}^+(\vec{E}) : \sum_{j=1}^m c_{kj}\,|\mu_j| = b_k, k = 1, \ldots, r \right\}$$

Vector potential equilibrium problem

$$\begin{cases} J(\mu) \to \min \\ \mu \in \mathrm{M}^+_{\mathcal{C},b}(\vec{E}) \end{cases}$$

Vector potential equilibrium problem

Theorem (A. I. Aptekarev, V. G. Lysov, 2010)

1. *There exists a unique measure λ, which solves the minimization problem.*
2. *Measure λ is the unique measure in $M^{+}_{\mathcal{C},b}(\vec{E})$ which satisfies the following equilibrium conditions:*

$$\begin{cases} W_k^{\lambda}(x) := \sum_{j=1}^{m} a_{kj} V^{\lambda_j}(x) \begin{cases} = \varkappa_k, & x \in S(\lambda_k), \\ \geqslant \varkappa_k, & x \in E_k \end{cases} & k = 1, \ldots, m, \\ (\varkappa_1, \ldots, \varkappa_m) \in \operatorname{Im} \mathscr{C}^T. \end{cases}$$

Weak asymptotics for $\mathcal{GN}$-systems

Interaction matrix from the graph Γ

$\mathcal{A} = \mathcal{A}(\Gamma) = (a_{\alpha\beta})$, $a_{\alpha\beta}$

- $= 2$ if $\alpha = \beta$ or if α and β come from the same vertex and come to the same vertex;
- $= 1$ if α and β come from the same vertex but not come into the same vertex or if α and β come out from diffetent vertices but come into the same vertex;
- $= -1$ if α follows β or β follows α
- $= 0$ if α and β do not have common vertices.

Measures masses restrictions

$$\left\{ \mu \in \mathrm{M}^+(\vec{E}) : \sum_{\alpha \in \mathscr{E}_{A-}} |\mu_\alpha| - \sum_{\beta \in \mathscr{E}_{A+}} |\mu_\beta| = v_A,\ A \in \mathring{\mathscr{V}} \right\}$$

Weak asymptotics for $\mathcal{GN}$-systems

Theorem (A. I. Aptekarev, V. G. Lysov, 2010)

Let ω be set set of edges that come out from the root vertex O; then

$$\frac{1}{|\vec{n}|}\chi(P_{\vec{n}}) \xrightarrow{*} \sum_{\alpha\in\omega} \lambda_\alpha .$$

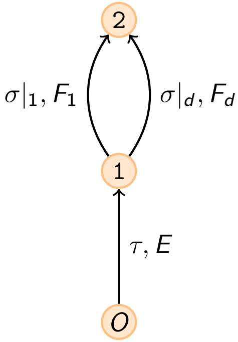
New scalar approach

S. P. Suetin, *On a New Approach to the Problem of Distribution of Zeros of Hermite-Padé Polynomials for a Nikishin System*, Proceedings of the Steklov Institute of Mathematics 301.**1** (2018), pp. 245–261

$$f_1(z) = \frac{1}{\sqrt{z^2-1}}, \quad f_2(z) = \int_{-1}^{1} \frac{\widehat{\sigma}(z)dx}{(z-x)\sqrt{1-x^2}},$$

where $\operatorname{supp}\sigma \subset F := \bigsqcup_{j=1}^{d} F_j \subset \mathbb{R}, F_j = [c_j, d_j]$; $\frac{d\sigma}{dx} > 0$ a.e. on F and $\operatorname{conv} F \cap [-1,1] = \varnothing$.

