Holographic RG flows of 3d $\mathcal{N} = 2$ supergravity model

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Gravity in 3-dimensional $AdS_3 \Leftrightarrow$ 2d conformal field theory.

The action of the 3-dimensional gravitational system:

$$S = \frac{1}{16\pi G_3} \int d^3x \sqrt{|g|} \left(R - \frac{1}{a^2} (\partial\phi)^2 - V(\phi) \right) + G.H.Y.,$$

where $G.H.Y.$ is the Gibbons–Hawking–York boundary term, and the potential is

$$V(\phi) = 2\Lambda \cosh^2 \phi \left[(1 - 2a^2) \cosh^2 \phi + 2a^2 \right],$$

Λ is a cosmological constant ($\Lambda = -1$), a is a finite constant.

(Sezgin & Sundell, '00; Deger '02)

Equations of motion

EOM in general view:

$$\mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{g}_{\mu\nu}\mathbf{R} = \mathbf{T}_{\mu\nu}, \quad T_{\mu\nu} = \frac{1}{a^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\sigma\phi\partial^\sigma\phi\right) - \frac{1}{2}g_{\mu\nu}V,$$

$$\Box\phi = \frac{\mathbf{a^2}}{\mathbf{2}}\frac{\partial\mathbf{V}(\phi)}{\partial\phi}, \qquad \Box = \frac{1}{\sqrt{|g|}}\partial_\mu\left(g^{\mu\nu}\sqrt{|g|}\partial_\nu\right).$$

Anzatz for the metric in domain wall coordinates:

$$ds^2 = e^{2A(w)}\left(-dt^2 + dx^2\right) + dw^2, \quad w \in (w_0, +\infty), \quad \phi = \phi(w).$$

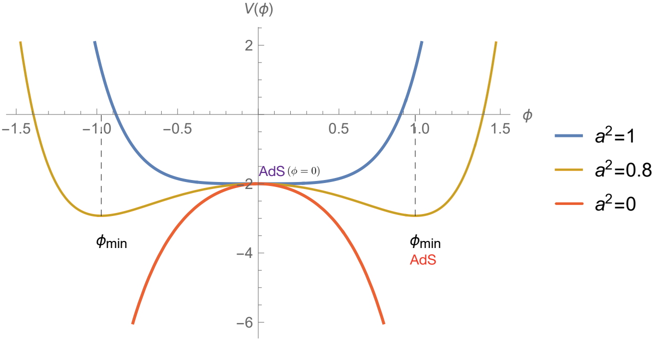
And EOM in terms of w-variable:

$$\boxed{V_\phi = \frac{dV}{d\phi}} \qquad \begin{cases} 2\dot{A}^2 + V - \frac{\dot{\phi}^2}{a^2} = 0, \\ \ddot{A} + \frac{\dot{\phi}^2}{a^2} = 0, \\ \ddot{\phi} + 2\dot{A}\dot{\phi} - \frac{a^2}{2}V_\phi = 0. \end{cases}$$

The behavior of the potential

$$\boxed{\mathrm{V}(\phi) = 2\Lambda \cosh^2\phi\left[(1-2a^2)\cosh^2\phi + 2a^2\right]}$$

$$0 < a^2 \leq 1/2, \quad 1/2 < a^2 < 1, \quad a^2 \geq 1 \quad \Rightarrow \quad \phi_1 = 0, \quad \phi_{2,3} = \frac{1}{2}\ln\left(\frac{1 \pm 2|a|\sqrt{1-a^2}}{2a^2-1}\right)$$



Dynamical system

Let's introduce variables X and Z:

$$X = \frac{\dot{\phi}}{\dot{A}}, \quad Z = e^{-\phi}, \quad Z \in (0, +\infty) \quad \forall \phi.$$

Kiritsis et.al. '09
Aref'eva, Golubtsova, Policastro, Skugoreva (work in progress)

Then we have the following representation of the Einstein equations:

$$\frac{dZ}{dA} = f(Z, X), \tag{1}$$
$$\frac{dX}{dA} = g(Z, X). \tag{2}$$

where the functions f and g are defined as:

$$\begin{aligned} f(Z,X) &= -ZX, \\ g(Z,X) &= \left(\frac{X^2}{a^2} - 2\right)\left(X + 2a^2 \cdot \frac{(2a^2(Z^8-1) - (Z^2-1)(Z^2+1)^3)}{(Z^2+1)^4 - 2a^2(Z^4-1)^2}\right). \end{aligned}$$

