Regular dessins with primitive automorphism groups

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Dessins

Belyĭ's Theorem: a compact Riemann surface X is defined (as an algebraic curve) over an algebraic number field if and only if there is a non-constant meromorphic function (a Belyĭ function)

$$\beta : X \to \Sigma := \mathbb{P}^1(\mathbb{C}) = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\},$$

with at most three critical values (wlog $0, 1, \infty$).

The evidence for this is a dessin $\mathcal{D}$, a combinatorial structure on X giving a simple 'picture' of β. Typically, $\mathcal{D}$ is either

- a bipartite map, i.e. a graph $\beta^{-1}([0, 1])$ embedded in X with black and white vertices over 0 and 1, or
- a tripartite triangular map, i.e. a graph $\beta^{-1}(\hat{\mathbb{R}})$ embedded in X with black, white and red vertices over $0, 1$ and ∞.

The red vertices are the face centres of the bipartite map.

Regular dessins

The automorphism group $G = \mathrm{Aut}\,\mathcal{D}$ is the group of covering transformations of β, the subgroup of $\mathrm{Aut}\,X$ preserving the coloured graph. It acts semi-regularly (freely) on the edges.

In the most symmetric case β is a regular covering, G acts regularly on the edges, and we say that $\mathcal{D}$ is regular.

Simple observation: Every dessin is the quotient of a regular dessin by a group of automorphisms.

This motivates the study of regular dessins and their automorphism groups.

Unfortunately, there are too many of them: every 2-generator finite group is the automorphism group of a regular dessin.
For example, this includes all the finite simple groups.

So, restrict attention to smaller classes of dessins $\mathcal{D}$ and groups G.

Primitivity

If $\mathcal{D}$ is regular, as I will assume from now on, G acts transitively on the sets of black, white and red vertices.

What if we impose a stronger condition, that G acts primitively (i.e. preserving no non-trivial equivalence relations) on one of these sets (wlog the set V of black vertices)?

Motivation (1): Every transitive finite group can be decomposed, by wreath product constructions (Kaloujnine–Krasner Theorem) into a "composition series" of finitely many primitive groups.

(2) Since the classification of finite simple groups much is now known about primitive groups (e.g. the O'Nan–Scott Theorem).

Define a regular dessin $\mathcal{D}$ to be primitive or faithful if G acts primitively or faithfully on the set V of black vertices.

Aims: Classify faithful primitive dessins, then all primitive dessins.

Example: The star dessin $\mathcal{S}t_m$ on the sphere Σ has one m-valent white vertex at 0, and m 1-valent black vertices where $z^m = 1$.

The Belyĭ function is $\beta : z \mapsto 1 - z^m$.

The automorphism group $G \cong \mathrm{C}_m$, generated by $z \mapsto e^{2\pi i/m}z$, is transitive and faithful on the black vertices, and is primitive on them if and only if m is prime.

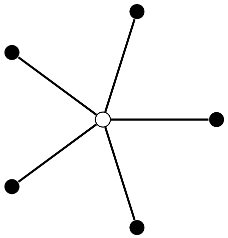


Figure: The star dessin $\mathcal{S}t_5$

Example: The Fermat dessin $\mathcal{F}_n$, on the curve $u^n + v^n = w^n$, is regular, of type (n, n, n) and genus $(n-1)(n-2)/2$, with

$$G = \langle x, y \mid x^n = y^n = [x, y] = 1 \rangle \cong \mathrm{C}_n \times \mathrm{C}_n.$$

This is transitive but not faithful on the black (and white and red) vertices, and is primitive on them if and only if n is prime.

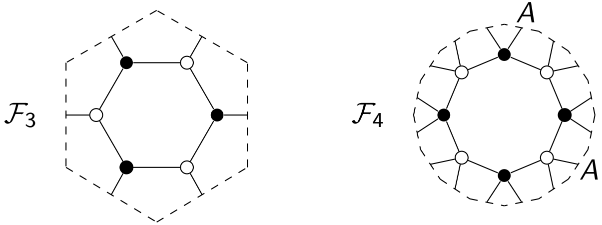


Figure: The Fermat dessins $\mathcal{F}_3$ and $\mathcal{F}_4$.

