OPERATIONS ON NON-DETERMINISTIC MATRICES AND THEIR USE

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What is a (propositional) Logic?

A (Tarskian) *consequence relation* for a language $\mathcal{L}$:

Reflexivity: $\psi \vdash \psi$.

Monotonicity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.

Transitivity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \varphi$ then $\mathcal{T}, \mathcal{T}' \vdash \varphi$.

A consequence relation $\vdash$ is called:

Structural: if $\mathcal{T} \vdash \psi$ then $\theta(\mathcal{T}) \vdash \theta(\psi)$.

Non-trivial: $\mathcal{T} \not\vdash \psi$ for some $\mathcal{T} \neq \emptyset$ and ψ.

Finitary: if $\mathcal{T} \vdash \psi$ then $\mathcal{T}' \vdash \psi$ for some finite $\mathcal{T}' \subseteq \mathcal{T}$.

A (propositional) *logic* is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where

- $\mathcal{L}$ is a propositional language, and
- $\vdash$ is a structural, non-trivial and finitary consequence relation for $\mathcal{L}$.

Ways of Defining Logics

- Semantically: $\Gamma \vdash_{\mathsf{S}} \psi$ if every "model" of Γ is a "model" of ψ in the semantics S.
- Syntactically: $\Gamma \vdash_{\mathsf{D}} \psi$ if ψ has a proof from Γ in the deduction system D.

Example: $\mathbf{CL}^+$

- $\mathcal{V} = \{0, 1\}$
- $\mathcal{D} = \{1\}$

$\mathcal{O}(\wedge)$	1	0
1	1	0
0	0	0

$\mathcal{O}(\vee)$	1	0
1	1	1
0	1	0

$\mathcal{O}(\supset)$	1	0
1	1	0
0	1	1

HCL$^+$

- Axiom schemata:

$$\begin{array}{rl}
\text{I1} & \varphi \supset (\psi \supset \varphi) \\
\text{I2} & (\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta) \\
\text{I3} & ((\psi \supset \varphi) \supset \psi) \supset \psi \\
\text{C1} & \varphi \wedge \psi \supset \varphi \\
\text{C2} & \varphi \wedge \psi \supset \psi \\
\text{C3} & \varphi \supset (\psi \supset \varphi \wedge \psi) \\
\text{D1} & \varphi \supset \varphi \vee \psi \\
\text{D2} & \psi \supset \varphi \vee \psi \\
\text{D3} & (\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)
\end{array}$$

- Inference Rule:

$$\frac{\psi \quad \psi \supset \varphi}{\varphi}\ MP$$

Matrices

Logical Matrices

A matrix **M** consists of:

- a nonempty set $\mathcal{V}$ of *truth values*
- a proper subset $\mathcal{D}$ of $\mathcal{V}$: the *designated* truth values
- for each connective $\diamond$ of arity n, a "truth table" $\mathcal{O}(\diamond) : \mathcal{V}^n \rightarrow \mathcal{V}$

M-valuations

$$v(\diamond(\varphi_1, \ldots, \varphi_n)) = \mathcal{O}(\diamond)(v(\varphi_1), , \ldots, v(\varphi_n))$$

When Does $\mathcal{T} \vdash_{\mathbf{M}} \varphi$?

For every **M**-valuation v:

$$v(\psi) \in \mathcal{D} \text{ for every } \psi \in \mathcal{T} \Longrightarrow v(\varphi) \in \mathcal{D}$$

Adding Negation

Suppose we want to conservatively add to $\mathbf{CL}^+$ a unary connective $\neg$ such that the following holds for every atomic formula P:

$$P \nvdash \neg P \text{ and } \neg P \nvdash P$$

(Such a connective may be called weak negation.)

In the framework of two-valued matrices there is exactly one way to introduce such a connective: classical negation.

Syntactically, this corresponds to adding to $\mathbf{HCL}^+$ the axioms:

$$\neg\varphi \vee \varphi \text{ and } \neg\varphi \supset (\varphi \supset \psi)$$

What should we do if we want just one of them?

Solution 1: Use of Three-valued Matrices

- $\mathcal{V} = \{0, 1/2, 1\}$
-

$\mathcal{O}(\wedge)$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\mathcal{O}(\vee)$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\mathcal{O}(\neg)$	
1	0
1/2	1/2
0	1

Łukasiewicz

- $\mathcal{D} = \{1\}$
-

$\mathcal{O}(\supset)$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

- $\not\vdash \neg P \vee P \quad \vdash \neg P \wedge P \supset Q$

$\mathbf{J}_3$

- $\mathcal{D} = \{1, 1/2\}$
-

$\mathcal{O}(\supset)$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1	1	1

- $\not\vdash \neg P \wedge P \supset Q \quad \vdash \neg P \vee P$

Problems with the Matrices-based Method

1. Solving problems of the above sort with this method seems to be done on a case by case basis.

   Can we solve such problems in a a more systematic way?

2. We get results we might not want. Thus $\neg\neg\varphi \supset \varphi$ is valid in Łukasiewicz 3-valued logic, while $\neg(\varphi \wedge \neg\varphi)$ is valid in $\mathbf{J}_3$.