System $\aleph_0$



- $d\tau(x) = \frac{dx}{\sqrt{1-x^2}}$, $E = [-1, 1]$
- σ: $\frac{d\sigma}{dx} > 0$ a.e. on $F := \bigsqcup_{j=1}^{d} F_j \subset \mathbb{R}$
- conv $F \cap [-1, 1] = \varnothing$

New scalar approach

HP type I for f_1 and f_2 and $\Omega(n) = (n, n)$

$$(Q_{\vec{n},0} + Q_{\vec{n},1} f_1 + Q_{\vec{n},2} f_2)(z) = O\left(\frac{1}{z^{2n+2}}\right), \quad z \to \infty$$

Potential and harmonic exterior field

$$P^{\mu}(z) := \int_F \log \frac{|1 - \varphi(z)\varphi(t)|}{|z-t|^2} d\mu(t), \quad \psi(z) := \log|\varphi(z)|$$

Energy with harmonic exterior field

$$J_{\psi}(\mu) := \int_F P^{\mu}(z) d\mu(z) + 2\int_F \psi(z) d\mu(z).$$

New scalar approach

Theorem (S. Suetin, 2018)

There exists the unique measure $\lambda \in M_1^{\circ}(F)$ with the following property:

$$J_\psi(\lambda) = \min_{\mu \in M_{\mathbf{1}}(F)} J_\psi(\mu).$$

It can be totally described with the equilibrium property

$$P^\lambda(z) + \psi(z) \begin{cases} \equiv w_F, & z \in S(\lambda), \\ \geq w_F, & z \in F \setminus S(\lambda). \end{cases}$$

There is the weak convergence

$$\frac{1}{n}\chi(Q_{\vec{n},2}) \xrightarrow{*} \lambda.$$

Scalar-vector equivalence

S. P. Suetin, *Equivalence of a Scalar and a Vector Equilibrium Problem for a Pair of Functions Forming a Nikishin System*, Math. Notes 106.**6** (2019), pp. 971–980

Vector equilibrium problem for f_1 and f_2

$$4U^{\lambda_1}(x) - U^{\lambda_2}(x) \equiv w_1 = \text{const}, \ x \in E,$$
$$-U^{\lambda_1}(t) + U^{\lambda_2}(t) \equiv w_2 = \text{const}, \ t \in F,$$

where $\operatorname{supp}\lambda_1 \subset E, \operatorname{supp}\lambda_2 \subset F$

Scalar equilibrium problem for f_1 and f_2

$$J_\psi(\lambda) = \min_{\mu \in M_1(F)} J_\psi(\mu)$$

Scalar-vector equivalence

Theorem (S. Suetin, 2019)

Scalar and vector equilibrium problems for f_1 and f_2 are equivalent in the sense that

$$\lambda = \beta_F(\lambda_1)$$

and, conversely,

$$\lambda_1 = \frac{\beta_E(\lambda) + 3\tau_E}{4}$$

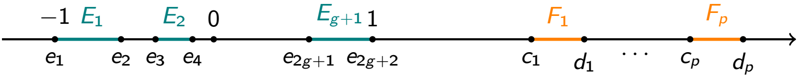
where τ_E is the Tchebyshev measure of E and β is the corresponding balayage of a measure.

A slite generalization of the new scalar approach

E. A. Lopatin, *On the generalization of the scalar approach to the problem of a limit distribution of zeros of Hermite-Padé polynomials for a Nikishin system*, Sbornik: Mathematics (2022), forthcoming

$$E := \bigsqcup_{j=1}^{g+1} E_j \subset \mathbb{R},\ E_j := [e_{2j-1}, e_{2j}],\ e_{2j-1} < e_{2j},\ e_1 = -1, e_{2g+2} = 1$$

$$F := \bigsqcup_{j=1}^{p} F_j \subset \mathbb{R},\ F_j = [c_j, d_j],\ c_j < d_j;\ \text{conv}\, E \cap \text{conv}\, F = \varnothing$$



Scalar approach on the Riemann surface for $\mathcal{S}_g$

$$d\sigma_1 = \frac{dx}{w(x+i0)r(x)}; \; w(z) := \sqrt{q(z)}, \; q(z) := \prod_{j=1}^{2g+2} (z - e_j);$$

