# Dynamic Epistemic Logic for Budget-Constrained Agents

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- $[?_i A]\varphi$ – semi-private (semi-public): the question is public, the answer is private (Dolgorukov & Gladyshev 2023)
- resource-based knowledge + common knowledge + group questions $[?_G A]\varphi$

# Example

Consider an example with two agents $i$ and $j$. Let $p_i$ stand for '$i$ is COVID-positive' and $p_j$ stands for '$j$ is COVID-positive'. Assume that the cost of the test is 20 resources ($c_{p_i} = 20 \wedge c_{p_j} = 20$). If we also assume that $i$ decides to make the test ($[?_i p_i]$),

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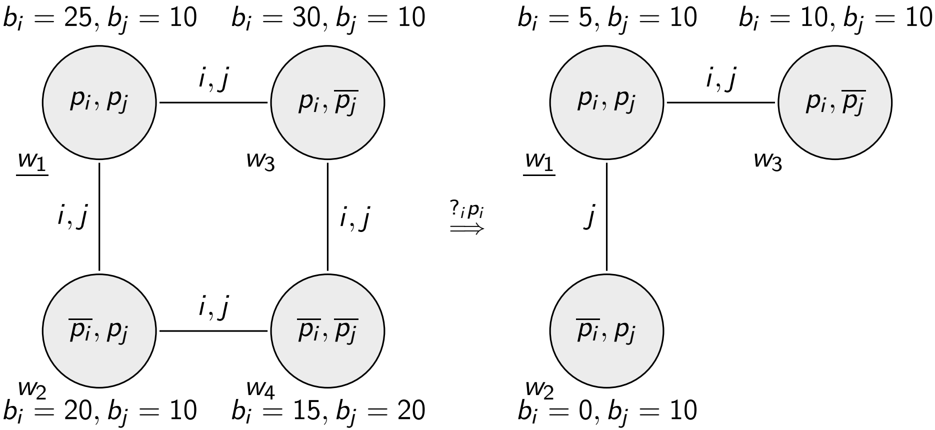


$p_i$ – "$i$ is COVID-positive", $p_j$ – "$j$ is COVID-positive", test's cost is 20$

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- $DEL^{C}_{bc}$ = dynamic epistemic logic for budget-constrained agents + common knowledge + group questions

# Syntax $\mathsf{EL}_{bc}$

- Let Prop $= \{p, q, \dots\}$ be a countable set of propositional letters
- Denote by $\mathcal{L}_{PL}$ the set of all propositional (non-modal) formulas
- Let $Ag = \{i, j, \dots\}$ be a finite set of agents.
- We fix a set of constants $Const = \{c_A \mid A \in \mathcal{L}_{PL}\} \cup \{b_i \mid i \in Ag\}$. It contains a constant $c_A$ for the *cost* of each propositional formula $A$ and a constant $b_i$ for the *budget* of each agent $i$.

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Formulas of the language $\mathrm{EL}_{\mathrm{bc}}$ are defined by the following grammar:

$$\varphi \ ::= \ p \mid (z_1 t_1 + \ldots + z_n t_n) \geq z \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi,$$

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- $\hat{K}_i\varphi := \neg K_i \neg\varphi$, $K_i^?\varphi := K_i\varphi \vee K_i\neg\varphi$

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$\mathfrak{M}$ is a class of S5-models for budget-constrained agents.

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We impose the following conditions on the function Cost:

(C1) $\text{Cost}(w, \bot) = \text{Cost}(w, \top) = 0$,

(C2) $A \approx B$ implies $\text{Cost}(w, A) = \text{Cost}(w, B)$, for all $A, B \in \mathcal{L}_{PL}$ and all $w \in W$.

# Semantics

**Definition**

The *truth* $\models$ of a formula $A$ at a state $w \in W$ of a model $\mathcal{M}$ is defined by induction:
$\mathcal{M}, w \models p$ iff $w \in V(p)$,
$\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$,
$\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$,
$\mathcal{M}, w \models K_i\varphi$ iff $\forall w' \in W$: $w \sim_i w' \Rightarrow \mathcal{M}, w' \models \varphi$,
$\mathcal{M}, w \models (z_1 t_1 + \cdots + z_n t_n) \geq z$ iff $(z_1 t'_1 + \cdots + z_n t'_n) \geq z$, where for $1 \leq k \leq n$,

$$t'_k = \begin{cases} \mathrm{Cost}(w, A), & \text{for } t_k = c_A, \\ \mathrm{Bdg}_i(w), & \text{for } t_k = b_i. \end{cases}$$

