# Logic of Combinatory Logic

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The present talk is based on joint work with Simona Kašterović

# Outline

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# Decision Problem - Background

- Gottfried Wilhelm Leibniz - Characteristica universalis

- David Hilbert and Wilhelm Ackerman (1928)
  - **Entscheidungsproblem**, or Decision Problem:

    "Given all the axioms of math,
    is there an algorithm that can tell if a proposition is universally valid,
    i.e. deducable from the axioms?"

- **Negative answers** (1935/36):
  - Alonzo Church - $\lambda$-calculus (equality)
  - Alan Turing - Turing Machines (halting problem)

  - Kurt Gödel - Incompleteness theorems (1931)

# $\lambda$-calculus and Combinatory Logic (1930s)

- Alonzo Church:
  - theory of functions - formalisation of mathematics (inconsistent)
  - successful model for computable functions - **$\lambda$-calculus**
  - **simply typed $\lambda$-calculus**
- Haskell Curry:
  - elimination of variables in logic - Moses Schönfinkel (1921)
  - successful model for computable functions - **Combinatory logic**
  - **Combinatory logic with types**
- Alan Turing :
  - formalisation of the concepts of algorithm and computation
  - **Turing Machines**

# $\lambda$-calculus and Combinatory Logic

- Expressiveness - Effective computability (mid 1930s)
  - **(Curry)** Equivalence of $\lambda$-calculus and Combinatory Logic
  - **(Kleene)** Equivalence of $\lambda$-calculus and recursive functions
  - **(Turing)** Equivalence of $\lambda$-calculus and Turing machines

# $\lambda$-calculus

Language $\qquad M \ ::= \ x \mid MM \mid \lambda x.M$

Reduction rule (operational semantics)

$$(\lambda x.M)N \ \longrightarrow M[x \to N]$$

$$f(x) = x^2 + x + 42$$

$$f \equiv \lambda x.(x^2 + x + 42)$$

$$\begin{array}{rcl} \lambda x.(x^2 + x + 42)5 & \longrightarrow & (x^2 + x + 42)[x \to 5] \\ & \equiv & 5^2 + 5 + 42 \\ & = & 72 \end{array}$$

# Combinatory logic

Language $M ::= S \mid K \mid I \mid MM$

Reduction rules (operational semantics)

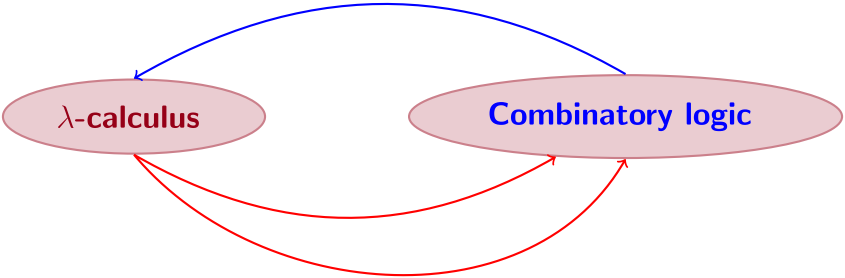
$$\begin{array}{lcl} IM & \longrightarrow & M \\ KMN & \longrightarrow & M \\ SMNP & \longrightarrow & MP(NP) \end{array}$$

$$I \equiv \lambda x.x$$

$$K \equiv \lambda xy.x$$

$$S \equiv \lambda xyz.xz(yz)$$

# Equivalence of expresiveness



# Simply typed $CL_{\to}$

**Terms:** $M, N ::= x \mid \mathsf{S} \mid \mathsf{K} \mid \mathsf{I} \mid MN$

**Types:** $\sigma, \tau := a \mid \sigma \to \tau$

**Type assignment statement**

$$M : \sigma$$

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \frac{}{\Gamma \vdash \mathsf{S} : (\sigma \to (\rho \to \tau)) \to (\sigma \to \rho) \to (\sigma \to \tau)}$$

$$\frac{}{\Gamma \vdash \mathsf{K} : \sigma \to (\tau \to \sigma)} \qquad \frac{}{\Gamma \vdash \mathsf{I} : \sigma \to \sigma}$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

