# The Milnor-Hirzebruch problem, complex cobordisms, and theta divisors

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Every complex manifold has a fixed complex structure in its tangent bundle.

Every smooth algebraic variety over $\mathbb{C}$ is a complex manifold.

Every symplectic manifold has a complex structure in its tangent bundle,
but in general a symplectic manifold is not a complex manifold,
as the symplectic form might not be integrable.

A manifold $M^{2n}$ with the tangent bundle $TM^{2n}$ is called almost complex if
there exist an $n$-dimensional complex vector bundle $\xi \to M^{2n}$ with
an isomorphism of real vector bundles $TM^{2n} \cong r\xi$.

Let $M^m$ be a smooth oriented real manifold with the tangent bundle $TM^m$. A stable complex structure (a $U$-structure) $(\xi, c)$ on $M^m$ is a complex $N$-dimensional vector bundle $\xi \to M^m$ with an orientation preserving isomorphism of real vector bundles

$$c\colon\ TM^m \oplus (2N-m)_{\mathbb{R}} \cong r\xi,$$

where $(2N-m)_{\mathbb{R}}$ is the trivial real vector bundle with the canonical orientation.

A manifold $M^m$ with a chosen $U$-structure is called a $U$-manifold.

Example. For any $k \in \mathbb{Z}$ there exists a $U$-structure on $\mathbb{C}P^1$ where $\xi = \eta^{2k} + \mathbb{C}^{N-1}$ and $\eta \to \mathbb{C}P^1$ is the tautological complex line bundle over the complex projective line with the first Chern class $c_1(\eta) = z \in H^2(\mathbb{C}P^1;\mathbb{Z}) = \mathbb{Z}$ and $(z, \langle\mathbb{C}P^1\rangle) = 1$.

Let $\zeta \to X$ be a real $2n$-dimensional oriented vector bundle over a finite $CW$-complex $X$.

**Theorem (E. Thomas, 1967)**

*The bundle $\zeta$ admits a complex structure if and only if it is stably isomorphic to $r\xi$ where $\xi \to X$ is a complex bundle such that the Chern class $c_n(\xi)$ is equal to its Euler class $e(\zeta)$.*

**Corollary**

*A U-manifold $M^{2n}$ with an isomorphism $TM^{2n} \oplus (2N - 2n)_{\mathbb{R}} \cong r\xi$ is almost complex if and only if the number $(c_n(\xi), \langle M^{2n} \rangle)$ is equal to the Euler characteristic $\chi(M^{2n})$, where $\langle M^{2n} \rangle \in H_{2n}(M^{2n}, \mathbb{Z}) = \mathbb{Z}$ is the fundamental cycle determined by the orientation of the manifold $M^{2n}$.*

Two closed $m$-dimensional $U$-manifolds $M_1$ and $M_2$ are called bordant if there exists a real $(m+1)$-dimensional $U$-manifold $W$ such that its boundary $\partial W$ is a disjoint union of $M_1$ and $M_2$, and the restriction of the $U$-structure of $W$ to $M_i$ coincides with the $U$-structure on $M_i$ for $i = 1, 2$.

Two closed $m$-dimensional $U$-manifolds $M_1$ and $M_2$ are called cobordant if there exists a real $(m+1)$-dimensional manifold $W \subset \mathbb{R}^{2N+m+1}$ with the complex normal bundle $\nu W$ such that the boundary $\partial W$ is a disjoint union of $M_1^m$ and $M_2^m$, and the restriction of the normal bundle $\nu W$ to $M_i$ coincides with the complex structure of the normal bundle $\nu M_i$ for $i = 1, 2$.

The bordism and cobordism classes of a $U$-manifold $M^n$ are denoted by $[M^n]$. (Note that by definition, the bordism class of $[M^n]$ has dimension $n$ and its cobordism class has dimension $-n$.)

The $U$-manifold $M^n \times [0,1]$ defines the bordism of the manifolds $M_1 = M^n$ and $M_2$, where $M_2$ differs from $M_1$ by the orientation of the fundamental cycle. Thus we can define the class $-[M^n]$ for every $U$-manifold $M^n$.

The sum of the bordism classes of two closed $U$-manifolds $M_1^m$ and $M_2^m$ is defined as

$$[M_1^m] + [M_2^m] = [M_1^m \sqcup M_2^m],$$

where $M_1^m \sqcup M_2^m$ is the disjoint union of $M_1^m$ and $M_2^m$.

The product of the bordism classes of $M_1^{m_1}$ and $M_2^{m_2}$ is defined by

$$[M_1^{m_1}][M_2^{m_2}] = [M_1^{m_1} \times M_2^{m_2}].$$

Thus we obtain the commutative graded ring $\Omega^U = \sum_{m\geqslant 0} \Omega_m^U$, where $\Omega_m^U$ is the group of bordism classes of $m$-dimensional $U$-manifolds.

Similarly, we obtain the graded ring $\Omega_U = \sum_{m\geqslant 0} \Omega_U^{-m}$, where $\Omega_U^{-m}$ is the group of cobordism classes of $m$-dimensional $U$-manifolds.

A continuous map $f\colon M^k \to X$ where $M^k$ is some closed $U$-manifold is called a $k$-dimensional $U$-cycle of the space $X$.

Two $U$-cycles $(M_1^k, f_1)$ and $(M_2^k, f_2)$ are called bordant if there are a $U$-manifold $W^{k+1}$ with boundary $\partial W^{k+1} = M_1^k \sqcup M_2^k$ and a continuous map $F\colon W \to X$ such that $F|_{\partial W} = f_1 \sqcup f_2$.

The set of bordism classes of $k$-dimensional $U$-cycles forms the bordism group $U_k(X)$ where $k = 0, 1, \ldots$.

The notions of $U$-cocycle and $U$-cobordism are more complicated. They use the Pontryagin–Thom construction and the theory of transversally regular mappings of smooth manifolds.

The set of cobordism classes of $k$-dimensional $U$-cocycles forms the cobordism group $U^k(X)$ where $k \in \mathbb{Z}$.

The homology theory $U_*(X)$ based on the groups $\{U_k(X),\ k = 0, 1, \ldots\}$ and the cohomology theory $U^*(X)$ based on the groups $\{U^k(X),\ k \in \mathbb{Z}\}$ are called the complex bordism theory and the complex cobordism theory respectively.

From the correspondence between stable complex structures in tangent and normal bundles of a manifold $M^m$, $m = 0, 1, 2, \ldots$, it follows that the groups $\Omega^U_m$ and $\Omega^{-m}_U$ are isomorphic.