# Axiomaization

| | |
|---|---|
| | **Axioms:** |
| (Taut) | All instances of propositional tautologies |
| (Ineq) | All instances of valid formulas about linear inequalities |
| (K) | $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$ |
| (T) | $K_i\varphi \to \varphi$ |
| (5) | $\neg K_i\varphi \to K_i\neg K_i\varphi$ |
| (Bd) | $b_i \geq 0$ |
| ($\geq_1$) | $c_A \geq 0$ |
| ($\geq_2$) | $c_\top = 0$ |
| ($\geq_3$) | $c_A = c_B$ if $A \approx B$, for all formulas $A, B \in \mathcal{L}_{PL}$ |
| | **Inference rules:** |
| (MP) | From $\varphi$ and $\varphi \to \psi$, infer $\psi$ |
| (Nec$_i$) | From $\varphi$ infer $K_i\varphi$ |

# Ineq axioms

Axioms for reasoning about linear inequalities (Fagin et al. 1990):

| | |
|---|---|
| (I1) | $(a_1 t_1 + \cdots + a_k t_k \geq c) \leftrightarrow (a_1 t_1 + \cdots + a_k t_k + 0 t_{k+1}) \geq c)$, |
| (I2) | $(a_1 t_1 + \cdots + a_k t_k \geq c) \rightarrow (a_{j_1} t_{j_1} + \cdots + a_{j_k} t_{j_k} \geq c)$, |
| | where $j_1, \ldots, j_k$ is a permutation of $1, \ldots, k$ |
| (I3) | $(a_1 t_1 + \cdots + a_k t_k \geq c) \wedge (a'_1 t_1 + \cdots + a'_k t_k \geq c') \rightarrow$ |
| | $\rightarrow (a_1 + a'_1) t_1 + \cdots + (a_k + a'_k) t_k \geq (c + c')$ |
| (I4) | $(a_1 t_1 + \cdots + a_k t_k \geq c) \leftrightarrow (d a_1 t_1 + \cdots + d a_k t_k \geq dc)$ for $d > 0$ |
| (I5) | $(t \geq c) \vee (t \leq c)$ |
| (I6) | $(t \geq c) \rightarrow (t > d)$, where $c > d$ |

# Completeness

**Theorem**

*The logic* $\mathsf{EL_{bc}}$ *is sound and (weakly) complete with respect to* $\mathfrak{M}$*, i.e.,*

$$\models_{\mathfrak{M}} \varphi \quad \Longleftrightarrow \quad \vdash_{\mathsf{EL_{bc}}} \varphi$$

$\mathsf{EL_{bc}}$ is not compact:

$$\{b_i > n \mid n \in \mathbb{N}\}$$

# $\mathsf{DEL_{bc}}$: **Syntax**

- The dynamic language $\mathsf{DEL_{bc}}$ extends the static language $\mathsf{EL_{bc}}$ with a dynamic operator $[?_i A]\varphi$.
- A formula $[?_i A]\varphi$ can be read as *"$\varphi$ is true after $i$'s question whether $A$ is true"*.

**Definition**

The *formulas* of $\mathsf{DEL_{bc}}$ are defined by the following grammar:

$$\varphi, \psi ::= p \mid (z_1 t_1 + \cdots + z_n t_n) \geq z) \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_i\varphi \mid [?_i A]\varphi,$$

where $p \in \mathrm{Prop}$, $A \in \mathcal{L}_{PL}$, $i \in Ag$, $t_1, \ldots, t_n \in Const$ and $z_1, \ldots, z_n, z \in \mathbb{Z}$.

# Semantics

## Definition

*Given a model* $\mathcal{M} = (W, (\sim_i)_{i \in Ag}, \mathrm{Cost}, \mathrm{Bdg}, V)$, *an updated model is a tuple* $\mathcal{M}^{?_i A} = (W^{?_i A}, (\sim_j^{?_i A})_j, \mathrm{Cost}^{?_i A}, \mathrm{Bdg}^{?_i A}, V^{?_i A})$, *where*

- $W^{?_i A} = \{w \in W \mid \mathcal{M}, w \models b_i \geq c_A\}$,
- $\sim_j^{?_i A} = (W^{?_i A} \times W^{?_i A}) \cap \sim_j^*$,