The equilibrium points

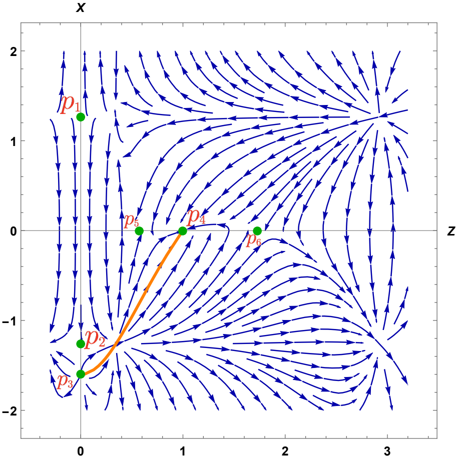
The equilibrium points of the dynamical system may be found as the result of the system:

$$\begin{cases} f(Z,X)\Big|_{Z_c,X_c} = 0, \\ g(Z,X)\Big|_{Z_c,X_c} = 0. \end{cases}$$

Then the points:

1. $Z_c = 0, \quad X_c = -a\sqrt{2},$
2. $Z_c = 0, \quad X_c = a\sqrt{2},$
3. $Z_c = 0, \quad X_c = -2a^2,$
4. $Z_c = 1, \quad X_c = 0,$

5 − 6. $Z_c = \sqrt{\dfrac{1 \pm 2|a|\sqrt{1-a^2}}{2a^2-1}}, \quad X_c = 0.$



Phase portrait for $a^2 = 0.8$.

Classification of the equilibrium points

Perturbation of a dynamical system (1)-(2) in the vicinity of the equilibrium points leads to equations:

$$\frac{d}{dA}\begin{pmatrix}\delta Z\\ \delta X\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta Z\\ \delta X\end{pmatrix} = \begin{pmatrix}\frac{\partial f}{\partial Z} & \frac{\partial f}{\partial X}\\ \frac{\partial g}{\partial Z} & \frac{\partial g}{\partial X}\end{pmatrix}\begin{pmatrix}\delta Z\\ \delta X\end{pmatrix}.$$

Eigenvalues λ in every equilibrium point may be found as a result of solving the characteristic equation:

$$\begin{vmatrix}\mathcal{M}_{11}-\lambda & \mathcal{M}_{12}\\ \mathcal{M}_{21} & \mathcal{M}_{22}-\lambda\end{vmatrix}_{Z=Z_c,X=X_c} = 0.$$

With respect to the λ and a^2 we obtained the following classification:

point	$0 < a^2 < \frac{1}{2}$	$a^2 = \frac{1}{2}$	$\frac{1}{2} < a^2 < 1$
1	$a < 0$: unst.node	$a = -\frac{1}{\sqrt{2}}$: need another approach	$a < 0$: saddle
	$a > 0$: saddle	$a = \frac{1}{\sqrt{2}}$: saddle	$a > 0$: saddle
2	$a < 0$: saddle	$a = -\frac{1}{\sqrt{2}}$: saddle	$a < 0$: saddle
	$a > 0$: unst.node	$a = \frac{1}{\sqrt{2}}$: need another approach	$a > 0$: saddle
3	saddle	need another approach	unst.node
4	stab.node	stab.node	stab.node
5-6	saddle	saddle	saddle

Asymptotic behavior near the critical points

1. $Z_c = 0,\ X_c = \sqrt{2}a$. The metric and the dilaton are given by

$$ds^2 \cong \left|\frac{2\dot{A}_0(w-w_0)+1}{2\dot{A}_0(w_1-w_0)+1}\right|(-dt^2+dx^2)+dw^2,$$
$$\phi = \frac{a}{\sqrt{2}}\ln\left|\frac{2\dot{A}_0(w-w_0)+1}{2\dot{A}_0(w_1-w_0)+1}\right|+\phi_0.$$

Since $Z_c = 0$, then $\phi \to +\infty$, so $a > 0$ and $w \to w_0 - \frac{1}{2\dot{A}_0}$, or $w \to +\infty$ and $a < 0$. The potential $\phi \to +\infty$: $V \to \pm\infty$, however from the EOM $V = 0, \quad \frac{dV}{d\phi} = 0$. $\Rightarrow$ NOT A SOLUTION TO EOM.

2. $Z_c = 0, \quad X_c = -\sqrt{2}a$. The metric and the dilaton are given by

$$ds^2 \cong \left|\frac{2\dot{A}_0(w-w_0)+1}{2\dot{A}_0(w_1-w_0)+1}\right|(-dt^2+dx^2)+dw^2,$$
$$\phi = -\frac{a}{\sqrt{2}}\ln\left|\frac{2\dot{A}_0(w-w_0)+1}{2\dot{A}_0(w_1-w_0)+1}\right|+\phi_0.$$

NOT A SOLUTION TO EOM.