For $X = \{pt\}$, this isomorphism can be considered as the Poincare duality isomorphism $\Omega^U_m = U_m(\{pt\}) = U^{-m}(\{pt\}) = \Omega^{-m}_U$ for the 0-dimensional $U$-manifold $\{pt\}$.

This isomorphism extends to the Poincare–Atiyah duality isomorphism between complex bordisms and cobordisms for any $U$-manifold $X^n$

$$D_U \colon U_m(X^n) \to U^{n-m}(X^n).$$

Let $\lambda = (i_1, \dots, i_k)$, $i_1 \geqslant \dots \geqslant i_k > 0$ be a partition of some integer number $n = i_1 + \dots + i_k = |\lambda|$.

Let $\xi \to M$ be a complex bundle giving a $U$-structure on $M$.

Using the splitting principle of complex vector bundles, one can define the class $c_\lambda(\xi) \in H^{2n}(M, \mathbb{Z})$ as the characteristic class corresponding to the monomial symmetric function $m_\lambda(t) = t_1^{i_1} \dots t_k^{i_k} + \dots$. Set $c_\lambda = c_{(n)}$ for $\lambda = (n)$ and $c_\lambda = c_n$ for $\lambda = (1, \dots, 1)$.

Let us define the tangent Chern class $c_\lambda(TM)$ of a $U$-manifold $M$ as $c_\lambda(\xi)$.

The tangent Chern number $c_\lambda(M^{2n})$, $|\lambda| = n$, of $U$-manifold $M^{2n}$ is defined as

$$c_\lambda(M^{2n}) := (c_\lambda(TM^{2n}), \langle M^{2n} \rangle).$$

We have $p(n)$ Chern numbers $c_\lambda(M^{2n})$, which depend only on the bordism class of $M^{2n}$. Here $p(n)$ is the number of the partitions of $n \in \mathbb{N}$.

The values $p(n)$ for $n = 1, 2, 3, 4, 5, 6, 7, \dots$ are $1, 2, 3, 5, 7, 11, 15, \dots$

The generating function of $p(n)$ is

$$1 + \sum_{n=1}^{\infty} p(n) x^n = \prod_{j=1}^{\infty} (1 - x^j)^{-1}.$$

In the case of cobordism classes, it is more convenient to use the normal Chern numbers $c^{\nu}_{\lambda}(M^{2n})$ for the normal bundle $\nu M^{2n}$ in the form

$$c^{\nu}_{\lambda}(M^{2n}) := (c_{\lambda}(\nu M^{2n}), \langle M^{2n} \rangle).$$

Since $TM^{2n} + \nu M^{2n}$ is a trivial bundle, the normal Chern numbers can be expressed through the tangent Chern numbers $c_{\lambda}(M^{2n})$ via the formula

$$\sum_{\lambda=(\lambda', \lambda'')} c_{\lambda'}(TM^{2n}) c^{\nu}_{\lambda''}(\nu M^{2n}) = 0, \ |\lambda| > 0.$$

The numbers $\{c_{\lambda}(M^{2n})\}$ and $\{c^{\nu}_{\lambda}(M^{2n})\}$ contain the same information about the $U$-manifold $M^{2n}$.

**Theorem (Milnor, Novikov, 1960)**

*The graded complex bordism ring $\Omega^U$ is isomorphic to the graded polynomial ring $\mathbb{Z}[y_1, \ldots, y_n, \ldots]$ in variables $y_n$, $n \in \mathbb{N}$, $\deg y_n = 2n$.*

*Two closed $2n$-dimensional $U$-manifolds $M_1$ and $M_2$ are $U$-bordant if and only if $c_\lambda(M_1) = c_\lambda(M_2)$ for all partitions $\lambda$ such that $|\lambda| = n$.*

The same is true for the complex cobordism ring $\Omega_U$ with the degrees of the multiplicative generators $-2n$.

In his celebrated monograph (1956), Hirzebruch introduced the characteristic class $Td(\xi)$ of complex vector bundles $\xi$. From his results, one can obtain

$$Td(\xi) = 1 + \sum_{n\geqslant 1} \frac{1}{\gamma_n} T_n(\xi),$$

where $\gamma_n = \prod_p p^{[\frac{n}{p-1}]}$, $p$ runs through all primes, and $T_n(\xi) = T_n(c_1, \dots, c_n)$ is a homogeneous integer polynomial of deg $2n$ in the characteristic classes $c_k(\xi)$.

First polynomials $T_n$ are

$$T_1 = c_1, \quad T_2 = c_2 + c_1^2, \quad T_3 = c_2 c_1, \quad T_4 = -c_4 + c_3 c_1 + 3c_2 c_1^2 - c_1^4.$$

The numbers $\gamma_n$ where $\gamma_1 = 2,\ \gamma_2 = 12,\ \gamma_3 = 24,\ \gamma_4 = 720, \dots$ form the sequence A091137, see the On-line Encyclopedia of Integer Sequences (OEIS).

Note that $\gamma_{2n+1} = 2\gamma_{2n},\ n > 0$.

The Todd genus $Td(M^{2n})$ of a $U$-manifold $M^{2n}$ is the number $Td(M^{2n}) = (Td(\xi), \langle M^{2n} \rangle)$, where $\xi \to M^{2n}$ is a complex vector bundle giving a $U$-structure on the manifold $M^{2n}$.

Combining the Hirzebruch's construction with the Milnor–Novikov theorem, one can obtain

**Theorem**

*The characteristic Todd class $Td(\xi)$ is uniquely determined by the fact that the Todd genus defines a unique ring homomorphism $Td\colon \Omega_U \to \mathbb{Z}$ such that $Td(\mathbb{C}P^n) = 1$, $n \in \mathbb{N}$.*

Let's put $\beta_\lambda(n) = \prod_{q=1}^{k} (i_q + 1)$, where $\lambda = (i_1 \geqslant \ldots \geqslant i_k)$, $|\lambda| = \sum_{q=1}^{k} \lambda_q = n$.