   Are there methods for getting exactly what we want?

3. Actually, while $\mathbf{J}_3$ is indeed conservative over $\mathbf{CL}^+$, Łukasiewicz 3-valued logic is not.

   Can we ensure conservativity in advance?

Yes, we can – by using non-deterministic matrices!

Matrices

Logical Matrices

A matrix **M** consists of:

- a nonempty set $\mathcal{V}$ of *truth values*
- a proper subset $\mathcal{D}$ of $\mathcal{V}$: the *designated* truth values
- for each connective $\diamond$ of arity n, a "truth table" $\mathcal{O}(\diamond) : \mathcal{V}^n \to \mathcal{V}$

M-valuations

$$v(\diamond(\varphi_1, \ldots, \varphi_n)) = \mathcal{O}(\diamond)(v(\varphi_1), , \ldots, v(\varphi_n))$$

When Does $\mathcal{T} \vdash_{\mathbf{M}} \varphi$?

For every **M**-valuation v:

$$v(\psi) \in \mathcal{D} \text{ for every } \psi \in \mathcal{T} \Longrightarrow v(\varphi) \in \mathcal{D}$$

Non-deterministic Matrices

Non-deterministic Logical Matrices

A Nmatrix **M** consists of:

- a nonempty set $\mathcal{V}$ of *truth values*
- a proper subset $\mathcal{D}$ of $\mathcal{V}$: the *designated* truth values
- for each connective $\diamond$ of arity n, a "truth table" $\mathcal{O}(\diamond) : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$

M-valuations

$$v(\diamond(\varphi_1, \ldots, \varphi_n)) \in \mathcal{O}(\diamond)(v(\varphi_1), , \ldots, v(\varphi_n))$$

When Does $\mathcal{T} \vdash_{\mathbf{M}} \varphi$?

For every **M**-valuation v:

$$v(\psi) \in \mathcal{D} \text{ for every } \psi \in \mathcal{T} \Longrightarrow v(\varphi) \in \mathcal{D}$$

Primal Infon Logic [Gurevich,Neeman'09]

- An extremely efficient logic
- The main logical engine in DKAL (Microsoft Research)

- $\mathcal{V} = \{0, 1\}$
- $\mathcal{D} = \{1\}$

$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0, 1\}$

$\mathcal{O}(\supset)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{1\}$	$\{0, 1\}$

- $v(p) = 0$, $v(q) = 0$, $v(p \vee q) = 0$ is an **M**-valuation
- $v(p) = 0$, $v(q) = 0$, $v(p \vee q) = 1$ is an **M**-valuation
- $p \vee p \not\vdash_{\mathbf{M}} p$
- No finite matrix

Batens' Logic CLuN

- Obtained from **HCL**$^+$ by adding $\neg\varphi \vee \varphi$ as an axiom.
- Paraconsistent: $\neg P \supset (P \supset Q)$ is not a theorem.
- Conservative extension of positive classical logic.
- No finite matrix.

Batens' Logic CLuN - Semantics

Finding semantics, stage 1:

- $\mathcal{V} = \{0, 1\}$
- $\mathcal{D} = \{1\}$

$\mathcal{O}(\wedge)$	1	0
1	{1}	{0}
0	{0}	{0}

$\mathcal{O}(\vee)$	1	0
1	{1}	{1}
0	{1}	{0}

$\mathcal{O}(\supset)$	1	0
1	{1}	{0}
0	{1}	{1}

	$\mathcal{O}(\neg)$
1	{0,1}
0	{0,1}

Finding semantics, stage 2:

	$\mathcal{O}(\neg)$
1	{0,1}
0	{1}

Some Other Possible Axioms for Negation

(c) $\neg\neg\varphi \supset \varphi$

(e) $\varphi \supset \neg\neg\varphi$

(m1) $\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$

(m2) $(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$

(ℓ) $\neg(\varphi \wedge \neg\varphi) \supset (\varphi \wedge \neg\varphi \supset \psi)$

CLuN[S], for $S \subseteq \{(c), (e), (m1), (m2), (\ell)\}$, is the system obtained from **CLuN** by adding the elements of S as axioms.

How can we get semantics for all these theories?

Rexpansions: Refined Expansions

$\mathbf{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$, $\mathbf{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$

Expansion

$\mathbf{M}_2$ is an expansion of $\mathbf{M}_1$ if $\exists\ F$ s.t.:

- $F(x) \neq \emptyset$ for every $x \in \mathcal{V}_1$.
- $\mathcal{V}_2 = \biguplus_{x \in \mathcal{V}_1} F(x)$
- $\mathcal{D}_2 = \biguplus_{x \in \mathcal{D}_1} F(x)$
- $\mathcal{O}_2(\diamond)(\vec{y}) = \biguplus_{z \in O_1(\diamond)(F^{-1}(\vec{y}))} F(z)$

Refinement

$\mathbf{M}_2$ is a refinement of $\mathbf{M}_1$ if:

- $\mathcal{V}_2 \subseteq \mathcal{V}_1$
- $\mathcal{D}_2 = \mathcal{V}_2 \cap \mathcal{D}_1$
- $\mathcal{O}_2(\diamond)(\vec{x}) \subseteq \mathcal{O}_1(\diamond)(\vec{x})$

Rexpansion

- $\mathbf{M}_2$ is a rexpansion of $\mathbf{M}_1$ if it is a refinement of some expansion of $\mathbf{M}_1$

Proposition

If $\mathbf{M}_2$ is a rexpansion of $\mathbf{M}_1$ then $\vdash_{\mathbf{M}_1} \subseteq \vdash_{\mathbf{M}_2}$.

