On the quantified version of the Belnap–Dunn modal logic and some extensions of it

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Intuitionistic logic and the strong negation

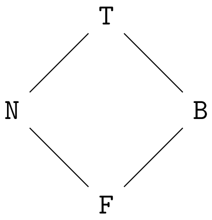
Intuitionistic logic, QInt, is one of the most important systems in mathematical logic.

- At the beginning of the 20th century the informal BHK semantics of QInt was proposed.
- After some time Stephen Kleene managed to transform the informal semantics of QInt into a formal one for intuitionistic arithmetic, HA. Such semantics are now known as the realizability semantics.
- Finally, Kleene's student David Nelson proposed a constructive arithmetic, NA, which expands HA by adding strong negation, $\sim$. The corresponding logics are now known as Nelson's logics, QN4 and QN3.

The Belnap–Dunn four-valued matrix

Locally, the truth in Nelson's logics is four-valued and given by a special matrix, BD4, with domain $\mathtt{T}$, $\mathtt{F}$, $\mathtt{B}$, and $\mathtt{N}$.

This matrix corresponds to a lattice that have the following Hasse diagram:



The strong negation on BD4 is defined as follows: $\sim\mathtt{T}$, $\sim\mathtt{B}$, $\sim\mathtt{N}$ are equal to $\mathtt{F}$, $\mathtt{B}$, $\mathtt{N}$ respectively and $\sim\sim$ is the identity operator.

Nelson's logics as the fragments of modal logics

It is well known that one can treat QInt as a fragment of the modal logic QS4:

$$
\begin{aligned}
\tau(\bot) &:= \bot; \\
\tau(P(t_1,\ldots,t_n)) &:= \Box P(t_1,\ldots,t_n); \\
\tau(\Phi \vee \Psi) &:= \tau(\Phi) \vee \tau(\Psi); \\
\tau(\Phi \wedge \Psi) &:= \tau(\Phi) \wedge \tau(\Psi); \\
\tau(\Phi \to \Psi) &:= \Box(\tau(\Phi) \to \tau(\Psi)); \\
\tau(\exists x\, \Phi) &:= \exists x\, \tau(\Phi); \\
\tau(\forall x\, \Phi) &:= \Box \forall x\, \tau(\Phi).
\end{aligned}
$$

The natural question arises: Can we define a similar translation for Nelson's logics?

The main problem

- S.P. Odintsov and H. Wansing proposed the Belnapian version of the logic K, denoted by BK. They proved that BK and some of its natural extensions are strongly complete w.r.t. suitable Kripke-style semantics.
- By expanding the propositional part of the translation above they also proved that N4 and N3 can be faithfully embedded into the appropriate BK-extensions.
- Our aim is to prove a strong completeness theorem for a quantified version QBK of BK, and to expand the result about faithful embeddings to the first-order setting.

System QBK

Fix a first-order signature σ. From now on by formulas and sentences we mean σ-formulas and σ-sentences respectively.

We begin by expanding BK. Everywhere below $\Phi \Leftrightarrow \Psi$ is an abbreviation for

$$(\Phi \leftrightarrow \Psi) \wedge (\sim\Phi \leftrightarrow \sim\Psi)\,.$$

In addition to the standard axiom schemata for K, our predicate calculus employs the following ones:

SN1. $\sim\sim\Phi \leftrightarrow \Phi$;

SN2. $\sim(\Phi \to \Psi) \leftrightarrow (\Phi \wedge \sim\Psi)$;

SN3. $\sim(\Phi \vee \Psi) \leftrightarrow (\sim\Phi \wedge \sim\Psi)$;

SN4. $\sim(\Phi \wedge \Psi) \leftrightarrow (\sim\Phi \vee \sim\Psi)$;

SN5. $\sim\bot$;

M1. $\neg\Box\Phi \leftrightarrow \Diamond\neg\Phi$;

M2. $\neg\Diamond\Phi \leftrightarrow \Box\neg\Phi$;

M3. $\Box\Phi \Leftrightarrow \sim\Diamond\sim\Phi$;

M4. $\Diamond\Phi \Leftrightarrow \sim\Box\sim\Phi$;

...