## Subject reduction in $CL_{\to}$ ✓

If $\Gamma \vdash M : \sigma$ and $M \to N$ then $M : \sigma$

**NB** Subject expansion fails ✗

$$KI\Omega \to I$$

$\vdash I : \sigma \to \sigma \quad KI\Omega$ not typable

# Equational theory $\mathcal{EQ}$ of CL

**Equational theory $\mathcal{EQ}$**

$$M = M \quad (id) \qquad \mathsf{S}MNL = (ML)(NL) \quad (S)$$

$$\mathsf{K}MN = M \quad (K) \qquad \mathsf{I}M = M \quad (I)$$

$$\frac{M = N}{N = M}\,(\text{sym}) \qquad \frac{M = N \qquad N = L}{M = L}\,(\text{trans})$$

$$\frac{M = N}{MP = NP}\,(\text{app-l}) \qquad \frac{M = N}{PM = PN}\,(\text{app-r})$$

Rule:
$$\frac{Mx = Nx \qquad x \notin FV(M) \cup FV(N)}{M = N}\,(\text{ext})$$

- Equational theory $\mathcal{EQ}$ + rule (ext) = Equational theory $\mathcal{EQ}^{\eta}$.

# Simply typed combinatory logic with equality $CL^{=}$

Extend simply typed combinatory logic $CL_{\rightarrow}$ with the rule

$$\frac{\Gamma \vdash_{\mathsf{CL}^=} M : \sigma \qquad M = N \text{ is provable in } \mathcal{EQ}^{\eta}}{\Gamma \vdash_{\mathsf{CL}^=} N : \sigma} \text{(eq)}$$

**Subject reduction and expansion in $CL^{=}$ ✓**

If $\Gamma \vdash_{\mathsf{CL}^=} M : \sigma$ and $M =_{(\beta\eta)} N$ then $N : \sigma$

# Outline

# *LCL* Syntax

- Logic of combinatory logic (*LCL*) - classical propositional logic over simply typed combinatory logic
- Simply Typed Combinatory logic

**Terms:** $M, N ::= x \mid \mathsf{S} \mid \mathsf{K} \mid \mathsf{I} \mid MN$

**Types:** $\sigma, \tau := a \mid \sigma \to \tau$

**Type assignment statement**

$$M : \sigma$$

**Syntax of *LCL***

$$\alpha, \beta := M : \sigma \mid \neg\alpha \mid \alpha \Rightarrow \beta$$

# *LCL* Axiomatisation

- Combination of the axiomatic system for classical propositional logic and type assignment system for simply typed combinatory logic
- Eight axiom schemes:

**(Ax** 1**)** $\mathsf{S} : (\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$

**(Ax** 2**)** $\mathsf{K} : \sigma \to (\tau \to \sigma)$

**(Ax** 3**)** $\mathsf{I} : \sigma \to \sigma$

**(Ax** 4**)** $(M : \sigma \to \tau) \Rightarrow ((N : \sigma) \Rightarrow (MN : \tau))$

**(Ax** 5**)** $M : \sigma \Rightarrow N : \sigma$, if $M = N$

**(Ax** 6**)** $\alpha \Rightarrow (\beta \Rightarrow \alpha)$

**(Ax** 7**)** $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$

**(Ax** 8**)** $(\neg\alpha \Rightarrow \neg\beta) \Rightarrow ((\neg\alpha \Rightarrow \beta) \Rightarrow \alpha)$

- Inference rule:

$$\frac{\alpha \Rightarrow \beta \qquad \alpha}{\beta}\ (\mathsf{MP})$$

# LCL Logic

$$\alpha_1, \ldots, \alpha_n \vdash \alpha \ (LCL)$$

$$T \vdash \alpha$$

$LCL$ is an extension of the simply typed combinatory logic $CL_{\rightarrow}$

Let $\Gamma$ be a basis. If $\Gamma \vdash M : \sigma$, then $\Gamma \vdash M : \sigma \ (LCL)$

# Related work


S. Feferman, "Constructive theories of functions and classes," in Proceedings Logic Colloquium '78, v. D. Boffa, D. and K. McAloon, Eds., vol. 97. Amsterdam: North-Holland, 1979, pp. 159–224, mons, Aug. 24–Sept. 1., 1978.