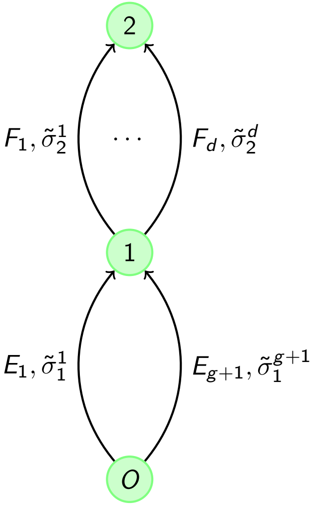
$$r(x) \in \mathbb{R}[x]\colon \forall x \in \operatorname{conv} E \; r(x) \neq 0; \; S(\sigma_1) = E;$$

$$\sigma_2 = \sigma\colon \frac{d\sigma}{dx} > 0 \text{ a.e. on } F; \; S(\sigma_2) = F$$

$$d\mu_1(x) = d\sigma_1(x), \; d\mu_2(x) = d\langle \sigma_1, \sigma_2 \rangle (x) = \widehat{\sigma}_2(x) d\sigma_1(x),$$
$$\mathcal{S}_g := \{f_1(z), f_2(z)\} = \{\widehat{\mu}_1(z), \widehat{\mu}_2(z)\}$$

$$f_1(z) = \frac{1}{w(z)r(z)} - r_0(z), \quad f_2(z) = \int_E \frac{\widehat{\sigma}_2(x)\, dx}{(z-x)w(x+i0)r(x)}$$

Why $\mathcal{S}_g$ is also a $\mathcal{GN}$ system



- $\tilde{\sigma}_1^j = \sigma_1|_{E_j}, \quad j = 1, \ldots, g+1$
- $\tilde{\sigma}_2^k = \sigma_2|_{F_k}, \quad k = 1, \ldots, p$

HP type I for $\mathcal{S}_g$ and $\Omega(n) = (n, n)$

$$(Q_{\vec{n},0} + Q_{\vec{n},1} f_1 + Q_{\vec{n},2} f_2)(z) = O\left(\frac{1}{z^{2n+2}}\right), z \to \infty$$

Scalar approach for $\mathcal{S}_g$

- Let $\mathcal{R}$ be the Riemann surface of $w(z)$;
- $\boldsymbol{z}$ denotes a point on $\mathcal{R}$;
- $E_j \leftrightarrow L_j$, $L := \bigsqcup_{j=1}^{g+1} L_j$
- $\pi : \mathcal{R} \to \overline{\mathbb{C}}$ denotes the natural projection; $\pi(\boldsymbol{z}) = z$;
- $\forall z \neq e_j$, $\pi^{-1}(z) = \{z^{(0)}, z^{(1)}\} := \{(z, \sqrt{p(z)}), (z, -\sqrt{p(z)})\}$;
- $\mathcal{R} = D^{(0)} \sqcup L \sqcup D^{(1)}$, $D := \overline{\mathbb{C}} \setminus E$;
- $\pi(D^{(0)}) = \pi(D^{(1)}) = D$, $\pi(L) = E$;
- $H_1(\mathcal{R}, \mathbb{Z}) = \langle a_j, b_j \rangle_{j=1}^{g}$, where $a_j \simeq L_j$, $j = 1, \dots, g$

Some notions from Riemann surfaces

- Let $d\Omega_k(z)$, $k = 1, \dots, g$ be normalized holomorphic abelian differentials on $\mathcal{R}$, $\int_{a_j} d\Omega_k = \delta_{kj}$
- Functions $\Omega_k(z) := \int\limits_{e_{2g+2}}^{z} d\Omega_k(\tau)$ are single-valued modulo periods on $\mathcal{R}$

Jacobi inversion problem

Consider the map
$A\colon S^g\mathcal{R} \to \operatorname{Jac}\mathcal{R}, \ \ (z_1, \dots, z_g) \to (A_1(z_1, \dots, z_g), \dots, A_g(z_1, \dots, z_g)),$
where

$$A_k(z_1, \dots, z_g) = \sum_{j=1}^{g} \Omega_k(z_j), \quad k = 1, \dots, g.$$

If some vector $(v_1, \dots, v_g) \in \operatorname{Jac}\mathcal{R}$ is given, then, the problem of determining the unordered set of points $(z_1, \dots, z_g) \in S^g\mathcal{R}$ such that

$$A_k(z_1, \dots, z_g) = v_k, \quad k = 1, \dots, g,$$

is called the *Jacobi inversion problem*.