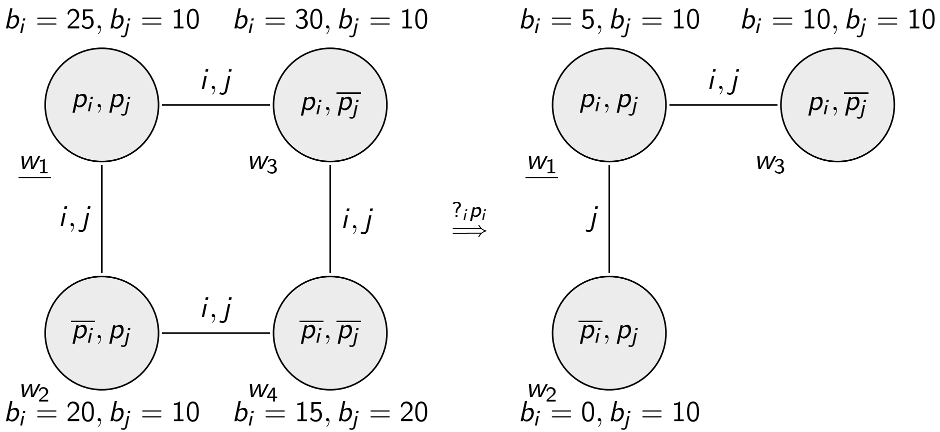
  *where* $\sim_j^* = \begin{cases} \sim_j \bigcap \Big( ([A]_{\mathcal{M}} \times [A]_{\mathcal{M}}) \bigcup ([\neg A]_{\mathcal{M}} \times [\neg A]_{\mathcal{M}}) \Big) & \textit{if } j = i, \\ \sim_j & \textit{otherwise}, \end{cases}$
- $\mathrm{Cost}^{?_i A} = \mathrm{Cost}$,
- $\mathrm{Bdg}_j^{?_i A}(w) = \begin{cases} \mathrm{Bdg}_j(w) - \mathrm{Cost}(w, A), & \textit{if } j = i, \\ \mathrm{Bdg}_j(w), & \textit{otherwise}, \end{cases}$
- $V^{?_i A}(p) = V(p) \cap W^{?_i A}$.

# Semantics

**Definition**

*Given a model* $\mathcal{M} = (W, (\sim_i)_{i \in Ag}, \mathrm{Cost}, \mathrm{Bdg}, V)$ *and a state* $w \in W$*,*

$$\mathcal{M}, w \models [?_i A]\varphi \quad \textit{iff} \quad \mathcal{M}, w \models (b_i \geq c_A) \Rightarrow \mathcal{M}^{?_i A}, w \models \varphi.$$




- $\mathcal{M}, w_1 \models \neg K_i p_i$,
- $\mathcal{M}^{?_i p_i}, w_1 \models K_i p_i$,
- $\mathcal{M}^{?_i p_i}, w_1 \models \neg K_j p_i$,
- $\mathcal{M}^{?_i p_i}, w_1 \models K_j K_i^? p_i$,

- $\mathcal{M}, w_1 \models \neg K_i (b_i \geq 20)$,
- $\mathcal{M}^{?_i p_i}, w_1 \models K_i (b_i \geq 0)$,
- $\mathcal{M}, w_1 \models \neg K_j (b_j = 10)$,
- $\mathcal{M}^{?_i p_i}, w_1 \models K_j (b_j = 10)$

# Some Validities

- $\models [?_i A] K_i^? A$
- $\models [?_i A] K_j K_i^? A$
- $\models [?_i A] K_l K_j K_i^? A$
- $\models \langle ?_i A \rangle \varphi \ \rightarrow \ [?_i A] \varphi$
- $\models (b_i \geq c_A) \ \leftrightarrow \ \langle ?_i A \rangle \top$

# Completeness via Reduction Axioms

| | |
|---|---|
| $(R_p)$ | $[?_i A]p \leftrightarrow (b_i \geq c_A) \rightarrow p$ |
| $(R_{\geq})$ | $[?_i A]\big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big) \leftrightarrow (b_i \geq c_A) \rightarrow$ |
| | $\rightarrow \big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big)^{[b_i \backslash (b_i - c_A)]}$ |
| $(R_{\neg})$ | $[?_i A]\neg\varphi \leftrightarrow (b_i \geq c_A) \rightarrow \neg[?_i A]\varphi$ |
| $(R_{\wedge})$ | $[?_i A](\varphi \wedge \psi) \leftrightarrow [?_i A]\varphi \wedge [?_i A]\psi$ |
| $(R_{K_j})$ | $[?_i A]K_j\varphi \leftrightarrow (b_i \geq c_A) \rightarrow K_j[?_i A]\varphi$, where $i \neq j$ |
| $(R_{K_i})$ | $[?_i A]K_i\varphi \leftrightarrow (b_i \geq c_A) \rightarrow \Big(\big(A \rightarrow K_i(A \rightarrow [?_i A]\varphi)\big) \wedge \big(\neg A \rightarrow K_i(\neg A \rightarrow [?_i A]\varphi)\big)\Big)$ |
| $(Rep)$ | From $\vdash \varphi \leftrightarrow \psi$, infer $\vdash [?_i A]\varphi \leftrightarrow [?_i A]\psi$ |

