On Modal Fragments of some Tense Logics

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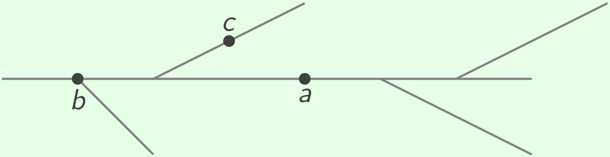
D1 $\Diamond A \equiv_{Df} A \vee FA$
$\Box A \equiv_{Df} A \wedge GA$

D2 $\Diamond A \equiv_{Df} PA \vee A \vee FA$
$\Box A \equiv_{Df} HA \wedge A \wedge GA$

D3 $\Diamond A \equiv_{Df} PFA$
$\Box A \equiv_{Df} HGA$



Some Smirnov's results

1. $\mathbf{T}$ is a modal fragment of $\mathbf{K_t} + D_1$
2. $\mathbf{S4}$ is a modal fragment of $\mathbf{K_t4} + D1$
3. $\mathbf{S4}.3$ is a modal fragment of $\mathbf{K_t4}.3 + D1$
4. $\mathbf{B}$ ($\mathbf{KTB}$) is a modal fragment of tense systems $\mathbf{K_t} + D2$ (likewise $\mathbf{K_t4} + D2$, $\mathbf{K_t4} + \textit{lin-p} + D2$)
5. $\mathbf{S5}$ is a modal fragment of $\mathbf{K_t4}.3 + D2$ (likewise $\mathbf{K_t4}.3 + \textit{noend} + \textit{nobeg} + D2$, $\mathbf{K_t4}.3 + \textit{noend} + \textit{nobeg} + \textit{dense} + D2$)
6. $\mathbf{B}$ is a modal fragment of $\mathbf{K_t} + \textit{nobeg} + D3$
7. $\mathbf{S5}$ is a modal fragment of $\mathbf{K_t4} + \textit{lin-p} + \textit{nobeg} + D3$ (and also $\mathbf{K_t4}.3 + D3$, $\mathbf{K_t4}.3 + \textit{noend} + \textit{nobeg} + D3$, $\mathbf{K_t4}.3 + \textit{noend} + \textit{nobeg} + \textit{dense} + D3$)

(T, R_F, R_P)
$R_F(x, y)$: x generates y
$R_P(x, z)$: z precedes x

(1) $R_F(x, y) \Rightarrow R_P(y, x)$

(2) $R_F(x, y) \wedge R_P(y, z) \Rightarrow R_F(x, z) \vee R_P(x, z) \vee x = z$

(3) $\exists y(R_P(x, y) \wedge R_F(y, x))$

(1*) $A \rightarrow GPA$

(2*) $FPA \rightarrow PA \vee A \vee FA$

(3*) $HA \rightarrow PA$

(4*) $A \rightarrow PFA$

- Interconnection schemata: $S = (1^*) + (2^*)$
- Logical system: $\langle KP, KF \rangle_S$

- Interconnection schemata: $R = (1^*) + (2^*) + (3^*) + (4^*)$
- Logical system: $\langle DP, KF \rangle_R$, $DP = KP + HA \rightarrow PA$

$$
\begin{array}{ll}
\textbf{(5)} & R_F(x,y) \wedge R_P(y,z) \Rightarrow R_F(x,z) \vee R_P(x,z) \vee x = z \\
\textbf{(6)} & R_P(x,y) \wedge R_F(y,z) \Rightarrow R_P(x,z) \vee R_F(x,z) \vee x = z \\
\textbf{(5}^*\textbf{)} & FPA \rightarrow PA \vee A \vee FA \\
\textbf{(6}^*\textbf{)} & PFA \rightarrow PA \vee A \vee FA
\end{array}
$$

- $Q = (5^*) + (6^*)$
- $\langle KP, KF \rangle_Q$

Embedding techniques

1. Translation: function $\phi \colon \mathcal{L}_{T_1} \mapsto \mathcal{L}_{T_2}$ s.t. for each formula $A \in \mathcal{L}_{T_1}$, $A \in T_1 \implies \phi(A) \in T_2$
2. ϕ is an embedding: $A \in T_1 \iff \phi(A) \in T_2$
3. $\psi \colon \mathcal{L}_{T_2} \mapsto \mathcal{L}_{T_1}$ s.t. for each formula $A \in \mathcal{L}_{T_2}$, $A \in T_2 \implies \phi(A) \in T_1$
4. If $A \equiv \psi(\phi(A)) \in T_1$, then ϕ embeds T_1 into T_2.