Generalized potential on a Riemann surface

Consider z to be a local coordinate now; recall that

$$\partial = dz\frac{\partial}{\partial z}, \quad \overline{\partial} = d\overline{z}\frac{\partial}{\partial \overline{z}}, \quad d^c := i(\overline{\partial} - \partial), \quad dd^c = 2i\overline{\partial}\partial.$$

Now, we can write the Poisson equation in the sense of de Rham currents:

$$dd^c v = \nu. \tag{1}$$

Degree 2 currents μ and $dd^c v$ act on test functions $\phi \in C_0^\infty(\mathcal{R})$ as $\nu(\phi) := \int_{\mathcal{R}} \phi \, d\nu$ and $v(dd^c\phi)$ correspondingly.

E. M. Chirka, *Potentials on a Compact Riemann Surface*, Proceedings of the Steklov Institute of Mathematics 301.**1** (2018), pp. 272–303

Definition (Green's bipolar function; Chirka, 2018)

The solution of the equation (1) for $\nu = \delta_{\boldsymbol{p}} - \delta_{\boldsymbol{q}}$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are two arbitrary distinct points in $\mathcal{R}$, is called *Green's bipolar function* with poles $\boldsymbol{p}$ and $\boldsymbol{q}$; it is denoted as $g(\boldsymbol{z}, \boldsymbol{p}, \boldsymbol{q})$.

Normalization of generalized potentials

By $\mathrm{sh}^{\delta}_{\rho}\mathcal{R}$ we will denote the Banach space of δ-subharmonic functions on $\mathcal{R}$, i.e. that can be locally represented as the difference of subharmonic functions, equipped with the following norm:

$$\|v\|_{\rho} := \left| \int_{\mathcal{R}} v\omega_{\rho} \right| + \|dd^c v\|_C .$$

Let ψ be a continuous linear functional on $\mathrm{sh}^{\delta}_{\rho}\mathcal{R}$ such that $\psi(\mathcal{R}) = 1$; now, we will call the space

$$\mathrm{Pot}_{\psi}(\mathcal{R}) := \{\nu \in \mathrm{sh}^{\delta}_{\rho}\mathcal{R} \mid \nu(\psi) = 0\}$$

the space of ψ-normalized potentials.

Scalar approach on the Riemann surface for $\mathcal{S}_g$

Set $\Phi(z) := \log|\phi_g(z)|$, where $\phi_g(z)\colon \mathcal{R} \to \overline{\mathbb{C}}$ is a multi valued conformal map with a single-valued modulus.

$\forall \mu \in \boldsymbol{M}^1_\circ(F^{(1)})$

$\forall \psi \in C(\mathrm{sh}^\delta_\rho \mathcal{R})$

$$P^{\mu}_{\psi}(z) := \int_{F^{(1)}} \left[\log \frac{1}{|z-t|} + g_\psi(z, \infty^{(0)}, t)\right] d\mu(t)$$

$$J^{\psi}_{\Phi}(\mu) := \int_{F^{(1)}} P^{\mu}_{\psi}(z)\, d\mu(z) + 2 \int_{F^{(1)}} \Phi(z)\, d\mu(z)$$

Scalar approach on the Riemann surface for $\mathcal{S}_g$

Let us take a normalized area measure ω of the zero sheet $D^{(0)}$ of $\mathcal{R}$ as ϕ; by $\boldsymbol{M}_1^{\circ}(F^{(1)})$ we will denote the set of all Borel probability measures supported on $F^{(1)}$ and with finite logarithmic energy.