**Lemma**

*The number $\gamma_n = \prod_p p^{[\frac{n}{p-1}]}$ is the least common multiple of the set of numbers $(\beta_\lambda(n), |\lambda| = n)$.*

**Corollary**

*1) For any $p(n)$-dimensional vector $(c_\lambda, |\lambda| = n) \in \mathbb{Z}^{p(n)}$, there exists a unique bordism class $[M^{2n}]$ such that $c_\lambda(TM^{2n}) = \gamma_n \cdot c_\lambda$ for any $\lambda$, $|\lambda| = n$.*
*This class $[M^{2n}]$ is an integer polynomial in $[\mathbb{C}P^1], \ldots, [\mathbb{C}P^n]$.*

*2) The number $\gamma_n$ is the minimum among all numbers $\gamma \in \mathbb{N}$ for which the statement 1) is true.*

**Theorem (Milnor, 1958)**

*Any bordism class $[M^{2n}] \in \Omega^U_{2n}$ contains a nonsingular complex algebraic variety (not necessarily connected).*

The proof of this theorem uses the construction by which Milnor showed that for any non-singular complex algebraic manifold $M^{2n}$ there exists a non-singular complex algebraic manifold $\widehat{M}^{2n}$ such that

$$[\widehat{M}^{2n}] = -[M^{2n}].$$

The following Milnor-Hirzebruch problem (1958) is still open:

*Which sets of $p(n)$ integers $c_\lambda$, $|\lambda| = n$, can be realised as the Chern numbers $c_\lambda(M^n)$ of some smooth complex irreducible algebraic variety?*

According to the Milnor–Novikov theorem, this problem is equivalent to the following problem:

*Describe all the bordism classes $[M^{2n}] \in \Omega^U_{2n}$ that contain as representatives some smooth complex irreducible algebraic varieties.*

For $n = 1$ the solution of this problem is classical:
The cobordism class $[M^2] \in \Omega^U_2$ contains some irreducible smooth complex algebraic variety if and only if $c_1(M^2) = 2 - 2g$, $g = 0, 1, \dots$

The multiplicative transformation of cohomology theories

$$ch_U : U^*(X) \to H^*(X, \Omega_U \otimes \mathbb{Q}),$$

which for $X = \{pt\}$ is the canonical embedding $\Omega_U \subset \Omega_U \otimes \mathbb{Q} \ : \ 1 \to 1 \otimes 1$, is called the Chern-Dold character in complex cobordism.

**Theorem (Buchstaber, 1970)**

*1) For every $n \in \mathbb{N}$, there is the unique cobordism class $[\mathcal{B}^{2n}] \in \Omega_U^{-2n}$ such that*

$$c_\lambda^\nu(\mathcal{B}^{2n}) = 0 \text{ for } \lambda \neq (n) \text{ and } c_{(n)}^\nu(\mathcal{B}^{2n}) = (n+1)!.$$

*2) $Td(\mathcal{B}^{2n}) = (-1)^n$.*

*3) The Chern-Dold character $ch_U$ is uniquely defined by the formula*

$$ch_U(u) = \beta(z) = z + \sum_{n=1}^{\infty} [\mathcal{B}^{2n}] \frac{z^{n+1}}{(n+1)!},$$

*where $u = c_1^U(\eta) \in U^2(\mathbb{C}P^\infty)$ and $z = c_1^H(\eta) \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ are the first Chern classes of the universal line bundle $\eta \to \mathbb{C}P^\infty$ in the complex cobordisms theory and cohomology theory respectively.*

# Formal group of geometric cobordisms

Consider the tensor product of universal bundles

$$\eta_1 \otimes_{\mathbb{C}} \eta_2 \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty.$$

The formal group of geometric cobordisms is given by the series

$$c_1^U(\eta_1 \otimes_{\mathbb{C}} \eta_2) = F_U(u, v) = u + v + \sum_{i,j} a_{i,j} u^i v^j, \; a_{i,j} \in \Omega_U^{-2(i+j-1)},$$

where $u = c_1^U(\eta_1), \; v = c_1^U(\eta_2)$ (S.P. Novikov, A.S. Mishchenko, 1967).

**Theorem (A.S. Mishchenko, 1967)**

*The logarithm $g(u)$ of the formal group $F_U(u, v)$ is given by the series*

$$g(u) = u + \sum_{n=1}^{\infty} [\mathbb{C}P^n] \frac{u^{n+1}}{n+1}, \quad \textit{i.e.} \; g\left(F_U(u, v)\right) = g(u) + g(v).$$

**Theorem (Buchstaber, 1970)**

$$F_U(u, v) = \frac{u + v + \sum_{i,j} [H_{i,j}] u^i v^j}{\mathbb{C}P(u) \mathbb{C}P(v)}, \; \textit{where} \; \mathbb{C}P(u) = 1 + \sum_{n=1}^{\infty} [\mathbb{C}P^n] u^n,$$

*and $H_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$ are smooth irreducible algebraic Milnor hypersurfaces. The coefficients $a_{i,j}$ of the series $F_U(u, v)$ generate the whole ring $\Omega_U$.*

Consider the ring $\mathcal{L} = \mathbb{Z}[\alpha_{i,j}],\ i,j \in \mathbb{N}$. Let us introduce the series

$$\mathcal{F}(u,v) = u + v + \sum_{i,j} \alpha_{i,j} u^i v^j \in \mathcal{L}[[u,v]]$$

and polynomials $p_{i,j,k} \in \mathcal{L}$ defined by the generating series

$$\sum_{i\geqslant 1, j\geqslant 1, k\geqslant 1} p_{i,j,k} u^i v^j w^k = \mathcal{F}(\mathcal{F}(u,v),w) - \mathcal{F}(u,\mathcal{F}(v,w)).$$

Let $\mathcal{J} \subset \mathcal{L}$ be the ideal generated by polynomials $\alpha_{i,j} - \alpha_{j,i}$ and $p_{i,j,k}$.
The universal one-dimensional commutative formal group is given by the series

$$F(u,v) = u + v + \sum_{i,j} b_{i,j} u^i v^j \ \text{ over the ring } L = \mathcal{L}/\mathcal{J},$$

where $b_{i,j}$ are the images of the coefficients $\alpha_{i,j}$ of the series $\mathcal{F}(u,v)$ under the projection $\mathcal{L} \to L$.

**Theorem (Lazard, 1955)**

*There exists an isomorphism* $L \cong \mathbb{Z}[b_1, \dots, b_n, \dots]$.

**Theorem (Quillen, 1969)**

*The ring homomorphism* $L \to \Omega_U$*, which determines the formal group* $F_U(u,v)$*, is an isomorphism, and therefore the universal formal group* $F(u,v)$ *can be identified with the formal group* $F_U(u,v)$ *over* $\Omega_U$.