Embedding of K into $\langle KP, KF \rangle_Q$

Define two translation functions: $\Phi\colon \mathcal{L}_{ML} \mapsto \mathcal{L}_{TL}$ and $\Psi\colon \mathcal{L}_{TL} \mapsto \mathcal{L}_{ML}$ as follows:

1. $\Phi(p) = p$, for all $p \in \mathtt{Var}$
2. $\Phi(\neg A) = \neg\Phi(A)$
3. $\Phi(A \circ B) = \Phi(A) \circ \Phi(B)$, for $\circ \in \{\rightarrow, \wedge, \vee\}$
4. $\Phi(\Diamond A) = PF\Phi(A)$
5. $\Phi(\Box A) = HG\Phi(A)$

1. $\Psi(p) = p$, for all $p \in \mathtt{Var}$
2. $\Psi(\neg A) = \neg\Psi(A)$
3. $\Psi(A \circ B) = \Psi(A) \circ \Psi(B)$, for $\circ \in \{\rightarrow, \wedge, \vee\}$
4. $\Psi(GA) = \Phi(A)$
5. $\Phi(HA) = \Box\Phi(A)$

Proposition

1. $\mathbf{K} \vdash A \implies \langle KP, KF \rangle_Q \vdash \Phi(A)$, *for each* $A \in \mathcal{L}_{ML}$
2. $\langle KP, KF \rangle_Q \vdash A \implies \mathbf{K} \vdash \Psi(A)$, *for each* $A \in \mathcal{L}_{TL}$
3. $\langle KP, KF \rangle_Q \vdash A \equiv \Phi(\Psi(A))$, *for each* $A \in \mathcal{L}_{TL}$

Corollary

Φ *is an embedding function*

Another method

1. Consider two theories T_1 and T_2 formulated in different languages
2. Expand them with definitions: $T_1 + D_{T_2}$, $T_2 + D_{T_1}$
3. Prove the following: $T_1 \subseteq T_2 + D_{T_1}$ and $T_2 + D_{T_1} \subseteq T_1 + D_{T_2}$
4. T_2 is a conservative extension of T_1
5. Any T_3 such that $T_2 + D_{T_1} \subseteq T_3 \subseteq T_1 + D_{T_2}$ is also a conservative extension of T_1

1. Q: what is a modal fragment of $TL + D_i$?
2. H: some ML is an appropriate candidate
3. Take expansion $ML + D_{TL}$
4. Prove $ML \subseteq TL + D_i$ and $TL + D_i \subseteq ML + D_{TL}$
5. If $TL + D_i \subseteq TL' \subseteq ML + D_{TL}$, then TL' is a c.e. of ML

1. $\langle KP, KF\rangle_Q + D_3$
2. **K** is a m.f. of $\langle KP, KF\rangle_Q + D_3$?
3. $\mathbf{K} + GA \equiv_{Df} \Box A + HA \equiv_{Df} \Box A$
4. $\mathbf{K} \subseteq \langle KP, KF\rangle_Q + D_3$,
   $\langle KP, KF\rangle_Q + D_3 \subseteq \mathbf{K} + GA \equiv_{Df} \Box A + HA \equiv_{Df} \Box A$,
   $D_3 : \Box A \equiv_{Df} HGA$
5. $\mathbf{K} + GA \equiv_{Df} A + HA \equiv_{Df} \Box A$

1. **K** is a modal fragment of any tense system TL such that

$$\langle KP, KF\rangle_Q + D3 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{K} + GA \equiv A + HA \equiv \Box A.$$

2. **S4** is a modal fragment of any tense system TL such that

$$\langle KP, KF4\rangle_Q + D1 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{S4} + GA \equiv \Box A + HA \equiv \Box A.$$

3. **S4** is a modal fragment of any tense system TL such that

$$\langle KP4, KF4\rangle_Q + D2 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{S4} + GA \equiv \Box A + HA \equiv \Box A.$$

4. **K4** is a modal fragment of any tense system TL such that

$$\langle KP, KF4\rangle_Q + D3 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{K4} + GA \equiv A + HA \equiv \Box A.$$

$$R_F(x,y) \wedge R_P(y,z) \to \\ (R_P(x,z) \vee x = z \vee (R_F(x,z) \wedge R_F(z,y))) \quad (\alpha)$$

$$R_P(x,y) \wedge R_F(y,z) \to \\ (R_F(x,z) \vee x = z \vee (R_P(x,z) \wedge R_P(z,y))) \quad (\beta)$$

$$R_P(x,y) \wedge R_F(x,z) \to R_F(y,z) \quad (\gamma)$$

(α') $F(B \wedge PA) \to PA \vee A \vee F(A \wedge FB)$

(β') $P(B \wedge FA) \to FA \vee A \vee P(A \wedge PB)$

(γ') $FA \wedge PB \to P(B \wedge FA)$

$Q_3 = (\alpha') + (\beta') + (\gamma')$

Theorem

K4.3 *is a modal fragment of any tense system TL such that*

$$\langle DP4.3, KF4.3\rangle_{Q3} + D3 \subseteq TL \text{ and}$$
$$TL \subseteq \mathbf{K4.3} + GA \equiv A + HA \equiv \Box A.$$