Is the inclusion proper?

Rexpansions: Refined Expansions

$\mathbf{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$, $\mathbf{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$

Expansion

$\mathbf{M}_2$ is an expansion of $\mathbf{M}_1$ if $\exists\, F$ s.t.:

- $F(x) \neq \emptyset$ for every $x \in \mathcal{V}_1$.
- $\mathcal{V}_2 = \biguplus_{x \in \mathcal{V}_1} F(x)$
- $\mathcal{D}_2 = \biguplus_{x \in \mathcal{D}_1} F(x)$
- $\mathcal{O}_2(\diamond)(\vec{y}) = \biguplus_{z \in O_1(\diamond)(F^{-1}(\vec{y}))} F(z)$

Refinement

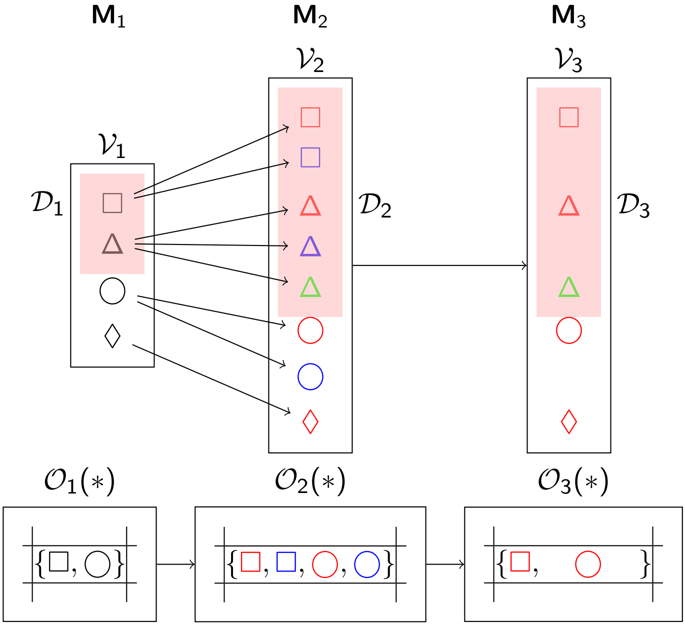
$\mathbf{M}_2$ is a refinement of $\mathbf{M}_1$ if:

- $\mathcal{V}_2 \subseteq \mathcal{V}_1$
- $\mathcal{D}_2 = \mathcal{V}_2 \cap \mathcal{D}_1$
- $\mathcal{O}_2(\diamond)(\vec{x}) \subseteq \mathcal{O}_1(\diamond)(\vec{x})$

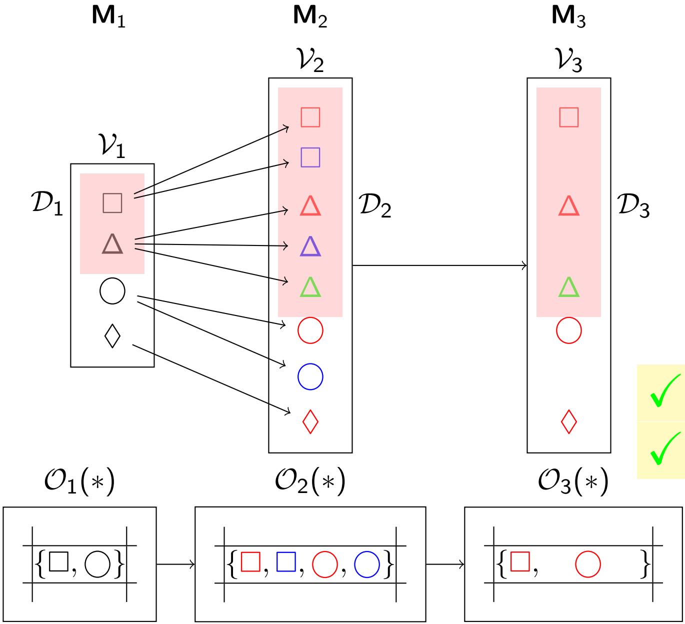
Rexpansion

- $\mathbf{M}_2$ is a rexpansion of $\mathbf{M}_1$ if it is a refinement of some expansion of $\mathbf{M}_1$
- $\mathbf{M}_2$ is a preserving rexpansion of $\mathbf{M}_1$ if $F(x) \cap \mathcal{V}_2 \neq \emptyset$ for every $x \in \mathcal{V}_1$
- $\mathbf{M}_2$ is a strongly preserving rexpansion of $\mathbf{M}_1$, if in addition:
  If $z \in O_1(\diamond)(\vec{x})$, and $\vec{y} \in F(\vec{x})$ then $F(z) \cap \mathcal{O}_2(\diamond)(\vec{y}) \neq \emptyset$.