The notation $\big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big)^{[b_i \backslash (b_i - c_A)]}$ means that all occurrences of $b_i$ in $(z_1 t_1 + \cdots + z_n t_n) \geq z$ are replaced with $(b_i - c_A)$.

# Completeness

**Theorem**

*Logic* $\mathsf{DEL_{bc}}$ *is sound and (weakly) complete w.r.t.* $\mathfrak{M}$, *i.e.* $\vdash_{\mathsf{DEL_{bc}}} \varphi \Leftrightarrow \ \models_{\mathfrak{M}} \varphi$

# $DEL_{bc!} = DEL_{bc} + PAL$

### Definition

The *formulas* of $\mathrm{DEL_{bc!}}$ are defined by the following grammar:

$$\varphi ::= p \mid (z_1 t_1 + \cdots + z_n t_n) \geq z) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid [?_i A]\varphi \mid [!\varphi]\varphi$$

where $p \in \mathrm{Prop}$, $A \in \mathcal{L}_{PL}$, $i \in Ag$, $t_1, \ldots, t_n \in Const$ and $z_1, \ldots, z_n, z \in \mathbb{Z}$.

### Definition

$$\mathcal{M}, w \models [!\varphi]\psi \iff \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}^{!\varphi}, w \models \psi$$

where $\mathcal{M}^{!\varphi}$ is a model $\mathcal{M}$ restricted to $\varphi$-worlds.

# Rational Question

- rational question = the agent doesn't know the answer to this question
- $[?^r_i A]\varphi := [!\neg K^?_i A][?_i A]\varphi$.
- $[?^r_i A]\varphi$ can be read as "$\varphi$ is true after $i$'s rational question whether $A$ is true".

# Reduction Axiom

$$[!\varphi]((z_1 t_1 + \cdots + z_n t_n) \geq z) \leftrightarrow (\varphi \rightarrow (z_1 t_1 + \cdots + z_n t_n) \geq z)$$

# Example

From a pack of three known cards $X, Y, Z$, Alice, Bob and Cath each draw one card. Initially, all agents has zero points. If an agent has $X$ or $Y$, then its score increases by one point. Also, from a pack of three known card $1, 0, 0$ each agent draws one card. If an agent has 1, then its score increases by one point, 0 does not change anything. An agent may ask a question publicly and get an answer (yes or no) privately. The cost of any question is 1 point.

- Bob asks: "Whether Cath has the card $Y$?".
- Alice says "I know that my points and Bob's points are different".
- Cath says "I know the cards".

The sequence of updates can be formalized as follows:

$$\langle ?^r_b Y_c \rangle \langle ! K_a (b_a \neq b_b) \rangle \langle ! K_c (XYZ)_? \rangle \top$$

- $K_i(XYZ)_? := K_i^? X_? \wedge K_i^? Y_? \wedge K_i^? Z_?$
- $K_i^? X_? := K_i^? X_a \wedge K_i^? X_b \wedge K_i^? X_c$ (similarly for $Y$ and $Z$).

