On the contribution of Soviet mathematicians to the development of nonlinear functional analysis in the 1920s–1960s.

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Purpose of the study

Reconstruction of a holistic picture of the formation and development of the NFA* consisting of:

- theory of functional spaces;
- the calculus of variations in large;
- topological methods;
- variational methods;
- the theory of positive operators;
- theories of branching and bifurcations;
- theoretical justification of approximate methods

within the framework of the Soviet mathematical school in the indicated period.

*NFA – non-linear functional analysis.

On the development of the NIE theory until the mid-1920s, as a prerequisite for the emergence of the NFA

A.M. Lyapunov (1905-1906)

St. Petersburg University

$$R\zeta - \frac{1}{4\pi}\int_S \frac{\zeta' d\sigma'}{D} = W(\zeta) - L\zeta + const, \quad (1.1)$$

$\zeta = \zeta(\theta,\psi)$; $R=R(\theta,\psi)$; $D=D(\theta,\psi)$.

$W(\zeta)$ - integro-power form

Purpose: search for new figures of equilibrium of a rotating fluid

Solution of (1.1): $\zeta = \sum_{n=1}^{\infty} \zeta_n \alpha^n$

Result: reduction of (1.1) to a transcendental system of equations containing the parameters of the bifurcation figure.

Lyapunov–Schmidt equation

E. Schmidt (1908)

$$u(x) - \int_a^b K(x,s)u(s)ds = f(x) +$$
$$+ \int_a^b K_1(x,s)v(s)ds - \sum_{m+n\geq 2} U_{mn}\binom{x}{u,v} \quad (1.2)$$

Method: application of the Fredholm resolvent + Puiseux's theorem on the branching of systems of equations.(1850)

Special case of (1.2):

$$u(x) + \int_a^b K(x,\xi)u(\xi)d\xi = v(x) +$$
$$+\int_0^1 \int_0^1 K(x,y_1,y_2)u^2(y_1)v^3(y_2)dy_1dy_2.$$

Lyapunov–Schmidt reduction.

Non-uniqueness of solutions!

Urysohn equation

$$y(x) = \lambda \int_a^b K(x,s,y(s))ds + \varphi(x), \qquad \lambda > 0 \quad (1.3)$$

а) $K(x,y,0) = 0$;

б) $K_u(x,y,u) > 0$ is continuous and decreases monotonically as u increases.

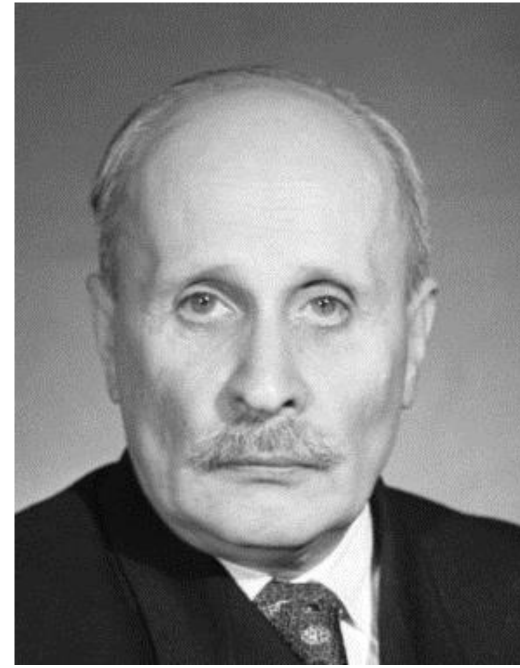
Theorem (P.S. Urysohn, 1918)

Positive eigenfunctions of (1.3) exits if $\lambda \in (\lambda_P ; \lambda_Q)$ where $P(x,y) < K_u(x,y,u) < Q(x,y)$; and λ_P , λ_Q are the largest eigenvalues of linear integral operators with kernels P and Q.

P.S. Urysohn, MSU

Урысон П. С. Об одном типе нелинейных интегральных уравнений //Матем. сб., 31:2 (1923), с. 236–255.