3. $Z_c = 0,\ X_c = -2a^2$. The metric and the dilaton are

$$ds^2 \cong \left|\frac{4a^2\dot{A}_0(w-w_0)+1}{4a^2\dot{A}_0(w_1-w_0)+1}\right|^{\frac{1}{2a^2}}(-dt^2+dx^2)+dw^2,$$

$$\phi = -\frac{1}{2}\ln\left|\frac{4a^2\dot{A}_0(w-w_0)+1}{4a^2\dot{A}_0(w_1-w_0)+1}\right|+\phi_0.$$

Since $Z_c = 0$, $\phi \to +\infty$, and the potential behaves as

$$V \sim \begin{cases} -\infty, & \text{for} \quad 0 \le a^2 \le \frac{1}{2}, \\ +\infty, & \text{for} \quad a^2 > \frac{1}{2}. \end{cases}$$

SOLVES EOM $\forall a$ and $w \to w_0 - \dfrac{1}{4a^2\dot{A}_0}$.

For any a the asymptotic solution describes a supersymmetric non-conformal vacuum and corresponds to the exact solution and near $w = 0$.

4. $Z_c = 1,\ X_c = 0$. The metric and the dilaton are given by

$$ds^2 \approx e^{-2\sqrt{-\Lambda}(w-w_0)}\left(-dt^2+dx^2\right)+dw^2, \quad \phi = 0, \quad V = 2\Lambda.$$

where w_0 is a constant of integration. SOLVES EOM $\forall a$.

The asymptotic metric with $\phi = 0$ represents a supersymmetric AdS_3 vacua for any a. It corresponds to the exact solution with $w \to \infty$.

5. $Z_c = \sqrt{\frac{1-2|a|\sqrt{1-a^2}}{2a^2-1}}, \quad X_c = 0$. The metric and the scalar field are

$$ds^2 \approx e^{2a^2\sqrt{-\frac{\Lambda}{2a^2-1}}(w-w_0)}(-dt^2+dx^2)+dw^2, \quad \phi = \ln\sqrt{\frac{1-2|a|\sqrt{1-a^2}}{2a^2-1}}$$

$$V = \frac{2a^4\Lambda}{2a^2-1},$$

where w_0 is a constant of integration. SOLVES EOM $\forall a^2 > \frac{1}{2}$. The solution is a non-susy AdS_3 and corresponds to the extremum of V: $\phi_3 = \frac{1}{2}\ln\left(\frac{1-2|a|\sqrt{1-a^2}}{2a^2-1}\right)$.

6. $Z_c = \sqrt{\frac{1+2|a|\sqrt{1-a^2}}{2a^2-1}}, \quad X_c = 0$. The metric and the scalar field are

$$ds^2 \approx e^{2a^2\sqrt{-\frac{\Lambda}{2a^2-1}}(w-w_0)}(-dt^2+dx^2)+dw^2, \quad \phi = \ln(\sqrt{\frac{1+2|a|\sqrt{1-a^2}}{2a^2-1}})$$

$$V = \frac{2a^4\Lambda}{2a^2-1},$$

where w_0 is a constant of integration. SOLVES EOM $\forall a^2 > \frac{1}{2}$. The solution is a non-susy AdS_3, related to the extremum of V: $\phi_2 = \frac{1}{2}\ln\left(\frac{1+2|a|\sqrt{1-a^2}}{2a^2-1}\right)$.

AdS/CFT correspondence

- $\phi_0(x)$ is a source of $\mathcal{O}(x)$ in d-dim CFT:

$$S = S_{\text{CFT}} + \int d^d x \phi_0(x)\mathcal{O}(x), \quad Z(\phi_0) = \langle e^{\int d^d x \phi_0 \mathcal{O}(x)} \rangle_{CFT}$$

AdS/CFT-correspondence:

$$\langle e^{\int d^d x \phi_0 \mathcal{O}(x)} \rangle_{CFT} = e^{-S_{AdS}(\tilde{\phi})}|_{\lim_{z\to 0}(\tilde{\phi}(x,z)z^{\Delta-d})=\phi_0(x)}$$

The Holographic Renormalization Group (RG) arises as a result of holographic renormalization.

$$S = \int d^{(d+1)} x \sqrt{-g}(R - \Lambda) \quad \Longrightarrow S = \int d^{d+1} x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right]$$