Theorem (Lopatin, 2021)

There is a unique measure $\boldsymbol{\lambda} := \boldsymbol{\lambda}_{F^{(1)}} \in \boldsymbol{M}_1^{\circ}(F^{(1)})$ *with the following property:*

$$J_{\Phi}^{\omega}(\boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda} \in \boldsymbol{M}_1^{\circ}(F^{(1)})} J_{\Phi}^{\omega}(\boldsymbol{\lambda}).$$

It can be described completely via the following equilibrium property:

$$P_{\omega}^{\boldsymbol{\lambda}}(\boldsymbol{z}) + \Phi(\boldsymbol{z}) \begin{cases} \equiv w_F, & \boldsymbol{z} \in S(\boldsymbol{\lambda}) \\ \geqslant w_F, & \boldsymbol{z} \in F^{(1)} \setminus S(\boldsymbol{\lambda}). \end{cases}$$

Moreover,

$$\frac{1}{n}\chi(Q_{n,2}) \to \lambda.$$

Recovering of a meromorphic function on $\mathcal{R}$ via its divisor

$2\partial g(\boldsymbol{z},\boldsymbol{p},\boldsymbol{q})$ is a meromorphic abelian differential of third kind on $\mathcal{R}$ with simple poles at the points $\boldsymbol{p}$ and $\boldsymbol{q}$ and purely imaginary periods; it has residues ± 1 at that points respectively.

Let $\sum_{j=1}^{n}\boldsymbol{q}_j-\sum_{j=1}^{n}\boldsymbol{p}_j$ be an arbitrary divisor satisfying Abel's theorem, i.e. indeed is the divisor of some function $F(\boldsymbol{z})$ meromorphic on $\mathcal{R}$. Consider the differential

$$U(\boldsymbol{z}):=d\log F(\boldsymbol{z})+\sum_{j=1}^{n}2\partial g(\boldsymbol{z},\boldsymbol{p}_j,\boldsymbol{q}_j);$$

it is holomorphic on the whole $\mathcal{R}$, and has purely imaginary periods; as is well known, it is equal to zero. Therefore,

$$F(\boldsymbol{z})=\exp\left\{-\sum_{j=1}^{n}\int_{\boldsymbol{e_{2g+2}}}^{\boldsymbol{z}}2\partial g(\boldsymbol{\tau},\boldsymbol{p}_j,\boldsymbol{q}_j)\right\}$$

Sketch of the proof

- Obtain the orthogonality condition for $Q_{n,2}$ on the zero sheet $D^{(0)}$ of $\mathcal{R}$;
- Lift them onto the first sheet $D^{(1)}$ of $\mathcal{R}$;
- Interpret the corresponding weight function as a function on the whole $\mathcal{R}$; as polynomials are single-valued function, their values at $\mathbf{z} = z^{(0)}$ and $\mathbf{z} = z^{(1)}$ coinside; therefore, we can consider the whole orthogonality conditions written on the whole $\mathcal{R}$;
- Treat the weight function via the procedure of recovering the meromorphic function on the Riemann surface from its divisor to obtain it in form expressed in terms of abelian integrals;
- Choose the normalization of the bipolar Green functions and rewrite the preceding form in the terms of g_ψ's;
- Treat the obtained orthogonality relations on $\mathcal{R}$ via Chirka's technique of analysis of potentials on Riemann surfaces to prove the existence and uniqueness of the energy minimizing measure and then use the GRS method to prove the equilibrium property.

Something about the proof

$P_{n,0}, P_{n,1} \not\equiv 0$: $\deg P_{n,1} \leq n$ and

$$H_n(z) = (P_{n,0} + P_{n,1} f_1)(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty$$

$$\int_{F^{(0)}} Q_{n,2}(x) \{q_{m,1}(x) H_{n+m+1}(x) + q_{m,2}(x) H_{n+m}(x)\} \, d\sigma(x) = 0$$

$$-(m-1)\infty^{(1)} - m\infty^{(0)} - \sum_{k=1}^{g} z^*_{n+m,k} + \sum_{j=1}^{n-1} a_{n,j} + \sum_{k=1}^{g} \widetilde{z}_{n,k}.$$