Nekrasov equation



A.I. Nekrasov, MSU

A.I. Nekrasov (1919–1922)

Problems of plane wave theory

$$\Phi(y) = \mu \int_0^{2\pi} \frac{K(x,y)\sin\Phi(y)dy}{1+\mu\int_0^y \sin\Phi(t)dt} \qquad (1.4)$$

Linearization:

$$\Phi(\theta) = -\frac{\mu}{12\pi}\int_0^{2\pi} \Phi(\varepsilon)\, ln\left|\frac{1 - \cos(\varepsilon - \theta)}{1 - \cos(\varepsilon + \theta)}\right| d\varepsilon \qquad (1.6)$$

$$f(x) = \lambda\int_a^b f(y)K(x,y)dy + \varepsilon\lambda\int_a^b R[\lambda, y, f(y)]K(x,y)\,dy. \quad (1.5)$$

Solution (1.5) – expansion into a series in fractional powers of the parameter; Fredholm's theory.

2 The history of some functional spaces in the context of the NFA

V.V. Nemytskii, MSU

D. Gilbert (1907) spaces l^2 and L^2.

F. Riesz (1909) space L^p

V.V. Nemytskii (1934) Hammerstein's equation in L^p

$$u(x) = \lambda \int_G K(x,y) f(y,u(y)) dy$$

Hammerstein operator splitting: *Hu=Af(u).*

S. Banach, H. Hahn (1920s) duality theory of normed spaces.

W. Orlicz (1936) space L_M^* of functions integrable with a convex M-function.(Example: M(u)= $e^{|u|}-1$).

A. Zaanen (1946) solvability of linear IEs with unbounded kernels K(x,y).

The history of Orlicz spaces in the context of the NFA

A growth condition for M(u) under which the Orlicz space is separable and reflexive (Orlicz): $\Delta_2-condition$:

$\exists k>0$: $M(2u)\leq kM(u)$ $\forall u\geq u_0>0$.

Kyiv, RIMM; Voronezh, VSU; VIBI
M.A. Krasnosel'skii, Ya.B. Rutitskii

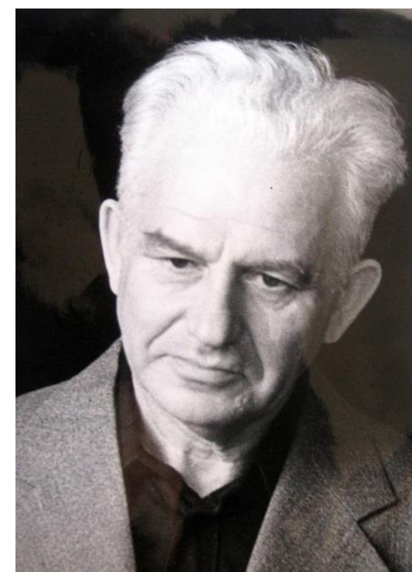
New classification of convex M – functions (1951–54).

$\Delta_3-condition$: functions growing faster than any power function;

$\Delta^2-condition$: functions having exponential nonlinearities.

Example. $f(x,u)=e^{c|u|}$

M.A. Krasnosel'skii



Ya.B. Rutitskii

The history of Kantorovich spaces in the context of NFA

1. If $y>0$, then it is impossible that $y=0$.
2. *If* $y_1>0$ *и* $y_2>0$ *then* $y_1+y_2>0$.
3. Whatever y *there exists* $y_1>0$, such that $y_1-y>0$.
4. If $\lambda>0$, $\lambda\in\mathbf{R}$ and $y>0$ then $\lambda y>0$.
5. Any subset Y, bounded from above, has supremum.

K- spaces (1935-37)

Majority principle

Solvability of equations:

$$y = Ay+y_0, \quad y, y_0 \in Y;$$

A is a monotonic positive operator ($y > 0 \Rightarrow Ay > 0$).

L.V. Kantorovich, LSU

3. On the development of the calculus of variations in large

The works of Lyapunov, Schmidt – a local approach.
Global approach: the problem of three geodesics (C.G. Jacobi (1839); A. Poincaré (1905)).
J. Birkhoff (1917) Minimax points x^*:
$C=f(x^*) \Rightarrow$ the set $f(x)<C$ is disconnected.

The presence of r minimum points on an m-connected surface S implies the existence of an (m+r-1)th minimax point of f on S.