Domain wall solution:

$$ds^2 = e^{2A(w)} \eta_{ij} dx^i dx^j + dw^2, \quad \phi = \phi(w)$$

E. Akhmedov'98; de Boer, Verlinde, Verlinde'98, Skenderis et. al.'98; Skenderis'99

Holographic RG flows

As a result we have the following holographic dictionary:

- e^{A} corresponds to the energy scale E of the dual field theory
- $\lambda = e^{\phi(w)}$ must be identified as running coupling of the field theory
- Connection with β-function

$$\beta = \frac{d\lambda}{d\log E}\bigg|_{QFT} = \frac{d\phi}{dA}\bigg|_{Holo}$$

- conformal symmetry is recovered only in fixed points
- The holographic RG is geometrized

Then physical meaning of the dynamical system:

- Variable $X = \frac{\dot{\phi}}{\dot{A}}$ is associated with overstretched β-funtion (Kiritsis et.al. '09)
- Variable $Z = e^{-\phi}$ is related with the potential (Aref'eva, Golubtsova, Policastro, Skugoreva (work in progress))
- EOM (we assume that $A(w)$ decreases along the flow):

$$\frac{dZ}{dA} = f(Z,X),$$
$$\frac{dX}{dA} = g(Z,X).$$

Holographic RG flows

	$V(\phi)$	Type according to energy scale	UV/IR
p_3	$V \to -\infty,\ a^2 \in (0; \frac{1}{2})$ $V \to +\infty,\ a^2 \in (\frac{1}{2}; 1]$	unstable (saddle, $a^2 \in (0; \frac{1}{2})$) stable (stable node, $a^2 \in (\frac{1}{2}; 1]$)	IR
p_4	const	unstable (unstable node for all a)	UV
p_5	const	unstable (saddle for all a)	IR/UV, $a^2 \in (\frac{1}{2}; 1)$
p_6	const	unstable (saddle for all a)	IR/UV, $a^2 \in (\frac{1}{2}; 1)$

Таблица: The fixed points and their characteristics.

Holographic RG flows for $a^2 \leq 1/2$

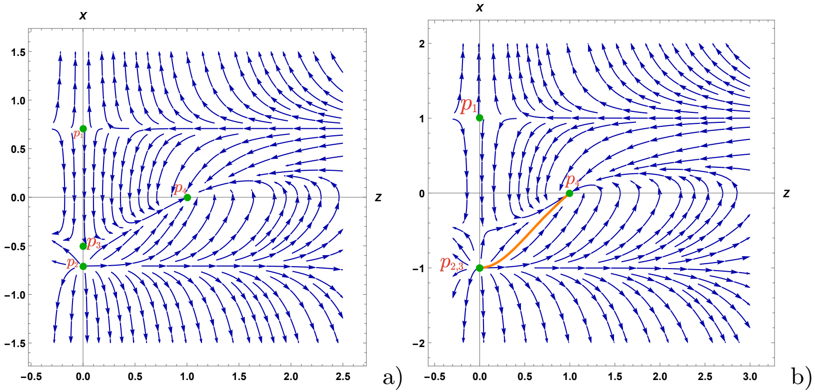
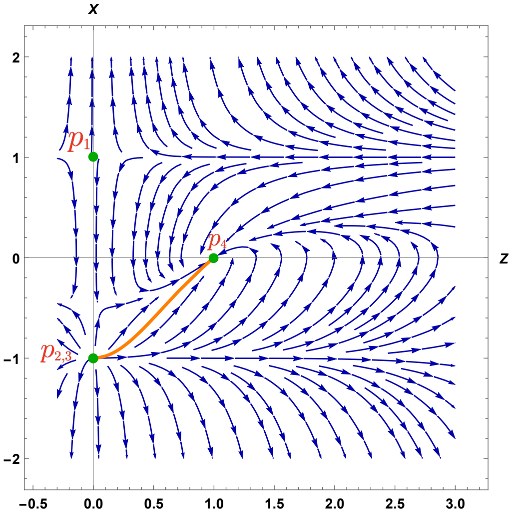


Рис.: a) Phase portrait for $a^2 = 0.25$; b) Phase portrait for $a^2 = 0.5$.