**Theorem (Buchstaber, 1970)**

*The series*

$$ch_U(u) = \beta(z) = z + \sum_{n=1}^{\infty} [\mathcal{B}^{2n}] \frac{z^{n+1}}{(n+1)!}$$

*is the exponential of the formal group of geometric cobordisms*

$$F(u,v) = u + v + \sum_{i,j} a_{i,j} u^i v^j, \quad i.e. \quad \beta(z+w) = F(\beta(z), \beta(w)).$$

**Corollary**

*The inverse of the series $\beta(z)$ is*

$$\beta^{-1}(u) = u + \sum_{n=1}^{\infty} [\mathbb{C}P^n] \frac{u^{n+1}}{n+1} = g(u), \quad i.e. \quad F(u,v) = \beta\big(g(u) + g(v)\big).$$

*The question of the existence of a smooth irreducible algebraic variety in the cobordism class $[\mathcal{B}^{2n}]$ had remained open since 1970.*

Let $A^{n+1} = \mathbb{C}^{n+1}/\Gamma$ be a principally polarised abelian variety (ppav).
The line bundle $L$, which polarizes it, has one-dimensional space of holomorphic sections generated by the classical Riemann $\theta$-function

$$\theta(z,\tau) = \sum_{m \in \mathbb{Z}^{n+1}} \exp[\pi i(m,\tau m) + 2\pi i(m,z)],\ z \in \mathbb{C}^{n+1},\ \tau^t = \tau,\ \mathrm{Im}\,\tau > 0.$$

**Theorem (Andreotti & Mayer, 1967)**

*For a generic ppav, the theta divisor $\Theta^n \subset A^{n+1}$ given by $\theta(z,\tau) = 0$ is irreducible smooth algebraic variety of general type.*

*The cobordism class $[\Theta^n]$ does not depend on the choice of the abelian variety $A^{n+1}$.*

For example,

- $\Theta^1 \cong \mathcal{C} \subset A^2 = J(\mathcal{C})$ for a smooth hyperelliptic curve $\mathcal{C}$ of genus 2;
- $\Theta^2 \cong S^2(\mathcal{C}) \subset A^3 = J(\mathcal{C})$ for a smooth non-hyperelliptic curve $\mathcal{C}$ of genus 3.

*For $n \geqslant 3$, a general ppav is not Jacobian.*

**Theorem (Buchstaber & Veselov, 2020)**

*As a representative of $[\mathcal{B}^{2n}]$, one can take a smooth theta divisor $\Theta^n$ of a general principally polarised abelian variety $A^{n+1}$, i.e. the exponential of the universal formal group can be written in the form*

$$\beta(z) = z + \sum_{n=1}^{\infty} [\Theta^n] \frac{z^{n+1}}{(n+1)!}.$$

Set $\Theta^\lambda = \Theta^{i_1} \times \cdots \times \Theta^{i_k}$, $|\lambda| = i_1 + \ldots + i_k = n$, and $(\lambda+1)! = (i_1+1)! \cdots (i_k+1)!$.

**Corollary**

*For any smooth irreducible algebraic manifold $\Theta^\lambda$, $\lambda = (i_1, \ldots, i_k)$, $|\lambda| = n$, we have*

1. $c^\nu_{\lambda'}(\Theta^\lambda) = 0$ *for* $\lambda' \neq \lambda$ *and* $c^\nu_\lambda(\Theta^\lambda) = (\lambda+1)!$;
2. $Td(\Theta^\lambda) = (-1)^n$.

Set $\mathbb{C}P_U = \mathbb{Z}\big[[\mathbb{C}P^1], \dots, [\mathbb{C}P^n], \dots\big] \subset \Omega_U, \quad \Theta_U = \mathbb{Z}\big[[\Theta^1], \dots, [\Theta^n], \dots\big] \subset \Omega_U.$

**Theorem**

*The rings $\Theta_U$ and $\mathbb{C}P_U$ are $S$-modules, where $S$ is the Landweber–Novikov algebra in the theory $U^*(\cdot)$.*

Using formulas $\beta(g(u)) = u$ and $\frac{d}{du}g(u) = 1 + [\mathbb{C}P^1]u + \ldots \in \mathbb{C}P_U[[u]]$ we obtain:

$$-[\Theta^1] = [\mathbb{C}P^1], \qquad -[\Theta^n] = n![\mathbb{C}P^n] + \sum_{k=1}^{n-1}[\Theta^{n-k}]P_{n,k}, \; n > 1,$$

where $P_{n,k} \in \mathbb{C}P_U$ are homogeneous polynomials of degree $k$ with positive coefficients.

**Theorem**

$\Theta_U \subset \mathbb{C}P_U.$

Examples:

$$-[\Theta^2] = 2[\mathbb{C}P^2] + 3[\Theta^1][\mathbb{C}P^1],$$
$$-[\Theta^3] = 6[\mathbb{C}P^3] + 6[\Theta^2][\mathbb{C}P^1] + [\Theta^1]\big(3[\mathbb{C}P^1]^2 + 8[\mathbb{C}P^2]\big).$$

From the formula $F(u,v) = u + v + \sum a_{i,j} u^i v^j = \beta\big(g(u) + g(v)\big)$, we obtain:

**Corollary.**

Any cobordism class $a_{i,j} \in \Omega_U^{-2n}$, $i+j = n+1$, $1 \leqslant i,j \leqslant n$, can be written in the form

$$i!j!a_{i,j} = [\Theta^n] + \sum_{k=1}^{n-1} [\Theta^{n-k}] P_{i,j}^k,$$

where $P_{i,j}^k$ are homogeneous polynomials of degree $k$ in $[\mathbb{C}P^1], \ldots, [\mathbb{C}P^k]$ with integer positive coefficients.

Examples: $a_{1,1} = [\Theta^1]$,

$$2a_{2,1} = [\Theta^2] + [\Theta^1][\mathbb{C}P^1],$$

$$6a_{1,3} = [\Theta^3] + 3[\Theta^2][\mathbb{C}P^1] + 2[\Theta^1][\mathbb{C}P^2],$$

$$4a_{2,2} = [\Theta^3] + 2[\Theta^2][\mathbb{C}P^1] + [\Theta^1][\mathbb{C}P^1]^2.$$

**Lemma (Buchstaber, 1970)**

*The Todd class* $Td_U(\xi)$ *of a complex vector bundle* $\xi$ *over a CW-complex* $X$ *with values in* $H^*(X, \Omega_U \otimes \mathbb{Q})$ *is uniquely defined by the following properties:*