Example: The quaternion group Q_8 is the automorphism group of a regular dessin $\mathcal{Q}$ of type $(4, 4, 4)$ and genus 2.

It is primitive on the two black vertices, the two white vertices, and the two face-centres.

The central involution (a half-turn) acts trivially on all of them.

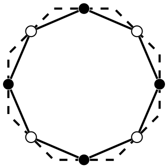


Figure: The dessin $\mathcal{Q}$ (identify opposite sides of the outer octagon).

Standard generators

If $\mathcal{D}$ is regular then G has standard generators x, y, z, rotations around incident black, white and red vertices, satisfying $xyz = 1$. Conversely, any 2-generator group $G = \langle x, y \rangle$ determines a regular dessin, with standard generators x, y and $z := (xy)^{-1}$.

$\mathcal{D}$ is primitive if and only if $\langle x \rangle$ is a maximal subgroup of G. (G-invariant equivalence relations on V correspond to subgroups of G containing the black vertex stabiliser $\langle x \rangle$.)

$\mathcal{D}$ is faithful if and only if $\langle x \rangle$ is a core-free subgroup of G, i.e. the core $\cap_{g \in G} \langle x \rangle^g$ of $\langle x \rangle$ in G, the kernel of the action on V, is $\{1\}$.

Hence look for primitive permutation groups G with a cyclic point-stabiliser $\langle x \rangle$, and choose any $y \in G \setminus \langle x \rangle$.
Then $G = \langle x, y \rangle$ by maximality, so we get a faithful primitive regular dessin $\mathcal{D}$, and every such dessin arises in this way.

The main theorem

Theorem (Mačaj and J, 2023)

Let $\mathcal{D}$ be a regular dessin with black vertex set V and $G = \mathrm{Aut}\,\mathcal{D}$. Then the following are equivalent:

1. *$\mathcal{D}$ is primitive and faithful;*
2. *G is a primitive permutation group on V with cyclic point stabilisers;*
3. *$\mathcal{D}$ is a generalised Paley dessin.*

I have explained (1) $\iff$ (2). (2) $\iff$ (3) is technical, omitted.
To explain (3) I need to define generalised Paley dessins.
They are generalisations of:

- Paley graphs, introduced by Sachs and Erdős & Rényi $\sim$ 1961;
- generalised Paley graphs, introduced by Lim & Praeger, 2009;
- generalised Paley maps, introduced by J in 2013.

Lemma

Let G be a subgroup of the affine group $\mathrm{AGL}_1(q)$ for some prime power $q = p^d$. Then G acts primitively on the field $\mathbb{F}_q$ if and only if

$$G = G_S := \{t \mapsto at + b \mid a \in S,\ b \in \mathbb{F}_q\}$$

for some subgroup S of order n in the multiplicative group $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, where either $n = d = 1$ (so $G \cong \mathrm{C}_p$), or $n > 1$ and S satisfies the following equivalent conditions:

(a) *S acts on $\mathbb{F}_q$ as an irreducible subgroup of $\mathrm{GL}_d(p)$;*

(b) *S generates the additive group of $\mathbb{F}_q$;*

(c) *n divides $p^d - 1$ but not $p^i - 1$ for $0 < i < d$;*

(d) *p has multiplicative order $d \bmod (n)$.*

G_S is a semidirect product $T \rtimes S$, where T ($\cong \mathbb{F}_q$) is the group of translations ($a = 1$) and S ($\cong \mathrm{C}_n$) is the stabiliser of 0 ($b = 0$).

Generalised Paley dessins

Let $G = G_S$ be a primitive subgroup of $\mathrm{AGL}_1(q)$, where $q = p^d$ and $S \leq \mathbb{F}_q^*$, so $S \cong \mathrm{C}_n$ for some $n \mid q-1$.