System QBK

Q1. $\forall x\,\Phi \to \Phi\,(x/t)$, where t is free for x in Φ;

Q2. $\Phi\,(x/t) \to \exists x\,\Phi$, where t is free for x in Φ;

Q3. $\sim\forall x\,\Phi \leftrightarrow \exists x \sim\Phi$;

Q4. $\sim\exists x\,\Phi \leftrightarrow \forall x \sim\Phi$.

Thus we have the axioms for QK and the axioms that describe interaction of Boolean connectives and modalities with strong negation plus generalized De Morgan's Laws.

System QBK

As for the rules, we have *modus ponens*, the monotonicity rules for $\Box$ and $\Diamond$, and the Bernays rules:

$$\frac{\Phi \quad \Phi \to \Psi}{\Psi}\ (\mathtt{MP})\,; \qquad \frac{\Phi \to \Psi}{\Box\Phi \to \Box\Psi}\ (\mathtt{MB})\,; \qquad \frac{\Phi \to \Psi}{\Diamond\Phi \to \Diamond\Psi}\ (\mathtt{MD})\,;$$

$$\frac{\Phi \to \Psi}{\Phi \to \forall x\, \Psi}\ (\mathtt{BR1})\,; \qquad \frac{\Psi \to \Phi}{\exists x\, \Psi \to \Phi}\ (\mathtt{BR2})\,.$$

System QBK

Denote by Form and Sent the set of all formulas and the set of all sentences respectively.

Denote by QBK the least subset of Form containing the axioms of our calculus and closed under its rules of inference.

Given $\Gamma \subseteq \text{Sent}$ and $\Delta \subseteq \text{Form}$, we write $\Gamma \vdash \Delta$ iff there is finite $\Delta' \subseteq \Delta$ such that the disjunction of Δ' can be obtained from elements of $\Gamma \cup \text{QBK}$ by means of MP, BR1, and BR2.

By a logic is meant simply a subset of Form closed under the five rules above and substitutions. Each logic that includes QBK will be called a QBK-extension. Given a QBK-extension L, we define

$$\Gamma \vdash_L \Delta \quad :\Longleftrightarrow \quad L \cup \Gamma \vdash \Delta.$$

Replacement rules

From a syntactic point of view QBK has some interesting features. For example, it is not closed under the usual replacement rule, i.e. under

$$\frac{\Psi \leftrightarrow \Phi}{\Theta\,(\Phi/\Psi) \leftrightarrow \Theta}\ (\mathtt{R}).$$

Nevertheless, QBK is closed under the so-called weak replacement rule, i.e. under

$$\frac{\Psi \Leftrightarrow \Phi}{\Theta\,(\Phi/\Psi) \Leftrightarrow \Theta}\ (\mathtt{WR}).$$

Negative normal form

By a negative normal form (a nnf for short) we mean a formula in which strong negation stands only before atomic subformulas.

Proposition

For every formula Φ there exists a nnf $\overline{\Phi}$ such that

$$\Phi \Leftrightarrow \overline{\Phi} \in \mathrm{QBK}.$$

Moreover, there is an algorithm that, for a given formula Φ, obtains a suitable $\overline{\Phi}$.

Kripke-style semantics

By a frame we mean a pair $\mathcal{W} = \langle W, R \rangle$ where:

- W is a nonempty set of 'possible worlds';
- $R \subseteq W \times W$.