*1) For every two complex vector bundles* $\xi_1$ *and* $\xi_2$ *over* $X$*,*

$$Td_U(\xi_1 \oplus \xi_2) = Td_U(\xi_1)Td_U(\xi_2);$$

*2) For any U-manifold* $M^{2n}$*,*

$$(Td_U(TM^{2n}), \langle M^{2n} \rangle) = [M^{2n}].$$

**Theorem (Buchstaber & Veselov, 2020)**

*The Todd class of* $\xi$ *can be expressed in terms of theta divisors by the formula*

$$Td_U(\xi) = 1 + \sum_{\lambda} c_\lambda(-\xi) \frac{[\Theta^\lambda]}{(\lambda + 1)!},$$

*where the sum is over all partitions* $\lambda = (i_1, \dots, i_k)$ *such that* $0 < |\lambda| \leqslant \dim_{\mathbb{C}} \xi$*,* $\Theta^\lambda = \Theta^{i_1} \times \cdots \times \Theta^{i_k}$*, and* $(\lambda + 1)! = (i_1 + 1)! \cdots (i_k + 1)!$*.*

**Lemma**

*The number* $\gamma_n = \prod_p p^{\left[\frac{n}{p-1}\right]}$, $n \in \mathbb{N}$, *is the least common multiple of the set of numbers* $((\lambda+1)!, |\lambda| = n)$.

Set $b_\lambda = \frac{\gamma_n}{(\lambda+1)!} \in \mathbb{N}$. Then $(b_\lambda : |\lambda| = n)$ is the vector of mutually-prime numbers.

$$n = 1 : \lambda = (1),\ \gamma_1 = 2,\ b_\lambda = 1.$$
$$n = 2 : \lambda = ((2),(1,1)),\ \gamma_2 = 12,\ (b_\lambda) = (2,3).$$
$$n = 3 : \lambda = ((3),(2,1),(1,1,1)),\ \gamma_3 = 24,\ (b_\lambda) = (1,2,3).$$

Let us introduce the characteristic $\Theta_U$-polynomials of complex vector bundles $\xi \to X$ by

$$T_{U,n}(\xi) = \sum_{|\lambda|=n} b_\lambda c_\lambda(-\xi)[\Theta^\lambda] \in H^{2n}(X;\Theta_U),\ n \geqslant 1.$$

**Theorem**

$$Td_U(\xi) = 1 + \sum_{n\geqslant 1} \frac{1}{\gamma_n} T_{U,n}(\xi).$$

**Theorem**

*For any $p(n)$-dimensional vector $(c_\lambda, |\lambda| = n) \in \mathbb{Z}^{p(n)}$, there exists a $U$-manifold $M^{2n}(c_\lambda)$ whose cobordism class is uniquely determined by the condition*

$$c_\lambda^\nu(M^{2n}(c_\lambda)) = \gamma_n \cdot c_\lambda \quad \text{for any } \lambda,\ |\lambda| = n.$$

*Thus:*

1. *For any vector $(c_\lambda, |\lambda| = n) \in \mathbb{Z}^{p(n)}$,*

$$\sum_{|\lambda|=n} b_\lambda c_\lambda [\Theta^\lambda] = [M^{2n}(c_\lambda)] \in \Theta_U.$$

2. *For any $U$-manifold $M^{2n}$,*

$$\gamma_n [M^{2n}] = \sum_{|\lambda|=n} b_\lambda c_\lambda^\nu(M^{2n}) [\Theta^\lambda] \in \Theta_U.$$

Example:

$$12[\mathbb{C}P^2] = 2c^\nu_{(2)}(\mathbb{C}P^2)[\Theta^2] + 3c^\nu_{(1,1)}(\mathbb{C}P^2)[\Theta^1]^2,$$

where $c^\nu_{(2)}(\mathbb{C}P^2) = -3$, $c^\nu_{(1,1)}(\mathbb{C}P^2) = 6$. Thus $2[\mathbb{C}P^2] = -[\Theta^2] + 3[\Theta^1]^2$.

A quasitoric manifold $M^{2n}$ is a $U$-manifold with a chose of

1. The effective action of a compact torus $T^n$ whose set of fixed points is finite.
2. An equivariant isomorphism

$$c\colon TM^{2n} \oplus (2N-2n)_{\mathbb{R}} \to r\xi,$$

where $\xi$ is a fixed $N$-dimensional complex vector $T^n$-bundle over $M^{2n}$.

3. Mappings

$$\psi\colon V \to \{\Lambda_i \;:\; i = 1, \ldots, q\}, \quad \varepsilon\colon [q] \to \{\pm\},$$

where $V = (v_1, \ldots, v_q) \subset M^{2n}$ is the set of the fixed points of the $T^n$-action, $\psi(i) = \Lambda_i = (\Lambda_i^1, \ldots, \Lambda_i^n)$ is the set of non-trivial weights of the representation $\varrho_i$ of the torus $T^n$ in $TM^{2n}_{v_i}$, and $\varepsilon(i)$ is the sign of the fixed point $v_i$.

For projective toric manifolds and also for the quasi-toric manifolds with equivariant almost complex structures, we have $\varepsilon(i) = 1$ for all $i \in [q]$.

**Theorem (Buchstaber–Panov–Ray, 2010)**

*Let $M^{2n}$ be some quasitoric manifold. Then*

$$\sum_{i=1}^{q} \varepsilon(i) \prod_{j=1}^{n} \frac{1}{\beta(\langle \Lambda_i^j, x\rangle)} = \mathcal{L}(M^{2n})(x),$$

*where $x = (x_1, \ldots, x_n)$, and $\mathcal{L}(M^{2n})(x)$ is the symmetrical series in $x$ such that $\mathcal{L}(M^{2n})(0) = [M^{2n}]$.*

**Corollary**

*For any quasitoric manifold $M^{2n}$, the formula*

$$\gamma_n \mathcal{L}(M^{2n})(0) = \sum_{|\lambda|=n} b_\lambda c_\lambda^\nu(M^{2n})[\Theta^\lambda]$$

*gives the local formulas for the Chern numbers $c_\lambda^\nu(M^{2n})$ for any $\lambda$, $|\lambda| = n$.*

Example. Set $a = x_1 - x_2,\ b = x_2 - x_1,\ c = x_3 - x_1$. Then

$$\mathcal{L}(\mathbb{C}P^2)(x) = \frac{1}{\beta(-a)\beta(c)} + \frac{1}{\beta(-b)\beta(a)} + \frac{1}{\beta(-c)\beta(b)} = \frac{1}{2}\left(3[\Theta_1^1]^2 - [\Theta^2]\right) + o(x).$$

We obtain

$$\frac{1}{2}\left(3[\Theta^1]^2 - [\Theta^2]\right) = c^{\nu}_{(1,1)}(\mathbb{C}P^2)\frac{[\Theta^1]^2}{4} + c^{\nu}_{(2)}(\mathbb{C}P^2)\frac{[\Theta^2]}{3!},$$

i.e. $c^{\nu}_{(1,1)}(\mathbb{C}P^2) = 6$ and $c^{\nu}_{(2)}(\mathbb{C}P^2) = -3$.