(Strongly) Preserving Rexpansions

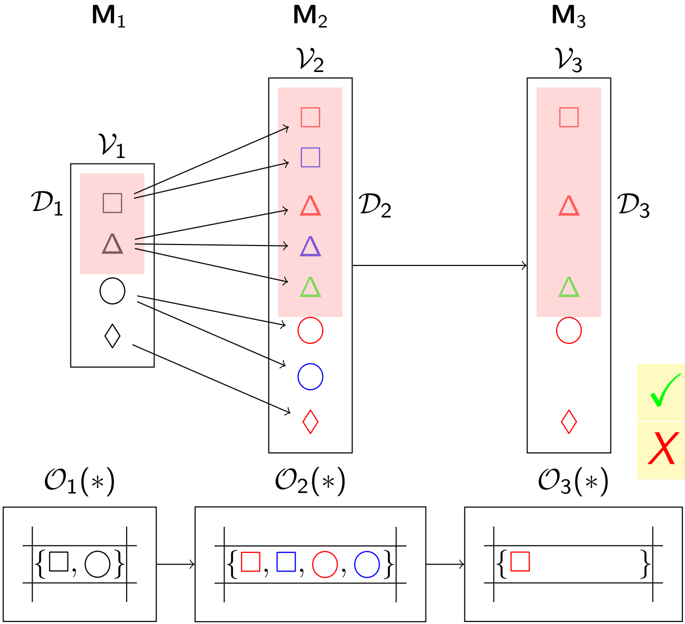


(Strongly) Preserving Rexpansions



✓ preserving

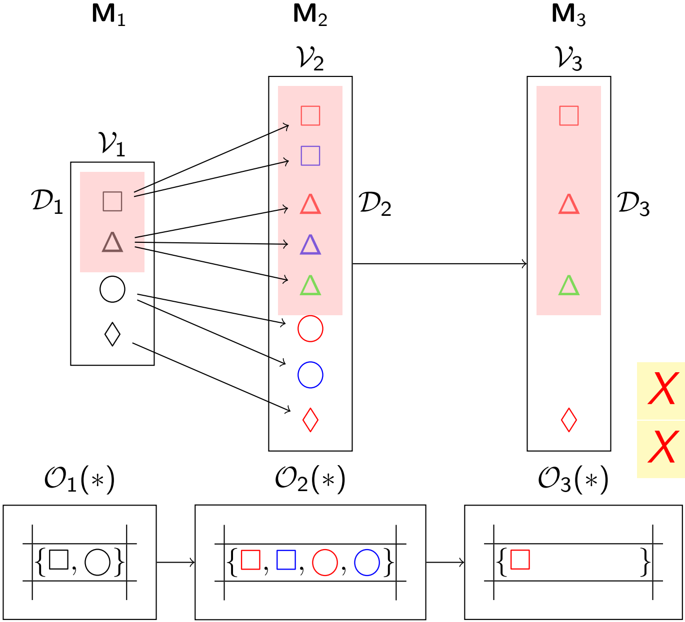
✓ strongly preserving



✓ preserving

✗ strongly preserving

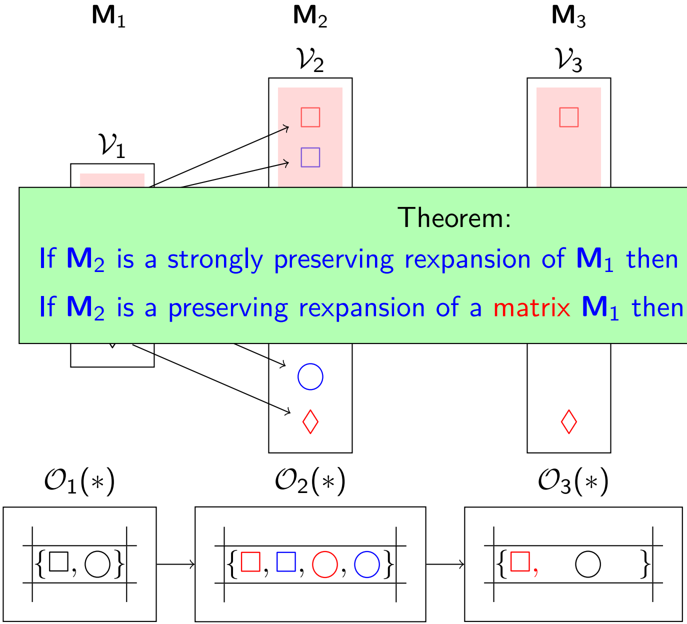
(Strongly) Preserving Rexpansions



X preserving

X strongly preserving

(Strongly) Preserving Rexpansions



Theorem:

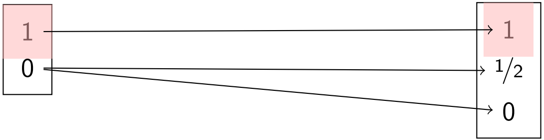
If $\mathbf{M}_2$ is a strongly preserving rexpansion of $\mathbf{M}_1$ then $\vdash_{\mathbf{M}_1} = \vdash_{\mathbf{M}_2}$.