Let x generate S, let $y \in G \setminus S$, and define $z = (xy)^{-1}$.

Then $G = \langle x, y, z \rangle$ with $xyz = 1$, so these are standard generators for a faithful primitive regular dessin $\mathcal{D}$, with

- edge set G, permuted regularly by G,
- black vertex set $V = G/S = T = \mathbb{F}_q$, permuted primitively,
- white vertex set $G/\langle y \rangle$, permuted transitively,
- red vertex (= face centre) set $G/\langle z \rangle$, permuted transitively.

This is a generalised Paley dessin. It has type (n, m, l) where these are the orders of x, y and z. By the main theorem, every faithful primitive regular dessin has this form.

Example: $n = 2$

Every prime $p > 2$ has multiplicative order $d = 1 \bmod (2)$, so $q = p^d = p$ and $G = T \rtimes S \cong \mathrm{C}_p \rtimes \mathrm{C}_2 \cong \mathrm{D}_p$ with $\langle x \rangle = \mathrm{C}_2$.

If we choose a second generator $y \in \mathrm{C}_p$ then x, y and z have orders $2, p$ and 2, so the dessin $\mathcal{D}$ has type $(2, p, 2)$ and genus 0. It is a spherical beachball $\mathcal{B}_p$, with p equatorial black vertices of valency 2, and 2 polar white vertices of valency p.

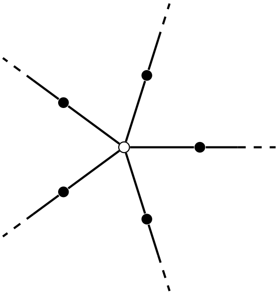


Figure: The dessin $\mathcal{B}_5$ (with a white vertex at infinity).

If instead we choose $y \in G \setminus \mathrm{C}_p$ then x, y and z have orders $2, 2$ and p, so the dessin $\mathcal{D}$ has type $(2, 2, p)$ and genus 0.
It has p black and p white vertices of valency 2, alternating around the equator.
It is the white/red dual of the beachball dessin $\mathcal{B}_p$.

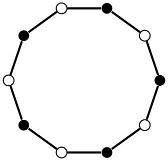


Figure: The white/red dual of the dessin $\mathcal{B}_5$.

Example: $n = 3$

A prime $p \neq 3$ has order $d = 1$ or 2 mod (3) as $p \equiv \pm 1$ mod (3), so $q = p^d = p$ or p^2 and $G = T \rtimes S \cong \mathbb{F}_q \rtimes \mathrm{C}_3$ with $\langle x \rangle = \mathrm{C}_3$.

If we choose $y \in T$ then $\mathcal{D}$ has type $(3, p, 3)$ and genus $g = (p-1)/2$ or $(p-1)(p-2)/2$ as $p \equiv \pm 1$ mod (3).

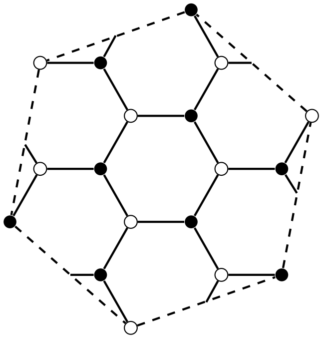
Example: if $p = 2$ then $d = 2$, $G \cong \mathrm{A}_4$ and $\mathcal{D}$ is the tetrahedron, with white vertices at the midpoints of its edges.

Example: if $p = 7$ then $d = 1$, $G \cong \mathrm{C}_7 \rtimes \mathrm{C}_3$ and $\mathcal{D}$ has genus 3. It embeds the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ in Klein's quartic curve, with the black and red vertices representing the 7 points and 7 lines.

$n = 3$, continued

If we choose $y \in Tx$ then $\mathcal{D}$ has type $(3,3,3)$ and genus 1.