Morse Inequalities (1928)

$$M_0 \geq R_0; M_0 - M_1 \leq R_0 - R_1$$

$$M_0 - M_1 + M_2 \geq R_0 - R_1 + R_2$$

$$M_0 - M_1 + \ldots + (-1)^n M_n = R_0 - R_1 + \ldots + (-1)^n R_n$$

$$d^2f(P) = -y_1^2 - y_2^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2$$

M_k is the number of critical points of index k;
R_k: k-dimensional Betty number of the surface S.

Category theory

L.G. Shnirelman, MSU

(1929–1930)

Estimation of critical points of functions and functionals on a manifold

L.A. Lyusternik, MSU

The category of a closed set A with respect to the space $M \supset A$ ($cat_M A$) is the minimum number of closed sets covering A in sum, each of which is contractible to a point. $cat_M M \geq N$

The result is the solution of the three geodesic problem.

4. Development of topological methods

L. Brouwer (1911): finite-dimensional results.

The mapping degree is the multiplicity of the surface S coverage by its image f(S), taking into account the orientation.

Brouwer FPT. If deg $f \neq (-1)^n$, then $\exists x \in S^n : f(x) = x$.

Consequence. The continuous mapping of the ball B_n into itself has a fixed point.

Birkhoff-Kellogg FPT. (1922) Exit to ∞-dimensional sp-s:

Let $M_f \subset E$ convex compact set of functions;

$A : M_f \to M_f$: continuous operator. Then there exists f_0: $Af_0 = f_0$. ($E = C[0,1]$, $C^n[0,1]$ или $L^2[0,1]$).

If in a Banach space with a basis a continuous operator transforms a closed convex body into its compact part, then there exists a fixed point (Schauder FPT, 1927)

Development of topological methods

S. Banach (1922) – R. Caccіopoli (1930)

BCP: Let’s E – B.S., $F: E \to E$ and $\rho(Fx,Fy) <$
$< \alpha\rho(x,y)$ for $\alpha<1 \Rightarrow \exists!\, x\in E: Fx=x.$

V.V. Nemytskii (1934)

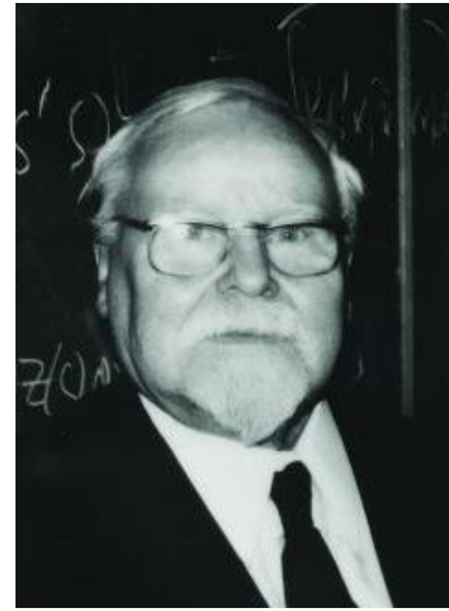
$u(x) = \lambda \int_G K(x,y) f(y,u(y))dy.$

Schauder FPT+BCP+decomposition

A.N. Kolmogorov (1934);
A.N. Tikhonov, J. von Neumann (1935)

Locally convex spaces (LCS)

Tikhonov– Schauder FPT (1935) $\to$
for F(x) in the LCS



A.N. Tikhonov, MSU

A.N. Kolmogorov, MSU

Leray–Schauder mapping degree

Leray and Schauder (1934)

$$F(x)=x \quad <=> \quad (I-F)x=\theta$$

$x \in U \subset E$ *(Banach space)*, F is compl. continuous.

Deg $(I-F,U,0)$ – topological invariant.

Leray and Schauder's line of reasoning:

1. Approximation of F by a finite-dimensional operator F_n with values in E_n (Schauder projection).
2. Using Brouwer mapping degree for $\Phi_n = I_n - F_n$:

$$\mathrm{Deg}\,(\Phi,U,z)=\deg[\Phi_n,U\cap E_n,z].$$

Fixed point principle :

$\mathrm{Deg}\,(\Phi,U,0) \neq 0$, то $\exists\, x_0 \in U$: $Fx_0 = x_0$

Rotation of vector fields in B.S.