M. Beeson, "Lambda logic," in Automated Reasoning - Second International Joint Conference, IJCAR 2004, Cork, Ireland, July 4-8, 2004, Proceedings, ser. Lecture Notes in Computer Science, D. A. Basin and M. Rusinowitch, Eds., vol. 3097. Springer, 2004, pp. 460–474. [Online]. Available: https://doi.org/10.1007/978-3-540-25984-8 34

H. Barendregt, M. W. Bunder, and W. Dekkers, "Systems of illative combinatory logic complete for first-order propositional and predicate calculus," J. Symb. Log., vol. 58, no. 3, pp. 769–788, 1993. [Online]. Available: https://doi.org/10.2307/2275096

W. Dekkers, M. W. Bunder, and H. Barendregt, "Completeness of two systems of illative combinatory logic for first-order propositional and predicate calculus," Arch. Math. Log., vol. 37, no. 5-6, pp. 327–341, 1998. [Online]. Available: https://doi.org/10.1007/s001530050102

# LCL Semantics

Applicative structure for $LCL$

$$\mathcal{M} = \langle \mathbf{D}, \{A^\sigma\}_\sigma, \cdot, \mathbf{s}, \mathbf{k}, \mathbf{i} \rangle$$

$D$ is a non-empty set, called *domain*

$\cdot$

It is

if $(\forall e \in$

It hold

$\mathbf{i}$ is an element of the domain $D$, such that

- for every $\sigma \in$ Types,

$$\mathbf{i} \in A^{\sigma \to \sigma}$$

- for every $d \in D$,

$$\mathbf{i} \cdot d = d$$

# LCL Semantics

Applicative structure for $LCL$

$$\mathcal{M} = \langle D, \{A^\sigma\}_\sigma, \cdot, \mathbf{s}, \mathbf{k}, \mathbf{i} \rangle$$

An environment $\rho$ for $\mathcal{M}$ is a map from the set of term variables to the domain of the applicative structure $\mathcal{M}$, $\rho : V \to D$.

A $LCL$-model is a tuple $\mathcal{M}_\rho = \langle \mathcal{M}, \rho \rangle$, where $\mathcal{M}$ is an applicative structure and $\rho$ is an environment for $\mathcal{M}$.


S. Ghilezan, S. Kašterović. Semantics for Combinatory Logic With Intersection Types, Frontiers in Computer Science, volume 4, 2022. doi: 10.3389/fcomp.2022.792570.

Kašterović, S., Ghilezan, S., *Kripke semantics and completeness for full simply typed lambda calculus,* Journal of Logic and Computation, Volume 30, issue 8 (2020).

Mitchell, J. C., and E. Moggi, *Kripke-style models for typed lambda calculus,* Annals of Pure and Applied Logic, vol. 51, pp. 99–124, 1991.

## The interpretation of a term

- $[\![x]\!]_\rho = \rho(x)$;
- $[\![\mathsf{S}]\!]_\rho = \mathbf{s}$;
- $[\![\mathsf{K}]\!]_\rho = \mathbf{k}$;
- $[\![\mathsf{I}]\!]_\rho = \mathbf{i}$;
- $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$.

## Satisfiability

- $\mathcal{M}_\rho \models M : \sigma$ if and only if $[\![M]\!]_\rho \in A^\sigma$;
- $\mathcal{M}_\rho \models \alpha \Rightarrow \beta$ if and only if $\mathcal{M}_\rho \not\models \alpha$ or $\mathcal{M}_\rho \models \beta$;
- $\mathcal{M}_\rho \models \neg\alpha$ if and only if it is not true that $\mathcal{M}_\rho \models \alpha$.

# Outline

# Soundness and Completeness results

- Soundness and Completeness of $LCL$ ✓ ✓
- Soundness of $CL_{\rightarrow}$ ✓ (no completeness ✗ )
- Soundness and Completeness of $CL^{=}$ ✓ ✓

# 1. Soundness and completeness of *LCL*

**Soundness of *LCL***

If $T \vdash \alpha$, then $T \models \alpha$

Proof. By induction on the proof length of $T \vdash \alpha$

**Completeness of *LCL***

If $T \models \alpha$, then $T \vdash \alpha$.

Proof. Adaptation of the Henkin-style completeness method.

The proof of completeness comprises the following steps:

- the proof of Deduction theorem,
- the proof that every consistent set of $LCL$-formulas can be extended to a maximal consistent set,
- the construction of a canonical model using maximal consistent set,
- the proof that a canonical model is an $LCL$-model,
- the proof that every consistent set is satisfiable,
- the proof of: If $T \models \alpha$, then $T \vdash \alpha$.

# 2. Soundness of $CL_{\to}$

**Soundness of $CL_{\to}$**

If $\Gamma \vdash_{\mathsf{CL}} M : \sigma$, then $\Gamma \models M : \sigma$

Proof. Follows directly from the soundness of $LCL$ and the fact that $LCL$ is an extension of $CL_{\to}$

Completeness of $CL_{\to}$ does not hold.