For any frame $\mathcal{W}$, and families of structures

$$\mathscr{A}^{+} := \left\langle \mathfrak{A}_w^{+} : w \in W \right\rangle, \qquad \mathscr{A}^{-} := \left\langle \mathfrak{A}_w^{-} : w \in W \right\rangle$$

we call a triple

$$\mathcal{M} = \langle \mathcal{W}, \mathscr{A}^{+}, \mathscr{A}^{-} \rangle$$

a QBK-model if for any $u, v \in W$ the following holds:

- $A_u^{+} = A_u^{-}$;
- $c^{\mathfrak{A}_u^{+}} = c^{\mathfrak{A}_u^{-}}$ for all $c \in \text{Const}$;
- if uRv, then $A_u^{+} \subseteq A_v^{+}$ and $c^{\mathfrak{A}_u^{+}} = c^{\mathfrak{A}_v^{+}}$ for all $c \in \text{Const}$.

Kripke-style semantics

The semantics for QBK is locally four-valued. It can be described using two relations:

- $\Vdash^+$, which intuitively stands for verifiability;
- $\Vdash^-$, which intuitively stands for falsifiability.

In general, for any formula Φ, QBK-model $\mathcal{M}$, and world w the following four cases are possible:

T : $\mathcal{M}, w \Vdash^+ \Phi$ and $\mathcal{M}, w \nVdash^- \Phi$;

F : $\mathcal{M}, w \nVdash^+ \Phi$ and $\mathcal{M}, w \Vdash^- \Phi$;

B : $\mathcal{M}, w \Vdash^+ \Phi$ and $\mathcal{M}, w \Vdash^- \Phi$;

N : $\mathcal{M}, w \nVdash^+ \Phi$ and $\mathcal{M}, w \nVdash^- \Phi$.

The semantic consequence, $\vDash$, is defined in the standard way, using the relation $\Vdash^+$.

Strong completeness of QBK

By modifying the canonical models method for first-order modal logics, we prove the following

Theorem

For any $\Gamma \subseteq \text{Sent}$ and $\Delta \subseteq \text{Form}$,

$$\Gamma \vdash \Delta \quad \Longleftrightarrow \quad \Gamma \models \Delta.$$

In fact, this theorem is an analog of the strong completeness theorem for BK obtained by Odintsov and Wansing.

Natural extensions

Strong completeness results can also be obtained for the following two types of QBK-extensions:

1. those obtained by excluding either $\mathtt{N}$ or $\mathtt{B}$ or both;
2. those obtained by imposing restrictions (expressible by modal formulas) on accessibility relations in Kripke frames.

Syntactically, the following axiom schemata correspond to the exclusion of $\mathtt{N}$ and $\mathtt{B}$ respectively:

$\mathtt{ExM}$. $\Phi \vee \sim \Phi$;

$\mathtt{Exp}$. $\sim \Phi \to (\Phi \to \Psi)$.

Natural extensions

Denote QBK $+ \{\texttt{Exp}\}$ and QBK $+ \{\texttt{ExM}\}$ by QB3K and QBK$^\circ$ respectively.

Theorem

QB3K and QBK$^\circ$ are strongly complete w.r.t. appropriate Kripke-style semantics.

We also obtain similar results for extensions of the second kind. In particular, this can be done for the Belnapian version of QS4, denoted by QBS4, and its three-valued extension QB3S4.

A faithful embedding

In conclusion, using the strong completeness theorems for QBS4 and QB3S4 and those for the first-order Nelson's logics, we generalize the result of Odintsov and Wansing about faithful embeddings.

Let $\tau^\star$ be the translation that extends the translation τ by only one new case:

$$\tau^\star(\sim P(t_1, \ldots, t_n)) := \Box \sim P(t_1, \ldots, t_n).$$

Next, for an arbitrary formula Φ we define $\tau^\star(\Phi)$ as the $\tau^\star(\overline{\Phi})$.

A faithful embedding

Theorem

For every formula Φ of the language of Nelson's logics,

$$\Phi \in \text{QN4 (QN3)} \iff \tau^{\star}(\Phi) \in \text{QBS4 (QB3S4)}.$$

Some references

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