Example: If $p = 7$ then $d = 1$, $G \cong \mathrm{C}_7 \rtimes \mathrm{C}_3$ and $\mathcal{D}$ has genus 1. It embeds the Fano plane in the hexagonal torus, with black and white vertices representing the 7 points and 7 lines.



An example with $n = 6$

Example: If $p = 13$ then $d = 1$ and $G \cong \mathrm{C}_{13} \rtimes \mathrm{C}_6$.
If we choose $y \in Tx^3$ then $\mathcal{D}$ has type $(6, 2, 3)$ and genus 1.
Ignoring white vertices of valency 2 gives a torus embedding of the Paley graph P_{13}.

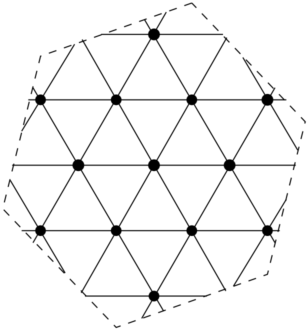


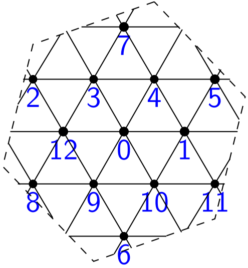
Figure: A torus embedding of the Paley graph P_{13}.

Paley graphs

More generally, the Paley graph P_q (Raymond Paley, 1907–33) has vertex set $\mathbb{F}_q$ where $q \equiv 1 \bmod (4)$, with vertices v, w adjacent if and only if $v - w$ is a quadratic residue (non-zero square) in $\mathbb{F}_q$. These graphs are important in several areas of mathematics.

If $\mathcal{D}$ is a generalised Paley dessin with $q \equiv 1 \bmod (4)$ and $n = (q-1)/2$ (so S is the group of quadratic residues in $\mathbb{F}_q$), and with $y \in Tx^{(q-1)/4}$ (so $y^2 = 1$), then $\mathcal{D}$ has type $(n, 2, l)$ and (ignoring 2-valent white vertices) it embeds the Paley graph P_q.

P_{13}



Enumeration

Generating pairs x, y and x', y' for G determine isomorphic dessins if and only if $x \mapsto x'$ and $y \mapsto y'$ for some automorphism of G.

For generalised Paley dessins, with $G = G_S$ and $\mathrm{Aut}\, G = \mathrm{A\Gamma L}_1(q) = \mathrm{AGL}_1(q) \rtimes \mathrm{Gal}\, \mathbb{F}_q$, this is equivalent to

- x, x' ($\in S \le \mathbb{F}_q^*$) are equivalent under $\mathrm{Gal}\, \mathbb{F}_q \cong \mathrm{C}_d$;
- y, y' ($\in G \setminus S$) lie in the same coset Tx^c of T in G.

Using this, one can show that if p is coprime to n there are $n\phi(n)/d$ generalised Paley dessins of valency n with $|V| = q = p^d$.
$\phi(n)$ is the number of choices for x generating $S \cong \mathrm{C}_n$;
$d = |\mathrm{Gal}\, \mathbb{F}_q|$, inducing automorphisms permuting generators x;
$n = |G : T|$ is number of choices for a coset Tx^c in G containing y.

Example If $n = 3$ then $d = 1$ or 2 as $p \equiv \pm 1 \bmod (3)$, so for each $p \neq 3$ there are six or three generalised Paley dessins $\mathcal{D}$ resp.

Real dessins

A dessin $\mathcal{D}$ is real if the Belyĭ pair (X, β) is defined over a real number field.

For a regular dessin, this is equivalent to some automorphism of $G = \mathrm{Aut}\,\mathcal{D}$ inverting the standard generators x and y.

For a generalised Paley dessin, this is equivalent to the condition:

$$d \text{ is even and } p^{d/2} \equiv -1 \bmod (n).$$

(This represents the involution $t \mapsto t^{\sqrt{q}} = t^{p^{d/2}}$ in $\mathrm{Gal}\,\mathbb{F}_q$ $(q = p^d)$ inducing $t \mapsto t^{-1}$ on $S \leq \mathbb{F}_q^*$ and thus inverting x and y in G.)