## Example

As a counterexample we take statements $x : \sigma$ and $\mathsf{K}xy : \sigma$
In the equational theory $\mathcal{EQ}^{\eta}$: $\mathsf{K}xy = x$
Thus, $[\![\mathsf{K}xy]\!]_\rho = [\![x]\!]_\rho$ and as consequence

$$x : \sigma \models \mathsf{K}xy : \sigma$$

However, in $CL_{\to}$:

$$x : \sigma \not\vdash_{\mathsf{CL}} \mathsf{K}xy : \sigma$$

# Soundness and completeness of equational theory $\mathcal{EQ}^\eta$

## Soundness of $\mathcal{EQ}^\eta$

If $M = N$ in provable in $\mathcal{EQ}^\eta$, then $[\![M]\!]_\rho = [\![N]\!]_\rho$ for any $LCL$-model $\mathcal{M}_\rho = \langle \mathcal{M}, \rho \rangle$

Proof. By induction on the length of derivation $M = N$

## Completeness of $\mathcal{EQ}^\eta$

If $[\![M]\!]_\rho = [\![N]\!]_\rho$ in every $LCL$-model, then $M = N$ is provable in $\mathcal{EQ}^\eta$.

Proof. By defining an $LCL$-model such that $[\![M]\!]_\rho = [M]$, where $[M]$ is the equivalence class of term $M$ with respect to the equivalence relation induced by $\mathcal{EQ}^\eta$, i.e.

$$[M] = \{N \mid M = N \text{ is provable in } \mathcal{EQ}^\eta\}$$

# 3. Soundness and completeness of $CL^=$

**Completeness of $CL^=$**

If $\Gamma \models M : \sigma$, then $\Gamma \vdash_{\mathsf{CL}^=} M : \sigma$

$LCL$ is a conservative extension of $CL^=$.

Simona Kašterović and SG. Logic of combinatory logic.
https://arxiv.org/abs/2212.06675

# Outline

# Motivation for LCL

- Automated reasoning in type systems
  $\hookrightarrow$ SAT solvers

- Probabilistic reasoning in type systems
  $\hookrightarrow$ Probabilistic logic

# Motivation

- probabilistic computation, probabilistic programming - mainstream

$$M \ \rightarrow_p \ N$$

- probabilistic reasoning in type systems

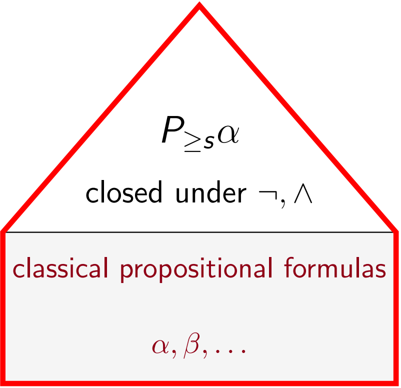
$$(M : \sigma)_p$$

# Probability logic

Zoran Ognjanović, Miodrag Rašković, and Zoran Marković
Probability Logics - Probability-Based Formalization of Uncertain Reasoning
Springer, 2016

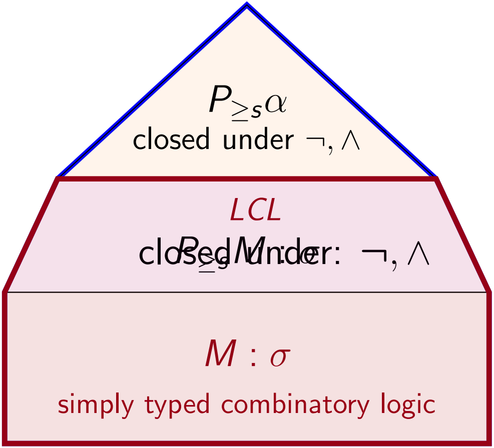
# Idea

$LPP_2$




$\alpha, \beta := p \mid \neg\alpha \mid \alpha \wedge \beta$

$\varphi, \psi := P_{\geq s}\alpha \mid \neg\varphi \mid \varphi \wedge \psi$




**Logic of combinatory logic**

# Ongoing and further work

- probabilistic reasoning in simply typed lambda calculus and combinatory logic
- probabilistic reasoning in different typed calculi such as polymorphic types, intersection types, higher-order types, etc.
- probabilistic reasoning in type theory
- intuitionistic reasoning at the basic level

# Hvala! ХВАЛА!


obrigado
Dank U
Merci
mahalo
Köszi
спасибо
Grazie
Thank you
mauruuru
Takk
Gracias
Dziękuję
Děkuju
danke
Kiitos