The group $S_{n+1}$ acts on $\mathbb{R}^{n+1}$ by permuting the coordinates. Denote by $\sigma_i \in S_{n+1}$ the permutation of the $i$-th and $(i+1)$-th coordinates for $i = 1, \dots, n$.

The regular $n$-dimensional permutohedron $\Pi^n \subset \mathbb{R}^{n+1}$, $n = 0, 1, \dots$, is the convex hull of the $S_{n+1}$-orbit $(\sigma v_*, \sigma \in S_{n+1})$ of the point $v_* = (0, 1, \dots, n)$.

The polyhedron $\Pi^n$ is simple. Its vertex $v_*$ is connected by edges to $n$ vertices $\sigma_i v_*$, $i = 1, \dots, n$.

Consider the standard action of the algebraic torus $(\mathbb{C}^*)^{n+1} \subset \mathbb{C}^{n+1}$ on the manifold $Fl(\mathbb{C}^{n+1})$ of complete complex flags in $\mathbb{C}^{n+1}$.

The permutohedral variety $M^n_\Pi$ is the compactification of the orbit $(\mathbb{C}^*)^{n+1}w$ of a general point $w \in Fl(\mathbb{C}^{n+1})$.

The toric variety $M^n_\Pi$ is a smooth irreducible projective algebraic manifold with the effective action of a compact torus $T^n = T^{n+1}/T^1_*$, where $T^{n+1} \subset (\mathbb{C}^*)^{n+1}$ and $T^1_* \subset T^{n+1}$ is the diagonal subgroup.

The image of the moment map $\mu\colon M^n_\Pi \to \mathbb{R}^{n+1}$ is the permutohedron $\Pi^n$ whose vertices correspond to the fixed points $\mu^{-1}(\sigma v_*)$ under the action of the torus $T^n$ on $M^n_\Pi$, where $v_* = (0, 1, \dots, n)$.

It follows from the theory of toric manifolds that the vectors $\Lambda^1_* = (1, -1, 0, \dots, 0), \dots, \Lambda^n_*(0, \dots, 0, 1, -1)$ are the weights under the action of the torus $T^n$ in the tangent space $T(M^n_\Pi)_{w_*}$ where $w_* = \mu^{-1}(v_*)$.

All fixed points are of sign "$+$".

**Corollary**

$$\sum_{\sigma \in S_{n+1}} \sigma \prod_{i=1}^{n} \frac{1}{\beta(x_i - x_{i+1})} = \mathcal{L}(M^n_\Pi)(x), \ \ \textit{where} \ \ \mathcal{L}(M^n_\Pi)(0) = [M^n_\Pi].$$

Examples: $[M^1_\Pi] = -[\Theta^1]$, $[M^2_\Pi] = [\Theta^2]$.

Set $\Delta_n = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$. The Weil operator $W_n$ is the linear operator

$$W_n \colon \mathbb{Q}[[x_1, \ldots, x_n]] \to \mathrm{Sym_n}, \qquad \mathrm{W_n(x^\xi) = \frac{1}{\Delta_n} \sum_{\sigma \in S_n} sign(\sigma)\sigma(x^\xi)},$$

where $\mathrm{Sym_n} \subset \mathbb{Q}[[\mathrm{x_1, \ldots, x_n}]]$ is the ring of symmetric power series,
$\xi = (j_1, \ldots, j_n)$, and $x^\xi = x_1^{j_1} \cdots x_n^{j_n}$.

From the definition of the Schur polynomials $Sh_\lambda(x_1, \ldots, x_n)$,
$\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$, it follows that

$$W_n(x^{\lambda+\delta}) = Sh_\lambda(x_1, \ldots, x_n), \quad \delta = (n-1, n-2, \ldots, 1, 0).$$

We have $W_n(x^\delta) = 1$, $W_n(\Delta_n(x)) = n!$. Moreover

- $W_n(x^\xi) = 0$ if $j_1 \geqslant \cdots \geqslant j_n \geqslant 0$ and $\xi \neq \lambda + \delta$ for some $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$;
- $W_n(x^\xi) = sign(\sigma) W_n(\sigma x^\xi)$ where $\xi = (j_1, \ldots, j_n)$ and $\sigma\xi = \xi'$,
  $\xi' = (j_1' \geqslant \cdots \geqslant j_n' \geqslant 0)$;
- $W_n$ is homomorphism of $\mathrm{Sym_n}$-modules.

Choose a series $a(z) = 1 + \sum_{m\geqslant 1} a_m z^m$ and consider the series $b(z) = z + \sum_{m\geqslant 1} b_m z^{m+1}$ such that $a(z)b(-z) = -z$.

Set $\widetilde{a}(z) = 1 + \sum_{m=1}^{n} a_m z^m$ and let's introduce the polynomials

$$\pi_n(x) = \widetilde{\Delta}_{n+1}(x) \prod_{i=1}^{n} \widetilde{a}(x_i - x_{i+1}) \quad \text{where} \quad \widetilde{\Delta}_{n+1}(x) \prod_{i=1}^{n} (x_i - x_{i+1}) = \Delta_{n+1}(x).$$

**Theorem (*)**

*For any $n \geqslant 1$, $W_{n+1}(\pi_n(x))\big|_{x=0} = (n+1)!b_n$.*

Examples:

$x^\delta = x_1 : \pi_1(x) = \widetilde{a}(x_1 - x_2) = 1 + a_1(x_1 - x_2)$

$W_2(\pi_1(x)) = 2a_1 = 2b_1$

$x^\delta = x_1^2 x_2 : \pi_2(x) = (x_1 - x_3)\widetilde{a}(x_1 - x_2)\widetilde{a}(x_2 - x_3) =$

$= 1 + \ldots + a_1^2(x_1 - x_2)(x_2 - x_3)(x_1 - x_3) + a_2(x_1 - x_3)[(x_1 - x_2)^2 + (x_2 - x_3)^2] + \ldots$

$W_3(\pi_2(x)) = 6(a_1^2 - a_2) = 6b_2$

Consider the series $\beta(z) = z + \sum\limits_{n\geqslant 1}[\Theta^n]\frac{z^{n+1}}{(n+1)!}$ and take the series $a(z)$ such that $a(z)\beta(-z) = -z$. Then from the Buchstaber–Panov–Ray Theorem (2010),

$$\mathcal{L}(M^n_\Pi)(x) = W_{n+1}\left(\widetilde{\Delta}_{n+1}\prod_{i=1}^{n} a(x_i - x_{i+1})\right),$$

where $W_{n+1}$ is the Weyl operator.