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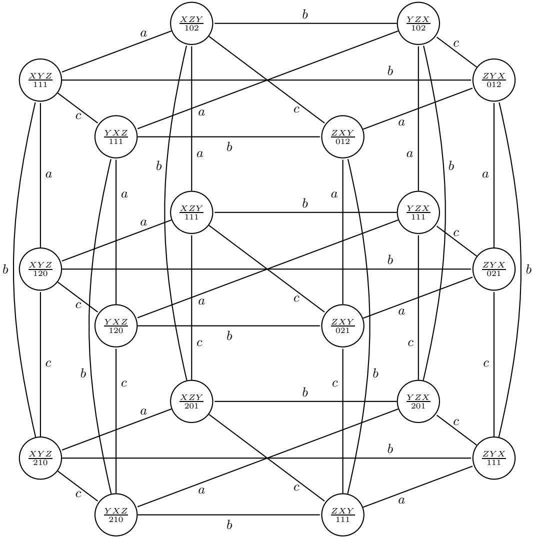
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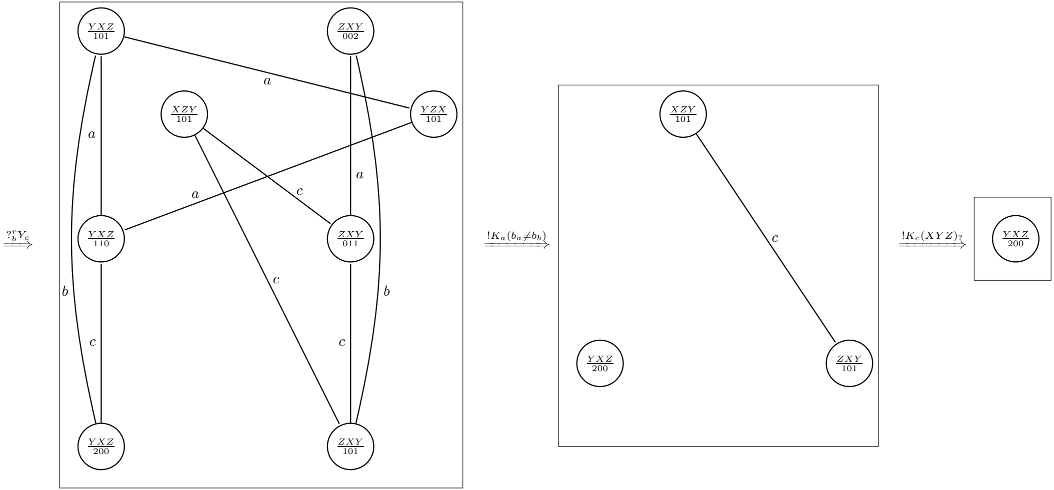
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# $\mathrm{DEL_{bc}}$ + Common Knowledge + Group Questions

**Definition (Language $DEL_{bc}^{C}$)**

$$\varphi \ ::= \ p \mid (z_1 t_1 + \ldots + z_n t_n) \geq z \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid C_G\varphi \mid [?_G A]\varphi$$

# Budget Constraint

**Definition**

$\mathrm{BCS}(G, A) := \bigwedge_{i \in G} (b_i \geq \frac{c_A}{|G|})$

"the Budget Constraint for the query $A$ for $G$ is Satisfied"

# Updated model

**Definition**

Given a model $\mathcal{M}$, a group $G \subseteq Ag$ and a formula $A \in \mathcal{L}_{PL}$ an updated model $\mathcal{M}^{?_G A}$ is a tuple $\mathcal{M}^{?_G A} = (W^{?_G A}, (\sim_j^{?_G A})_{j \in Ag}, \mathrm{Cost}^{?_G A}, \mathrm{Bdg}^{?_G A}, V^{?_G A})$ where

- $W^{?_G A} = \{w \in W \mid \mathcal{M}, w \vDash \mathrm{BCS}(G, A)\}$
- $\sim_j^{?_G A} = (W^{?_G A} \times W^{?_G A}) \cap \sim_j^*$, where $\sim_j^* = \sim_j$ if $j \notin G$ and $\sim_j^* = \sim_j \bigcap \Big(([A]_{\mathcal{M}} \times [A]_{\mathcal{M}}) \bigcup ([\neg A]_{\mathcal{M}} \times [\neg A]_{\mathcal{M}})\Big)$ if $j \in G$
- $\mathrm{Cost}_j^{?_G A}(w, B) = \mathrm{Cost}_j(w, B)$, for all $B \in \mathcal{L}_{PL}$
- $\mathrm{Bdg}_j^{?_G A}(w) = \begin{cases} \mathrm{Bdg}_j(w) - \frac{\mathrm{Cost}_j(w,A)}{|G|}, & \text{if } j \in G, \\ \mathrm{Bdg}_j(w), & \text{if } j \notin G, \end{cases}$
- $V^{?_G A}(p) = V(p) \cap W^{?_G A}$.