If $\mathbf{M}_2$ is a preserving rexpansion of a matrix $\mathbf{M}_1$ then $\vdash_{\mathbf{M}_1} = \vdash_{\mathbf{M}_2}$.

Kleene Three-valued Logic

Classical Logic

Kleene Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

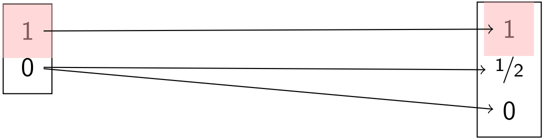
$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{0,1/2\}$	$\{0,1/2\}$
$1/2$	$\{0,1/2\}$	$\{0,1/2\}$	$\{0,1/2\}$
0	$\{0,1/2\}$	$\{0,1/2\}$	$\{0,1/2\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{0,1/2\}$	$\{0,1/2\}$
0	$\{1\}$	$\{0,1/2\}$	$\{0,1/2\}$

Kleene Three-valued Logic

Classical Logic | Kleene Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

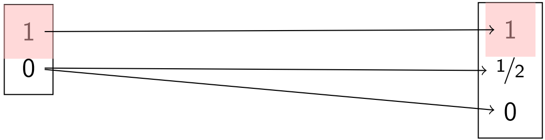
$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

Kleene Three-valued Logic

Classical Logic — Kleene Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

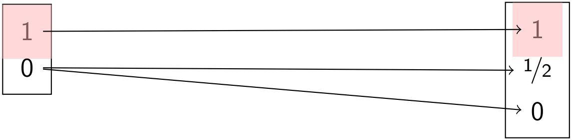
$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

	$\mathcal{O}(\neg)$
1	$\{0\}$
$1/2$	$\{1/2\}$
0	$\{1\}$

Therefore, Kleene logic is conservative over $\wedge\vee$-classical logic

Łukasiewich Three-valued Logic

Classical Logic | Łukasiewich Logic



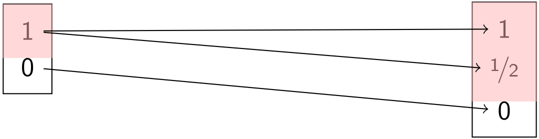
$\mathcal{O}(\supset)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{1\}$	$\{1\}$

$\mathcal{O}(\supset)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1\}$	$\{1\}$	$\{1/2\}$
0	$\{1\}$	$\{1\}$	$\{1\}$

Not a rexpansion of the two-valued matrix of $\mathbf{CL}^+$!

Therefore Łukasiewich Three-valued Logic is not conservative over $\mathbf{CL}^+$.

Classical Logic | Priest Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

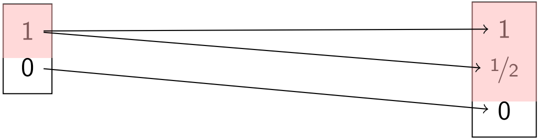
$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1, 1/2\}$	$\{1, 1/2\}$	$\{0\}$
$1/2$	$\{1, 1/2\}$	$\{1, 1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1, 1/2\}$	$\{1, 1/2\}$	$\{1, 1/2\}$
$1/2$	$\{1, 1/2\}$	$\{1, 1/2\}$	$\{1, 1/2\}$
0	$\{1, 1/2\}$	$\{1, 1/2\}$	$\{0\}$

Priest Three-valued Logics

Classical Logic | Priest Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

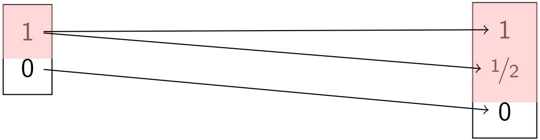
$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

Priest Three-valued Logics

Classical Logic | Priest Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

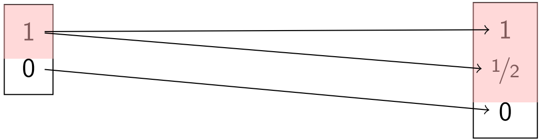
$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

	$\mathcal{O}(\neg)$
1	$\{0\}$
$1/2$	$\{1/2\}$
0	$\{1\}$

Priest Three-valued Logics

Classical Logic | Priest Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

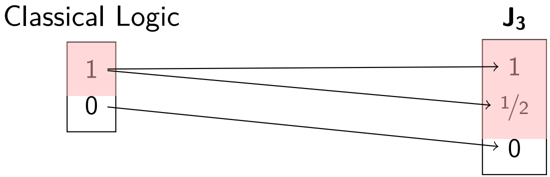
$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

	$\mathcal{O}(\neg)$
1	$\{0\}$
$1/2$	$\{1/2\}$
0	$\{1\}$

Therefore, Priest logic is conservative over $\wedge\vee$-classical logic

The Three-valued Logic $\mathbf{J_3}$



$\mathcal{O}(\supset)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{1\}$	$\{1\}$

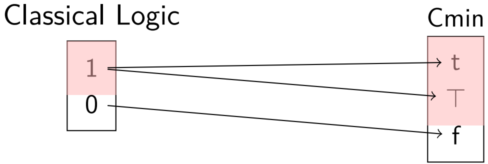
$\mathcal{O}(\supset)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{0\}$
0	$\{1\}$	$\{1\}$	$\{1\}$

A preserving rexpansion of the two-valued matrix of $\mathbf{CL}^+$

Therefore $\mathbf{J_3}$ is conservative over $\mathbf{CL}^+$.