**Theorem (Buchstaber & Veselov, 2023)**

*The cobordism classes of permutohedral manifolds $M^n_\Pi$ and $\Theta$-divisors $\Theta^n$ are related by the formula*

$$[M^n_\Pi] = (-1)^n[\Theta^n],\ n = 0, 1, \dots .$$

*Thus the Chern–Dold character $ch_U$ is uniquely determined by the formula*

$$ch_U(\Psi^{-1}(u)) = z + \sum_{n\geqslant 1}[M^n_\Pi]\frac{z^{n+1}}{(n+1)!},$$

*where $\Psi^{-1}(u) = -\bar{u}$ is the Adams–Novikov operation in the theory $U^*(\cdot)$ and $\bar{u} = \varphi(u)$ is the series such that $F_U(u, \bar{u}) = 0$.*

The first assertion follows from Theorem (*). The second assertion follows from the first one and the properties of the Chern–Dold character.

Consider an algebra $\mathcal{A}$ without additive torsion. The Hirzebruch genus of $U$-manifolds is given by the ring homomorphism $\Phi : \Omega_U \to \mathcal{A}$ determined by the characteristic power series

$$Q_\Phi(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A} \otimes \mathbb{Q}.$$

**Theorem (Buchstaber & Veselov, 2020)**

*The exponential generating function of Hirzebruch genera $\{\Phi(\Theta^n),\ n \in \mathbb{N}\}$ of theta divisors is*

$$\beta_\Phi(z) = z + \sum_{n=1}^{\infty} \Phi(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{z}{Q_\Phi(z)}.$$

Since the ring $\Omega_U$ is generated by the coefficients of series $F_U(u, v)$, we have

**Corollary**

*The series $Q(z) \in \mathcal{A} \otimes \mathbb{Q}$ defines the Hirzebruch genus $\Phi\colon \Omega_U \to \mathcal{A}$ if and only if all the coefficients of the formal group law*

$$F_\Phi(u, v) = \beta_\Phi(\beta_\Phi^{-1}(u) + \beta_\Phi^{-1}(v))$$

*belong to the ring $\mathcal{A}$.*

The classical Todd genus $Td : \Omega_U \to \mathbb{Z}$ is given by the series

$$Q(z) = \frac{z}{1 - e^{-z}} \in \mathbb{Q}[[z]],$$

which defines the exponential of the group $F_{Td}(u, v) = u + v - uv$. Therefore

$$z + \sum_{n=1}^{\infty} Td(\Theta^n) \frac{z^{n+1}}{(n+1)!} = 1 - e^{-z} = z + \sum_{n \in \mathbb{N}} (-1)^n \frac{z^{n+1}}{(n+1)!}.$$

*Thus, the Todd genus of the theta divisors $\Theta^n$ is $Td(\Theta^n) = (-1)^n$.*

For any $U$-manifold $M^{2n}$, the Todd genus is given by the formula

$$Td(M^{2n}) = (-1)^n \sum_{|\lambda|=n} \frac{1}{(\lambda+1)!} c_\lambda^\nu(M^{2n}) = \frac{(-1)^n}{\gamma_n} \sum_{|\lambda|=n} b_\lambda c_\lambda^\nu(M^{2n}) \in \mathbb{Z}.$$

**Example**

Let $c_\lambda^\nu(M^{2n}) = 0$ if $\lambda \neq \lambda_*$. Then $c_{\lambda_*}^\nu(M^{2n})$ is divisible by $(\lambda_* + 1)!$.

**Corollary**

*1. Let $M^{2n}$ be a projective toric manifold. Then*

$$\sum_{|\lambda|=n} b_\lambda c_\lambda^\nu(M^{2n}) = (-1)^n \gamma_n.$$

*2. Let $M^{2n}$ be a quasitoric manifold with an equivariant almost complex structure. Then*

$$(-1)^n \sum_{|\lambda|=n} b_\lambda c_\lambda^\nu(M^{2n}) > 0.$$

For any $U$-manifold $M^{2n}$, the integer

$$c_n(M^{2n}) = \big(c_n(TM^{2n}), \langle M^{2n} \rangle\big)$$

determines the Hirzebruch genus $c\colon \Omega_U \to \mathbb{Z}$.

If a $U$-manifold $M^{2n}$ is complex or almost complex, then the number $c_n(M^{2n})$ is equal to the Euler characteristic $\chi(M^{2n})$ of the manifold $M^{2n}$.

Since $\chi(\mathbb{C}P^n) = n + 1$, the Hirzebruch genus $c$ corresponds to the formal group

$$F_c(u, v) = \frac{u + v - 2uv}{1 - uv}$$

with the logarithm $\frac{u}{1-u}$ and the exponential $\beta_c(z) = \frac{z}{1+z}$.

Since $\Theta^n$ are complex manifolds and $Q(z) = 1 + z$, we have

$$z + \sum_{n=1}^{\infty} \chi(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{z}{1+z}.$$

*Hence the Euler characteristic of the theta divisor $\Theta^n$ is* $\chi(\Theta^n) = (-1)^n (n+1)!$

For $n \in \mathbb{N}$, denote by $\tau(M^{4n})$ the signature of the quadratic intersection form on the homology space $H_{2n}(M^{4n};\mathbb{Q})$ of an oriented manifold. Set $\tau(M^{4n-2}) = 0$.

Since $\tau(\mathbb{C}P^{4n}) = 1$ and $\tau(\mathbb{C}P^{4n-2}) = 0$, the map $\tau\colon \Omega_U^{-2n} \to \mathbb{Z}$ gives the Hirzebruch $L$-genus with the characteristic series $Q(z) = \frac{z}{\tanh z}$. The $L$-genus corresponds to the formal group

$$F_L(u,v) = \frac{u+v}{1+uv}$$

with the exponential $\beta_L(z) = \tanh z$. Since

$$\frac{z}{Q(z)} = \tanh z = \sum_{n=0}^{\infty} 2^{2n+2}(2^{2n+2}-1)B_{2n+2}\frac{z^{2n+1}}{(2n+2)!},$$

we have $\tau(\Theta^{2n-1}) = 0, \qquad \tau(\Theta^{2n}) = \frac{2^{2n+2}(2^{2n+2}-1)}{2n+2}B_{2n+2},$

where $B_n$ are the classical Bernoulli numbers: $B_{2n+1} = 0$, $n > 0$, and

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ldots$$

Example. $\tau(\Theta^2) = -2; \quad \tau(\Theta^4) = 16.$

The Betti numbers of the theta divisors $\Theta^n$ can be computed via the Lefschetz hyperplane theorem (Izadi–Wang, 2015).