Example If $n = 3$ then $\mathcal{D}$ is real if and only if $p \equiv -1 \bmod (3)$. If $p \equiv 1 \bmod (3)$ the dessins form chiral (mirror-image) pairs.

We have a characterisation of the pairs n, p satisfying the above condition, but it is too complicated to state here.

Hole operations and Galois conjugacy

Given a valency n, field characteristic p coprime to n, and $c \in \mathbb{Z}_n$ (with $y \in Tx^c$), there are $\phi(n)/d$ generalised Paley dessins $\mathcal{D}$, one for each orbit of $\mathrm{Gal}\,\mathbb{F}_q$ on the $\phi(n)$ generators x of S ($\cong \mathrm{C}_n$).

These $\phi(n)/d$ dessins are all equivalent under the hole operations

$$H_j : x \mapsto x^j, y \mapsto y^j \quad \text{for} \quad j \in \mathbb{Z}_n^*,$$

introduced by Coxeter and Wilson for maps.

These operations form a group of order $\phi(n)$, preserving the automorphism group, and in this case (though not in general) also the type and genus, of a regular dessin $\mathcal{D}$.

A theorem of Streit, Wolfart and J (Proc. LMS 2010) then implies that these dessins are defined over a subfield of the nth cyclotomic field, and are equivalent under the Galois group of that field.

Defining equations

Defining equations for a particularly simple subset of these dessins are already known (see Streit, Wolfart & J 2010, and the book on dessins by Wolfart & J 2016, for these and similar examples).

Let $p \equiv 1 \bmod (n)$, so $d = 1$ and $G = \mathrm{C}_p \rtimes \mathrm{C}_n$. If $c = 0$ then $\mathcal{D}$ has type (n, p, n) and genus $(n-2)(p-1)/2$; it embeds the complete bipartite graph $K_{p,n}$, with p black and n white vertices.

$\mathcal{D}$ is a p-sheeted regular covering of the spherical dessin $\mathcal{D}/T$, which has a black vertex of valency n at 0, and white vertices of valency 1 at the nth roots of 1, where the covering is branched.

This leads to an affine model

$$w^p = \prod_{j=1}^{n} \left(z - \zeta_n^j\right)^{u^j} \quad (\zeta_n := e^{2\pi i/n})$$

of X, with Belyĭ function $\beta : (w, z) \mapsto z^n$, where x acts on $\mathbb{F}_p$ by $t \mapsto ut$, $u \in \mathbb{Z}_n^*$. Automorphisms of order p are obvious, n less so.

Non-faithful primitive dessins

If G acts primitively but not faithfully, with kernel $K > 1$, on the black vertex set V of a regular dessin $\mathcal{D}$, then $\overline{G} := G/K$ acts primitively and faithfully on the black vertex set $\overline{V} = V$ of the regular dessin $\overline{\mathcal{D}} := \mathcal{D}/K$, so by the main theorem $\overline{\mathcal{D}}$ is a generalised Paley dessin.

Since K is the intersection of the black vertex stabilisers in G, and these are cyclic, K is also cyclic. Thus we have

Theorem (Mačaj & J, 2023)

If a regular dessin is primitive but not faithful then it is a cyclic regular cover of a generalised Paley dessin, branched over the black vertices (and possibly the white and red vertices).

Example Take a generalised Paley dessin $\overline{\mathcal{D}}$ with automorphism group $\overline{G} = T \rtimes \overline{S}$, and a cyclic group S with an epimorphism $S \to \overline{S}$ with kernel $K > 1$. Then $G := T \rtimes S$, with S acting on T via the action of $\overline{S}$ on T, yields the required cover $\mathcal{D}$ of $\overline{\mathcal{D}}$.

Thank you for listening !