M.A. Krasnoselskii (1950)

U – connected bounded domain in Banach space E;

Deg $(\Phi, U) \stackrel{\text{def}}{=} \gamma(\Phi, \partial U) \longrightarrow \gamma(\Phi_n, U_B)$ (4.1)

Φ_n – Schauder projector.

γ is vector field rotation.

Krasnoselskii fixed point principle:

$$\left.\begin{array}{r}\gamma(\boldsymbol{\Phi}, \partial U) \neq 0 \\ \boldsymbol{\Phi} - \text{c.c.}\,v.\,field\end{array}\right\} \Rightarrow \exists\ x_0 \in U\text{: } \boldsymbol{\Phi}(x_0) = 0.$$

Applications (1956):

1) Existence and uniqueness FPT theorems for completely continuous smooth vector fields $\boldsymbol{\Phi}$.

2) Theorems of the Hammerstein equation solvability:

a) in L^p; b) in L_M^*.

History of variational methods

L. Tonelli (1911–1924) use of the property of semicontinuity of functionals.

$\Phi(u) = \int_C F(x, u(x), u'(x))ds \to$ min; $u \in K$.

Ideas on the development of variational methods (1928..

1. Transferring the consideration of variat. problems to the field of F.A.
2. Restriction of the class K while preserving the form Φ.
3. Replacing $\Phi(u)$ with $G(u)$ while maintaining the class K.

M. Golomb (1935) Abstract form of Hammerstein equation: $y - H^2G(y) = 0$, H: $E \to E$ (Hilbert space)

$$G = grad\ F,\ \Psi(y) = (y, y) + 2F(Hy) \to min.$$

H is a completely continuous operator

Synthesis of variational and topological methods

Solutions $\psi = G\psi$ ($\Leftrightarrow A\varphi = \varphi$) are critical points of $\Phi(\psi)$.

L.A. Lyusternik (1939)

A theorem on the existence of a countable number of eigenvectors of an even homogeneous functional Φ on the spheres H.

M.A. Krasnoselskii (1953–57):

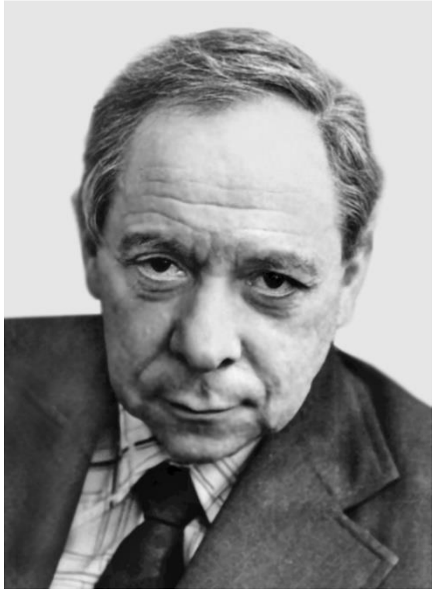
- General criterion for the existence of critical points of smooth increasing functionals $G = G_1 + G_2$ (1953);
- the existence of a continuum of eigenvectors of the potential operator A=grad F (1956);
- generalization of Lyusternik's theorem (homogeneity–>non-negativity) + resistance to perturbations Φ

Development of variational methods

M.M. Vainberg, 1952–1972:

- Solvability conditions $\lambda y - AF(y) = 0$; A – linear, F–potent.
- justification for the replacement $A_1 u = 0$ for $A_2 u = 0$
- replacement of complete continuity of A by monotonicity;
- renunciation of space $E = D(\Phi)$ completeness.

M.M. Vainberg, MSU



I.I. Vorovich, MSU

I.I. Vorovich (1955–57)

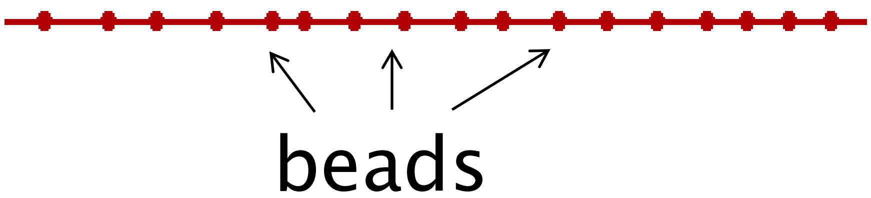
Mathematical model of shell deformation taking into account the influence of transverse deflections:

$$w = Aw.$$

6. History of positive operators

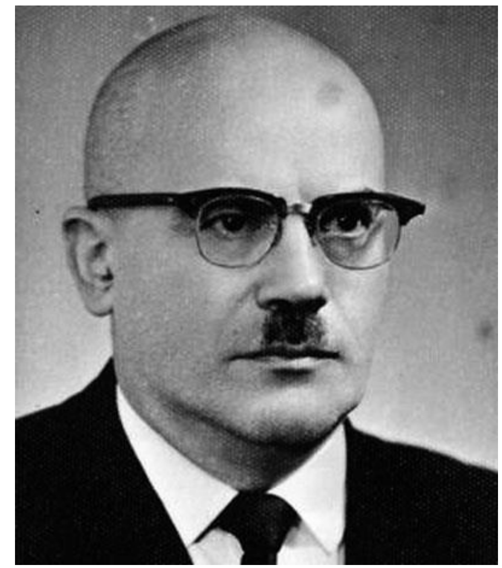
Motivation of M.G. Krein (1930s)

1. Theory of oscillations (positive matrices).
2. Theory of moments (positive functionals *F*: $y(x)>0 \Rightarrow Fy(x) > 0$).



Determine the masses of the beads and their arrangement so that the vibration frequencies of the thread have predetermined values.

Moment problem



M.G. Krein, Odessa

History of positive operators II

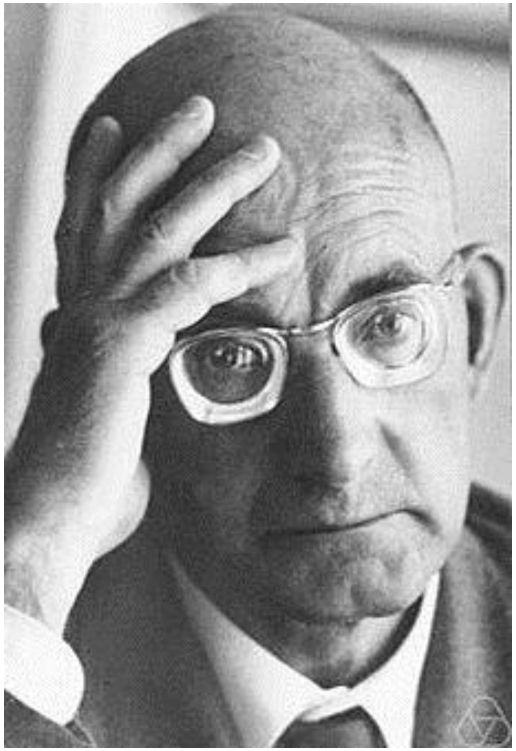
M.G. Krein (1937)

Positive functionals in E:

f(x)> 0 for $x \in K$.

1. $x \in K \Rightarrow \lambda x \in K$ при $\lambda \geq 0$.
2. $x, y \in K \Rightarrow (x + y) \in K$.
3. $x \in K; x \neq \theta \Rightarrow -x \in E \setminus K$.

The set of positive functionals is not empty and forms a cone in E^*.



P.S. Alexandrov, MSU

Reference point: Definition of positive operators: $AK \subset K$
P.S. Alexandrov, H. Hopf (1935).

Perron's theorem for $AK \subset K$; $K=\{x \in R^n : x_i \geq 0\}$ – cone!

Brower FPTs

History of positive operators III

Rutman M. A. (1948)

Generalization of Perron’s theorem to positive operators;

A is completely continuous, **monotone** in K and $\exists\, u \in K; c > 0, \varepsilon > 0$:

$$A(tu) \geqslant ctu \;\; \forall t \in [0, \varepsilon]. \qquad \text{(P)}$$

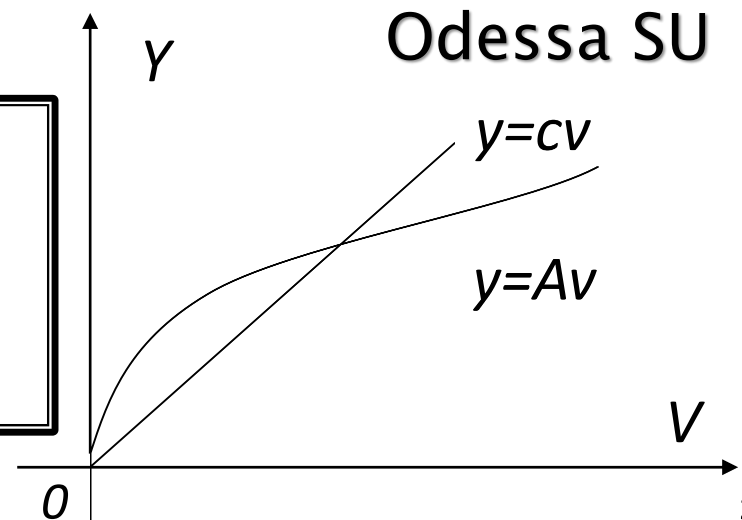
Then $\forall\, p > 0 \;\exists\, \lambda > 0$: $Ax = \lambda x$,

$$\|x\| = p, x \in K$$

Rutman M. A., Odessa SU

M.A. Krasnoselskii (1950s)

A) Generalization of Urysohn's results	B) Solving problems of nonlinear mechanics



Development of Urysohn's ideas

$$y = \lambda \mathrm{A}_0 y \qquad (6.1)$$

$$\mathrm{A}_0 y(x) = \int_a^b K(x,s,y(s))\,ds \qquad (6.2)$$

$K_y(x,s,y) > 0$; continuous; decreases in y.

$P(x,s) > K_y(x,s,y) > Q(x,s) \geq 0.$

Positive solutions of (6.1) exist for $\lambda \in (\lambda_P ; \lambda_Q)$.

M.A. Krasnoselskii

1. B is a monotone minorant *of A if*

$Ax \geq Bx$ $\qquad$ $(Ax - Bx) \in K$.

2. *A* *u_0 – bounded operator if*

$\exists\, u_0 \in K\ (u_0 \neq \theta)\ \forall \varphi \in K\ (\varphi \neq \theta)\ \exists\ (n \in \mathbb{N};\ \alpha, \beta > 0)$

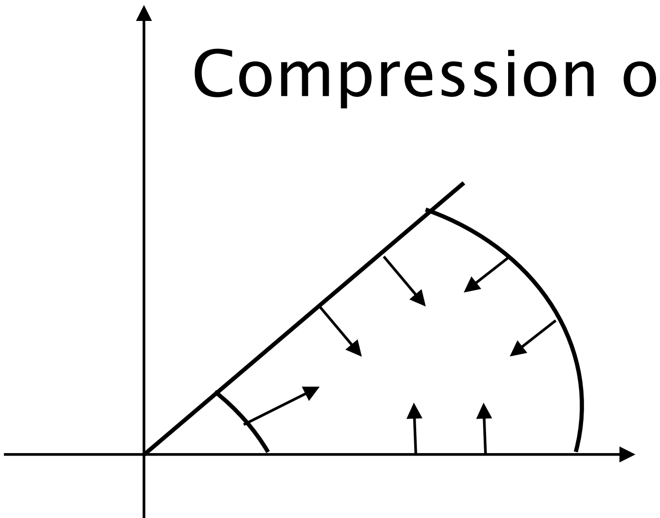
$$\alpha u_0 \leq A^n \varphi \leq \beta u_0.$$

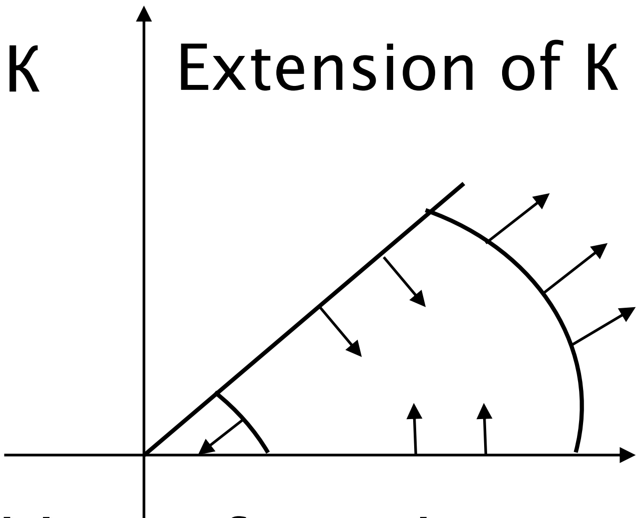
3. An analogue of Urysohn's th. on the structure of the spectrum (1954) $\qquad P\varphi \geq A\varphi \geq Q\varphi \qquad \varphi \in K \qquad (6.3)$

P, Q: u_0 – bounded, completely continuous

FPT Krasnoselskii cone theorem (1960)

Теорема. Let a completely continuous operator A compress or extend the cone K. Then A has at least one fixed point in the cone K.