**Theorem**

*The homomorphism*

$$i_* : H_k(\Theta^n, \mathbb{Z}) \to H_k(A^{n+1}, \mathbb{Z}), \quad i_* : \pi_k(\Theta^n) \to \pi_k(A^{n+1}),$$

*induced by the embedding* $i : \Theta^n \to A^{n+1}$, *is an isomorphism for* $k < n$ *and an epimorphism for* $k = n$.

Using Poincare duality, we obtain all Betti numbers of $\Theta^n$

$$b_k(\Theta^n) = b_k(A^{n+1}) = \binom{2n+2}{k} = b_{2n-k}(\Theta^n) \text{ for } k < n.$$

Using the formula for the Euler characteristic, we obtain

$$b_n(\Theta^n) = (n+1)! + \frac{n}{n+2}\binom{2n+2}{n+1} = (n+1)! + nC_{n+1},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n$-th Catalan number.

The group $H^*(\Theta^n; \mathbb{Z})$ is torsion-free. The multiplicative structure of the ring $H^*(\Theta^n; \mathbb{Z})$ has not yet been fully described.

Let $H^{p,q}(X)$ be the Dolbeault cohomology group of a complex $n$-dimensional manifold $X$, and set $h^{p,q}(X) = \dim H^{p,q}(X)$.
Following Hirzebruch, consider the index of the elliptic operator

$$\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$$

for fixed $p$ and the corresponding index

$$\chi^p(X) := \sum_{q=0}^{n} (-1)^q h^{p,q}(X).$$

For $p = 0$, this is the holomorphic Euler characteristic, which coincides with the Todd genus of $X$: $\chi^0(X) = Td(X)$ and is related to the arithmetic genus $\chi_a(X)$ by the formula

$$\chi_a(X) = (-1)^n(\chi^0(X) - 1).$$

To compute $\chi^p(X)$ for $p > 0$, introduce the generating polynomial

$$\chi_y(X) := \sum_{p=0}^{n} \chi^p(X) y^p.$$

**Theorem (Hirzebruch, 1956)**

*The value of $\chi_y(X)$ can be given by the Hirzebruch genus with the generating power series*

$$Q(x) = \frac{x(1 + ye^{-x(1+y)})}{1 - e^{-x(1+y)}}.$$

Applying now our general formula, we have

$$\sum_{n=0}^{\infty} \chi_y(\Theta^n) \frac{x^{n+1}}{(n+1)!} = \frac{1 - e^{-x(1+y)}}{1 + ye^{-x(1+y)}}.$$

Since

$$\frac{1-e^{-x(1+y)}}{1+ye^{-x(1+y)}} = \frac{e^{yx}-e^{-x}}{e^{yx}+ye^{-x}},$$

we obtain a particular case of the two-parameter Todd genus $Td_{s,t},\ s=y,\ t=-1$.
Thus we have the following result.

**Theorem**

*The $\chi_y$-genus of the theta divisor $\Theta^n$ can be written in the form*

$$\chi_y(\Theta^n) = (-1)^n A_{n+1}(-y),$$

*where $A_n(s) = \sum_{k=0}^{n-1} A_{n,k}s^k$ is the Eulerian polynomial. In particular,*

$$\chi^p(\Theta^n) = (-1)^{n-p} A_{n+1,p},$$

*where $A_{n,p}$ are the Eulerian numbers.*

The polynomials $A_n(s)$ were defined by Euler in 1755 by the relation

$$\sum_{k=1}^{\infty} k^n t^n = \frac{tA_n(t)}{(1-t)^{n+1}}.$$

Finally, we obtain all the Hodge numbers of the theta divisors:

**Theorem**

*The Hodge numbers $h^{p,q}(\Theta^n)$ of the theta divisor $\Theta^n$ are given by*

$$h^{p,q}(\Theta^n) = h^{p,q}(A^{n+1}) = \binom{n+1}{p}\binom{n+1}{q}, \quad p+q \leqslant n-1,$$

$$h^{p,q}(\Theta^n) = \binom{n+1}{n-p}\binom{n+1}{n-q} = \binom{n+1}{p+1}\binom{n+1}{q+1}, \quad p+q \geqslant n+1,$$

*while for $p+q=n$ we have*

$$h^{p,n-p}(\Theta^n) = A_{n+1,p} - S_{n,p},$$

*where $A_{n,p}$ are the Eulerian numbers and $S_{n,p}$ is given by*

$$S_{n,p} = (-1)^p \binom{n+2}{p+1}\left[(-1)^p \frac{2p-n}{n+2}\binom{n+1}{p} + \sum_{k=0}^{p-1}(-1)^k \binom{n+1}{k}\right].$$

Hodge symmetry: $h^{p,q} = h^{q,p}$; Serre duality: $h^{p,q} = h^{n-p,n-q}$.

The Hodge diamonds $\{h^{p,q} \ : \ p+q = 2n; 0 \leqslant p, q \leqslant n\}$ of the theta divisors $\Theta^n$ have the following form (with Betti numbers shown in the right column):

for $n = 2$ :

| | | | | | | |
|---|---|---|---|---|---|---|
| | | 1 | | | | 1 |
| | 3 | | 3 | | | 6 |
| 3 | | 10 | | 3 | | 16 |
| | 3 | | 3 | | | 6 |
| | | 1 | | | | 1 |

for $n = 3$ :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | 1 | | | | 1 |
| | | 4 | | 4 | | | 8 |
| | 6 | | 1 6 | | 6 | | 28 |
| 4 | | 2 9 | | 29 | | 4 | 66 |
| | 6 | | 1 6 | | 6 | | 28 |
| | | 4 | | 4 | | | 8 |
| | | | 1 | | | | 1 |

for $n = 4$ :

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | | 1 |
| | | | 5 | | 5 | | | | 10 |
| | | 10 | | 25 | | 10 | | | 45 |
| | 10 | | 50 | | 50 | | 10 | | 120 |
| 5 | | 66 | | 146 | | 66 | | 5 | 288 |
| | 10 | | 50 | | 50 | | 10 | | 120 |
| | | 10 | | 25 | | 10 | | | 45 |
| | | | 5 | | 5 | | | | 10 |
| | | | | 1 | | | | | 1 |

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Thank you for your attention!