# Semantics

**Definition**

$\mathcal{M}, w \models [?_G A]\varphi \quad$ iff $\quad \mathcal{M}, w \models \mathrm{BCS}(G, A)$ implies $\mathcal{M}^{?_G A}, w \models \varphi$

# Some Valid Formulas

1. $A \to [?_i A] K_i A$
2. $[?_i A] K_i^? A$
3. $A \to [?_G A] C_G A$
4. $[?_G A] C_G^? A$
5. $[?_i A] C_G K_i^? A$
6. $[?_G A] C_H C_G^? A$

# Axiomatization $DEL^C_{bc}$

- (Taut) All instances of propositional tautologies
- (Ineq) All instances of valid formulas about linear inequalities
- (K) $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
- (T) $K_i\varphi \to \varphi$
- (5) $\neg K_i\varphi \to K_i\neg K_i\varphi$
- (C) $C_G\varphi \to E_G(\varphi \wedge C_G\varphi)$
- ($\mathrm{B}^+$) $b_i \geq 0$
- ($\mathrm{c}^+$) $c_i(A) \geq 0$
- ($\mathrm{c}^\top$) $c_i(\top) = 0$
- ($\mathrm{c}^\approx$) $c_i(A) = c_i(B)$ if $A \approx B$

# Axiomatization

- ($\mathrm{r}_p$) $[?_G A]p \leftrightarrow \mathrm{BCS}(G, A) \rightarrow p$
- ($\mathrm{r}_\geq$) $[?_G A](\sum_{i=1}^{k} a_i t_i \geq z) \leftrightarrow \Big(\mathrm{BCS}(G, A) \rightarrow (\sum_{i=1}^{k} a_i t_i \geq z)^{(G,A)}\Big)$[1]
- ($\mathrm{r}_\neg$) $[?_G A]\neg\varphi \leftrightarrow \mathrm{BCS}(G, A) \rightarrow \neg[?_G A]\varphi$
- ($\mathrm{r}_\wedge$) $[?_G A](\varphi \wedge \psi) \leftrightarrow [?_G A]\varphi \wedge [?_G A]\psi$
- ($\mathrm{r}_{K1}$) $[?_G A]K_j\varphi \leftrightarrow \mathrm{BCS}(G, A) \rightarrow K_j[?_G A]\varphi$, where $j \notin G$
- ($\mathrm{r}_{K2}$) $[?_G A]K_i\varphi \leftrightarrow \mathrm{BCS}(G, A) \rightarrow \bigwedge_{A' \in \{A, \neg A\}} \Big((A' \rightarrow K_i(A' \rightarrow [?_G A]\varphi))\Big)$, where $i \in G$

[1] $(\sum_{i=1}^{k} a_i t_i \geq z)^{(G,A)}$ denotes $(\sum_{i=1}^{k} a_i t_i \geq z)$, in which all occurrences of $b_i$ for $i \in G$ among $t_1, \ldots, t_k$ are replaced with $(b_i - \frac{c_A}{|G|})$.

# Axiomatization

- (MP) from $\varphi$ and $\varphi \to \psi$, infer $\psi$
- (Nec$_i$) from $\varphi$ infer $K_i\varphi$
- (Rep) from $\varphi \leftrightarrow \psi$, infer $[?_G A]\varphi \leftrightarrow [?_G A]\psi$
- (RC1) from $\varphi \to E_G(\varphi \wedge \psi)$, infer $\varphi \to C_G\psi$
- (RC2)

$$\frac{\chi \to [?_G A]\psi \qquad (\chi \wedge \mathrm{BCS}(G,A)) \to \left( \bigwedge_{A' \in \{A, \neg A\}} (A' \to E_{H\cap G}(A' \to \chi)) \wedge E_{H\setminus G}\chi \right)}{\chi \to [?_G A] C_H \psi}$$

# Completeness

**Theorem**

*Logic* $DEL^{C}_{bc}$ *is sound and (weakly) complete w.r.t.* $\mathfrak{M}$, *i.e.*

$$\vdash_{DEL^{C}_{bc}} \varphi \Leftrightarrow \models_{\mathfrak{M}} \varphi$$

# Further Development

Modal formulas inside []

- $[?_i\varphi]\psi$, $\varphi \in \mathsf{DEL}_{\mathsf{bc!}}$

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Quantification over updates in APAL-style

- $\langle ?_i^n \rangle \varphi$ – there is a propositional formula, $A$, such that the cost of $A$ is at most, $n$, and it is true that $\langle ?_i A \rangle \varphi$
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Resource-Based Distributive Knowledge

- $M, x \models D_G^r \varphi := \forall i \in G\ \exists A_i \in PL\ \exists \varphi \in \mathsf{DEL}_{\mathsf{bc}}$ : 1) $M, x \models \langle ?_i A_i \rangle \varphi_i$ 2) $\models \bigwedge_{i \in G} \varphi_i \rightarrow \varphi$