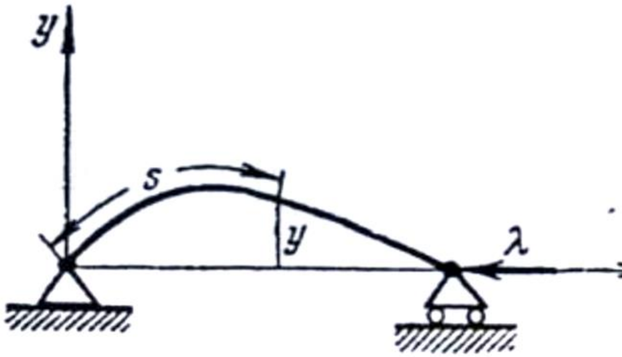




I.A. Bakhtin, VSU

The problem of buckling of a rod with variable stiffness (1955).

$$\varphi(s)=\lambda\rho(s)\int_0^1 K(s,t)\varphi(t)dt\times$$
$$\times\sqrt{1-\left[\int_0^1 K_s'(s,t)\varphi(t)dt\right]^2}$$



7 History of branching theory

Equilibrium theory of rotating fluids
A. Poincaré, A. M. Lyapunov, E. Schmidt

The number of small solutions of the Hammerstein equation?

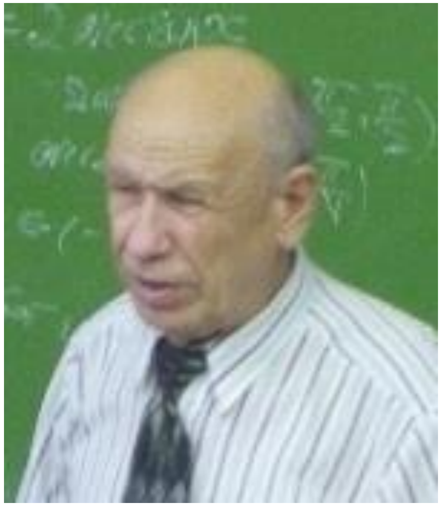
A. Stapan, 1950 (Newton's diagram)

Lyapunov–Schmidt equations

Operator's form: T. Shimizu (1948).

Topological methods: J. Cronin (1952)

$$(I + C + T)x = y \qquad (7.1)$$



V.A. Trenogin, MSU

Lyapunov–Schmidt analytical methods:

M.M. Vainberg, V.A. Trenogin, 1962

estimation of the number of solutions + asymptotics

Multidimensional branching

M.M. Vainberg, P. G. Aizengendler: 1965

History of branching

Bifurcation points $Au=\lambda u$ (7.2)

When $\lambda = \lambda^*$, in addition to the a priori known solution, one more (several) solutions arise. $\lambda = \lambda^*$ is bifurcation point.

Example: the appearance of surface waves at a certain critical force of influence on the liquid.

М.А. Красносельский (1956)

Local review + qualitative analysis

1) **Principle of index change**: if on the known solution curve of the equation $Au=\lambda u$ its index undergoes a jump, then a bifurcation takes place.

2) **Principle of linearization:** if k_0 is an eigenvalue of odd multiplicity of operator $A'(\theta)$, then $1/k_0$ is the bifurcation point of A.