Some Paraconsistent Logics

- $\mathbf{CLuN}[\{(c)\}] = C_{min} = \mathbf{HCL}^+ +$ the following axioms:
  - $(t)\ \neg\varphi \vee \varphi$
  - $(c)\ \neg\neg\varphi \supset \varphi$
- $\mathbf{CLuN}[\{(c), (m1)\}] = C_{min} +$ the following axiom:
  (m1) $\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$
- $\mathbf{CLuN}[\{(c), (\ell)\}] = C\ell = C_{min} +$ the following axiom:
  (ℓ) $\neg(\varphi \wedge \neg\varphi) \supset (\varphi \wedge \neg\varphi \supset \psi)$



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\wedge)$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$\mathcal{O}(\vee)$	t	f
1	$\{1\}$	$\{1\}$
f	$\{0\}$	$\{0\}$

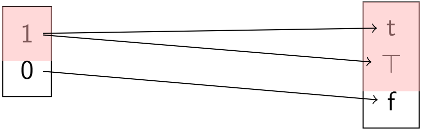
$\mathcal{O}(\vee)$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$

$\mathcal{O}(\supset)$	t	f
1	$\{1\}$	$\{0\}$
0	$\{1\}$	$\{1\}$

$\mathcal{O}(\supset)$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$

Paraconsistent Logics – C_{min}

Classical Logic   Cmin



C_{min} is conservative over positive classical logic

$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\wedge)$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

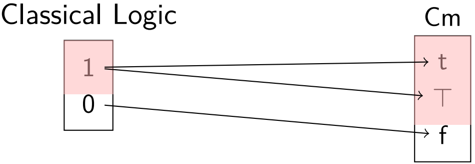
$\mathcal{O}(\vee)$	t	f
1	$\{1\}$	$\{1\}$
f	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$

	$\mathcal{O}(\neg)$
t	$\{f\}$
$\top$	$\{t,\top\}$
f	$\{t\}$

$\mathcal{O}(\supset)$	t	f
1	$\{1\}$	$\{0\}$
0	$\{1\}$	$\{1\}$

$\mathcal{O}(\supset)$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{t,\top\}$	$\{t,\top\}$	$\{t,\top\}$



$\mathcal{O}(\wedge)$	1	0
1	{1}	{0}
0	{0}	{0}

$\mathcal{O}(\wedge)$	t	⊤	f
t	{t, ⊤}	{t, ⊤}	{f}
⊤	{t, ⊤}	{t, ⊤}	{f}
f	{f}	{f}	{f}

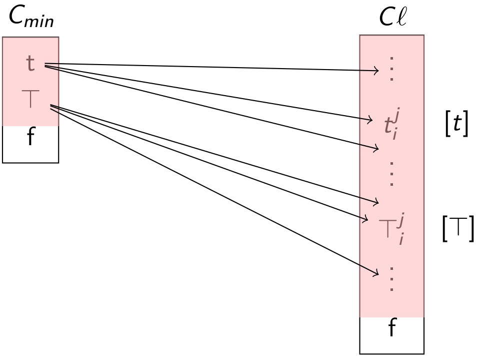
$\mathcal{O}(\vee)$	1	0
1	{1}	{1}
0	{1}	{0}

$\mathcal{O}(\vee)$	t	⊤	f
t	{t}	{t}	{t}
⊤	{t}	{t, ⊤}	{t, ⊤}
f	{t}	{t, ⊤}	{f}

	$\mathcal{O}(\neg)$
t	{f}
⊤	{t, ⊤}
f	{t}

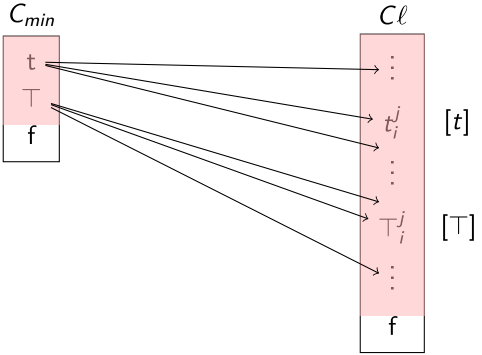
$\mathcal{O}(\supset)$	1	0
1	{1}	{0}
0	{1}	{1}

$\mathcal{O}(\supset)$	t	⊤	f
t	{t, ⊤}	{t, ⊤}	{f}
⊤	{t, ⊤}	{t, ⊤}	{f}
f	{t, ⊤}	{t, ⊤}	{t, ⊤}