Possible RG flows for $a^2 \leq \frac{1}{2}$

1. p_4 (UV, AdS_3, $\phi = 0$) to p_3 (IR, $\phi \to +\infty$)



At this phase portrait: $a^2 = 0.5$

Holographic RG flows for $a^2 > 1/2$

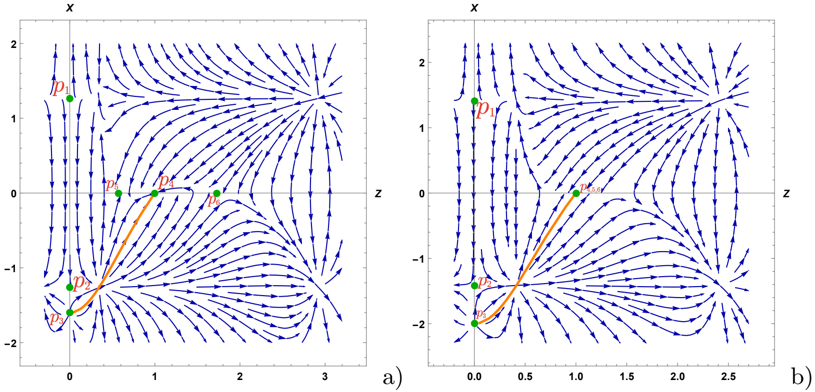
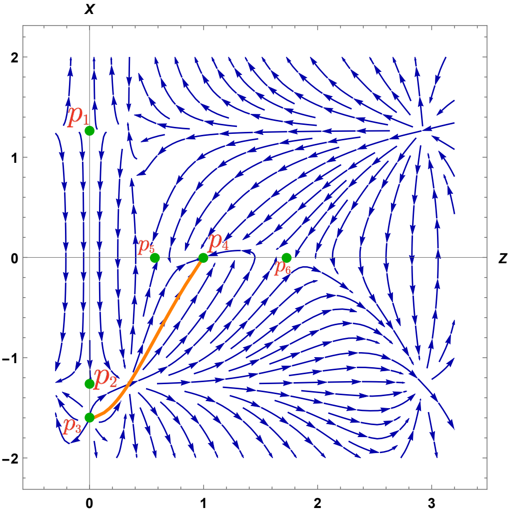


Рис.: a) Phase portrait for $a^2 = 0.8$; b) Phase portrait for $a^2 = 1$.

Possible RG flows for $a^2 > \frac{1}{2}$

1. p_4 (UV, AdS_3, $\phi = 0$) to p_3 (IR, $\phi \to +\infty$)
2. p_4 (UV, AdS_3, $\phi = 0$) to p_5, p_6 (IR, $\phi \to +\infty$)
3. p_5, p_6 (UV, AdS_3, $\phi = 0$) to p_3 (IR, $\phi \to +\infty$)



At this phase portrait: $a^2 = 0.8$.

The orange line represents the exact solution of (1)-(2), Deger'02.

Summary: Possible RG flows

- for $a^2 \leq \frac{1}{2}$:
  1. the flow starts at the unstable susy UV fixed point (p_4) with AdS metric and $\phi = 0$ and goes to the unstable susy IR fixed point (p_3) related to the hyperscaling violating metric and $\phi \to +\infty$, $V \to -\infty$; it corresponds to the exact solution.

- for $a^2 > \frac{1}{2}$:
  1. the RG flow starts at the unstable AdS UV fixed point (p_4) with $\phi = 0$ and flows to the stable IR fixed point (p_3) with $\phi \to +\infty$ and $V(\phi) \to +\infty$; it corresponds to the exact one; it is singular since V is unbounded from above.
  2. the flow begins at the unstable susy UV fixed point (p_4) with AdS_3 and $\phi = 0$ and flows to the unstable non-susy IR fixed point p_5 (p_6) with AdS and $\phi = const$.
  3. the flow can start from the unstable non-susy AdS fixed point p_5 (p_6) and goes to the stable IR $(p_3, \phi \to +\infty)$.

Conclusion

Results

- Stability analysis of the fixed points was done
- the metrics near the critical points, corresponding to a constant scalar field, are asymptotically AdS
- near the critical points with the scalar field tending to infinity we have obtained metrics with hyperscaling violation
- solutions near two critical points (with $\phi \to +\infty$) don't fulfill the equations of motion
- for three asymptotically AdS solutions we have found the constraint on the parameter: $a^2 > \frac{1}{2}$, so these solutions correspond to local minima of the potential
- Energy limits (UV/IR) were restored for each fixed point

Prospective questions

- Systematic description of the model for the case with non-zero finite T

  (for another potential: I.Ya.Aref'eva, A.A.Golubtsova, G.Policastro '19)
- Correlation functions in 3d holographic RG flows
- Investigation of a connection with phase transitions in a dual $N = 2$ $d = 2$ conformal field theory

Thank you for attention!