8. On the development of qualitative aspects of approximate methods for solving operator equations

L.V. Kantorovich (1939)

Iterative processes in K-spaces

U: $Y \to Y$, Y – B. space; V: $Z \to Z$, (monotone, positive)

Z – K-space; Y is normalized by elements of Z

$$y = U(y), \quad y \in Y \text{ (Banach-Kantorovich)} \qquad (8.1)$$

$$z = V(z), \quad z \in Z \qquad (8.2)$$

Majorization:

$$||U(0)|| \leq V(0); \quad ||\Delta U|| \leq \Delta V \; for \; ||y|| \leq \mathrm{z}, \; ||\Delta y|| \leq \Delta \mathrm{z} \quad (8.3)$$

Method of successive approximations:

$$y_n = U(y_{n-1}); \; ||y^*|| \leq \mathrm{z}^* \qquad (8.4)$$

Newton-Kantorovich method (1949-51)

Theoretical substantiation of approximate methods

L.V. Kantorovich (1948–1956)

$$Kx = y; \qquad K: X \to Y \qquad (8.5)$$

$$\bar{K}\bar{x} = \bar{y}; \qquad \bar{K}: \bar{X} \to \bar{Y} \qquad (8.6)$$

- Establishment of convergence and estimation of its speed;
- Efficient estimation of errors;
- Determining the stability of numerical schemes;
- Proof of the existence and uniqueness of the solution of the original and approximate problems;
- Obtaining properties of solutions to equation (8.5) based on data on an approximate solution (8.6).

M.A. Krasnoselskii (1956) $\bar{K}x \overset{\text{def}}{=} P_n K x$

Saving topological properties K.

Conclusions, part 1

1. The development of the NFA in the USSR is part of a global process.
2. Most of the results have been obtained by generalizing the corresponding finite-dimensional analogs that have come from topology or analysis.
3. Appearing as a result of the study of applied problems (figures of equilibrium of rotating fluids), the NFA does not cease to be the main tool for studying nonlinear systems.
4. The reduction of the equation $Au=0$ to an equation with a simpler operator $A_1u=0$, coming from Lyapunov and Schmidt, has become one of the basic methods for the qualitative study of operator equations.

Conclusions, part 2

5. The NFA turned out to be a necessary tool for qualitative analysis of approximate methods.
6. From the beginning of the 1950s the NFA was characterized by the mutual penetration of ideas from its various branches.
7. Various branches of the NFA form one whole (the general idea is the study of operator equations by qualitative methods). The most isolated section of the NFA is the calculus of variations in large.
8. The main driving forces in the development of the NFA: nonlinear mechanics + the internal mechanism of the evolution of mathematics.
9. The calling card of the Soviet school of NFA is the theory of positive operators.

Conclusions, part 3

10. The leading scientific schools of the NFA in the USSR are: Leningrad, Moscow, Kiev, Odessa, and Voronezh ones (10 monographs in 20 years).

11. Foreign schools had considerable weight:

- German mathematics school (E. Schmidt, L. Lichtenstein, W. Ritz, A. Gammerstein, M. Golomb, E. Rothe);
- Polish mathematical school (S. Banach, J. Schauder, W. Orlicz, S. Mazur, S. Ulam, K. Borsuk);
- American mathematical school (J. Birkgoff, O. Kellogg, M. Morse, J. von Neumann, T. Hildenbrandt, L. Graves);
- French mathematical school (M. Fréchet, J. Adamard, J. Leray, J. Dieudonné).

Published works (list № 1)

1. *Богатов Е.М.* Об истории развития нелинейных интегральных уравнений в СССР. Сильные нелинейности // Науч. вед. БелГУ. Сер. Матем., Физ. 2017, № 6, Вып. 46, С. 93-106.
2. *Богатов Е. М.* Об истории метода неподвижной точки и вкладе советских математиков (1920-е–1950-е гг.) // Чебышевский сборник, 2018, т. 19, вып. 2, с. 30–55.
3. *Богатов Е. М* О развитии качественных методов решения нелинейных уравнений и некоторых последствиях // Известия вузов. Прикладная нелинейная динамика, 27:1 (2019).
4. *Богатов Е.М.* Об истории положительных операторов (1900-е–1960-е гг.) и вкладе М.А. Красносельского // Науч. вед. БелГУ. Сер. Прикладная матем., Физ., 2020. Т. 52, № 2.
5. *Богатов Е.М.* Об истории уравнения Некрасова // Науч. вед. БелГУ. Сер. Прикладная матем., Физ., 2021. Т. 53, № 3.
6. *Богатов Е.М., Коренев А.В., Михайлов И.С*. О современных инструментах и методах ведения научных исследований по истории математики // Таврический вестник информатики и математики. 2021. -№ 3 (52), с. 35-57.

Published works (list № 2)

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