	$\mathcal{O}(\neg)$
t	{f}
⊤	{t, ⊤}
f	{t}

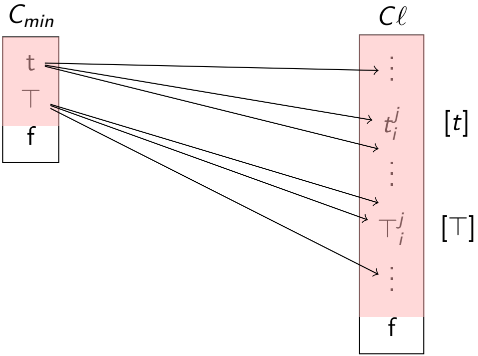
$$\mathcal{O}(\neg)(x) = \begin{cases} \{f\} & x = t_i^j \\ [t] \cup [\top] & x = \top_i^j \\ [t] & x = f \end{cases}$$



	$\mathcal{O}(\neg)$
t	{f}
$\top$	{t, $\top$}
f	{t}

$$\mathcal{O}(\neg)(x) = \begin{cases} \{f\} & x = t_i^j \\ \left\{\top_i^{j+1}, t_i^{j+1}\right\} & x = \top_i^j \\ [t] & x = f \end{cases}$$

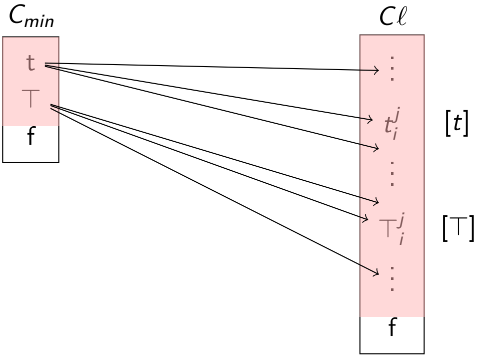
Paraconsistent Logics – $C\ell$



$\mathcal{O}(\wedge)$	t	⊤	f
t	{t, ⊤}	{t, ⊤}	{f}
⊤	{t, ⊤}	{t, ⊤}	{f}
f	{f}	{f}	{f}

$$\mathcal{O}(\wedge)(x,y) = \begin{cases} \{f\} & x = f \text{ or } y = f \\ [t] \cup [\top] & x = \top_i^j, y \in \{\top_i^{j+1}, t_i^{j+1}\} \\ [t] \cup [\top] & \textit{otherwise} \end{cases}$$

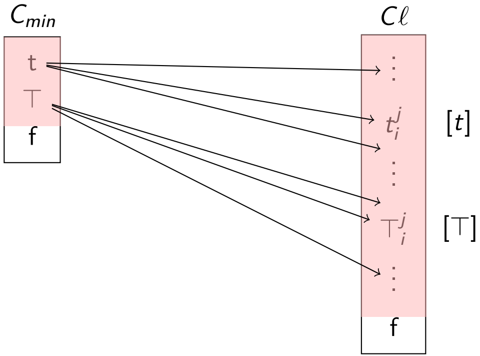
Paraconsistent Logics – $C\ell$



$\mathcal{O}(\wedge)$	t	$\top$	f
t	{t, $\top$}	{t, $\top$}	{f}
$\top$	{t, $\top$}	{t, $\top$}	{f}
f	{f}	{f}	{f}

$$\mathcal{O}(\wedge)(x,y) = \begin{cases} \{f\} & x = f \text{ or } y = f \\ [t] & x = \top_i^j, y \in \{\top_i^{j+1}, t_i^{j+1}\} \\ [t] \cup [\top] & \textit{otherwise} \end{cases}$$

Paraconsistent Logics – $C\ell$

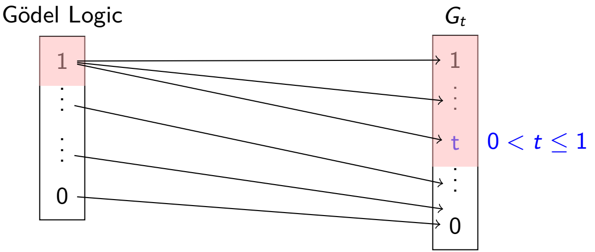


$C\ell$ is conservative over positive classical logic

$\mathcal{O}(\wedge)$	t	$\top$	f
t	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
$\top$	$\{t,\top\}$	$\{t,\top\}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$$\mathcal{O}(\wedge)(x,y) = \begin{cases} \{f\} & x = f \text{ or } y = f \\ [t] & x = \top_i^j, y \in \{\top_i^{j+1}, t_i^{j+1}\} \\ [t] \cup [\top] & \textit{otherwise} \end{cases}$$

Gödel Logic



$$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$$

$$\mathcal{O}(\wedge)(x,y) = \begin{cases} [t,1] & \min(x,y) \geq t \\ \{\min(x,y)\} & \textit{otherwise} \end{cases}$$

$$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \begin{cases} [t,1] & \max(x,y) \geq t \\ \{\max(x,y)\} & \textit{otherwise} \end{cases}$$

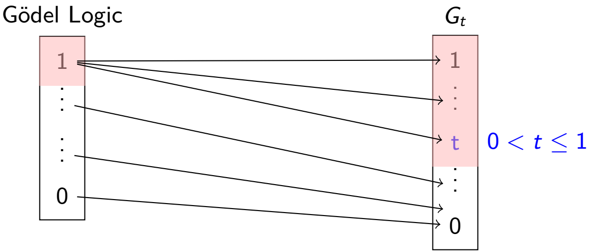
$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & \textit{otherwise} \end{cases}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & \textit{otherwise} \end{cases}$$

$$\mathcal{O}(\perp) = \{0\}$$

$$\mathcal{O}(\perp) = \{0\}$$

Gödel Logic



$\mathcal{O}(\wedge)(x, y) = \{\min(x, y)\}$

$\mathcal{O}(\vee)(x, y) = \{\max(x, y)\}$

$$\mathcal{O}(\supset)(x, y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

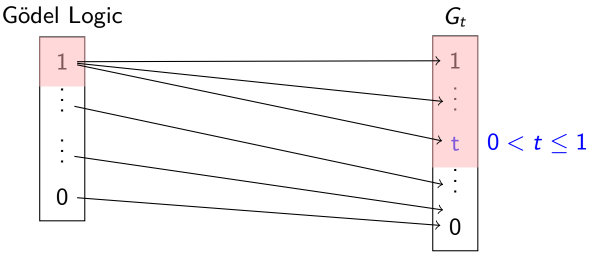
$\mathcal{O}(\perp) = \{0\}$

$\mathcal{O}(\wedge)(x, y) = \{min(x, y)\}$

$\mathcal{O}(\vee)(x, y) = \{max(x, y)\}$

$$\mathcal{O}(\supset)(x, y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$\mathcal{O}(\perp) = \{0\}$



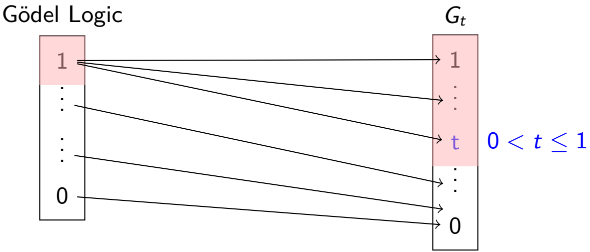
$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$ $\quad\quad$ $\mathcal{O}(\wedge)(x,y) = \{min(x,y)\}$

$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$ $\quad\quad$ $\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases} \qquad \mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$\mathcal{O}(\bot) = \{0\}$ $\quad\quad$ $\mathcal{O}(\bot) = \{0\}$

In Gödel logic, $\mathcal{D}$ may be $[t, 1]$



$$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$$

$$\mathcal{O}(\wedge)(x,y) = \{min(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

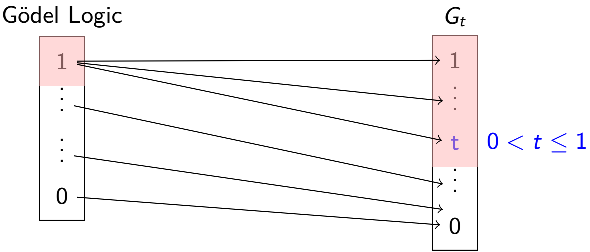
$$\mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\perp) = \{0\}$$

$$\mathcal{O}(\perp) = \{0\}$$

$$\mathcal{O}(\neg)(x) = 1 - x$$

Gödel Logic



$$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$$

$$\mathcal{O}(\wedge)(x,y) = \{min(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & otherwise \end{cases}$$

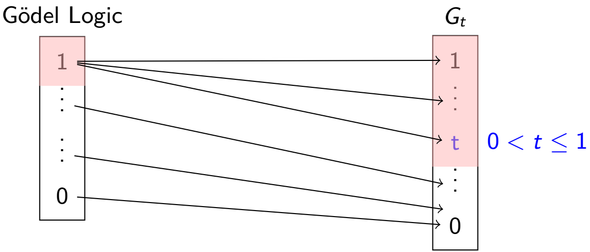
$$\mathcal{O}(\perp) = \{0\}$$

$$\mathcal{O}(\perp) = \{0\}$$

$$\mathcal{O}(\neg)(x) = 1 - x$$

All refinements are conservative!

Gödel Logic



$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$

$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$\mathcal{O}(\bot) = \{0\}$

$\mathcal{O}(\wedge)(x,y) = \{min(x,y)\}$

$\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & otherwise \end{cases}$$

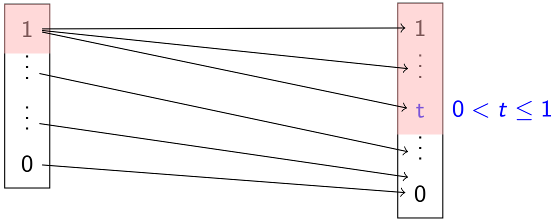
$\mathcal{O}(\bot) = \{0\}$

$\mathcal{O}(\neg)(x) = 1 - x$

Exactly three:

$\vdash_{G_{1/2}}, \vdash_{G_{3/4}}, \vdash_{G_1}$

Gödel Logic



$$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\} \qquad \mathcal{O}(\wedge)(x,y) = \{min(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\} \qquad \mathcal{O}(\vee)(x,y) = \{max(x,y)\}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases} \qquad \mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\bot) = \{0\} \qquad \mathcal{O}(\bot) = \{0\}$$

$$\mathcal{O}(\neg)(x) = 1 - x$$

for $t \leq 1/2$:
$\vdash_{G_t}$ is paraconsistent