Completeness of the logic HC in a special space

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Logic QHC

Crucial idea of logic QHC is to connect the classical predicate calculus and the intuitionistic predicate calculus.

The classical logic is used for reasoning about propositions (that can be true or false).

The intuitionistic logic is used for reasoning about problems (that require a solution).

HC is a propositional part of the QHC.

The language of the HC

Every formula of HC has one of two sorts:

proposition ($p, q, \dots$) and problem ($\alpha, \beta, \dots$).

Formulas of HC are built from variables using

- classical connectives ($\wedge, \vee, \to, \neg$), which can only be applied to propositions;
- classical 0-ary connective false (0);
- intuitionistic connectives ($\wedge, \vee, \to, \neg$), which can only be applied to problems;
- intuitionistic 0-ary connective false ($\bot$).

Connectives are type-specific (i.e. all of their input and output have the same type).

The language of the HC

The language of the HC contains two modalities ? and !:

- The unary connective ! inputs a proposition p and outputs a problem $!p$ with the intended reading «Find a proof of p».
- The unary connective ? inputs a problem α and outputs a proposition $?\alpha$ with the intended reading «There exists a solution of α».

Axioms and inference rules of HC

- All axiom schemes and inference rules of classical logic.
- All axiom schemes and inference rules of intuitionistic logic.
- Additional axiom schemes and inference rules:

$!(p \to q) \to (!p \to !q)$;

$\dfrac{p}{!p}$;

$?!p \to p$;

$\neg !0$.

$?(\alpha \to \beta) \to (?\alpha \to ?\beta)$;

$\dfrac{\alpha}{?\alpha}$;

$\alpha \to !?\alpha$;

Topological models

The triplet (X, τ, A) is named by topological frame.

- (X, τ) is an arbitrary nonempty topological space,
  A is its dense subset.
- The variables p of the proposition sort are interpreted as arbitrary subsets $|p| \subseteq A$.
- The variables α of the problem sort are interpreted as arbitrary open subsets $|\alpha| \subseteq X$.

The truth value of any formula of the proposition sort is some subset of A and the truth value of any formula of the problem sort is an open subset of X.

Topological models

The classical connectives and the constant 0:

$|p \wedge q| = |p| \cap |q|;$

$|p \vee q| = |p| \cup |q|;$

$|p \to q| = (A \setminus |p|) \cup |q|;$

$|0| = \varnothing.$

The intuitionistic connectives and the constant $\perp$:

$|\alpha \wedge \beta| = |\alpha| \cap |\beta|;$

$|\alpha \vee \beta| = |\alpha| \cup |\beta|;$

$|\alpha \to \beta| = \mathrm{Int}((X \setminus |\alpha|) \cup |\beta|);$

$|\perp| = \varnothing.$

Topological models

The modalities are interpreted as follows:
$|?\alpha| = A \cap |\alpha|$;
$|!p| = X \setminus \mathrm{Cl}(A \setminus |p|)$.

In other words, $|!p|$ is a union of all open sets U such that $U \cap A \subseteq |p|$.

As usual, a formula p of the proposition sort (a formula α of the problem sort) is true in this topological model of the HC if $|p| = A$ ($|\alpha| = X$).

Correctness and completeness

Theorem 1

1) If a formula is deducible in HC*, then it is true in any topological model of the logic* HC*.*
2) If a formula φ *is non deducible in the logic* HC*, then there exists a topological model of the logic* HC *in which the formula* φ *is not true.*

The topological models of the HC is an enrichment of the standart topological models of the intuitionistic logic.

If we put $A = X$ and consider just the classical part of the HC with derived modality $\Box = ?!$, then we obtain the standart topological models of the logic S4.

Logics S4 and H4(=IEL^+)

There are 2 derived modalities in HC:

$\Box$ =?! ($\Box p$ may be read as «p is provable»);

∇ =!? ($\nabla\alpha$ may be read as «prove that there is a solution of the problem α»).

The following principles are provable in HC:

$\Box p \to p$;

$\Box p \to \Box\Box p$;

$\dfrac{p}{\Box p}$;

$\Box(p \to q) \to (\Box p \to \Box q)$.

These are axioms of S4.

$\alpha \to \nabla\alpha$;

$\nabla\nabla\alpha \to \nabla\alpha$;

$\nabla\bot \to \bot$;

$\nabla(\alpha \to \beta) \to (\nabla\alpha \to \nabla\beta)$.

These are axioms of H4.

HC is a conservative extension of the intuitionistic logic, logic S4 and intuitionistic epistemic logic IEL^+ .

Topological models

Theorem 2

1) HC *is the logic of the class of all finite topological spaces with a dense subset.*
2) HC *has the effective finite model property with respect to the class of topological spaces with a dense subset.*

S4 is the logic of the class of all finite topological spaces.

McKinsey and Tarski's result: S4 is the logic of any dense-in-itself metric separable space.

S4 is complete in special spaces: the Cantor space $\mathbb{C}$, the rational line $\mathbb{Q}$ and the real line $\mathbb{R}$.

Audit set models

Definition

Audit set frame is a triplet $(W, \preccurlyeq, \mathrm{Aud})$, *where* $(W, \preccurlyeq)$ *is a standard intuitionistic frame (*W *is a nonempty set,* $\preccurlyeq$ *is a partial order),* $\mathrm{Aud} \subseteq W$ *is a confinal subset of audit states (i.e.* $\forall a \in W \exists b \in \mathrm{Aud}\ a \preccurlyeq b$*).*

The evaluation $\models$ for intuitionistic formulas is defined by standard way, for classical formulas is defined pointwisely only in audit states, and for modalities by this way:

$$a \models ?\alpha \Leftrightarrow a \models \alpha \text{ (for } a \in \mathrm{Aud}),$$

$$a \models !p \Leftrightarrow \forall b \in \mathrm{Aud}(a \preccurlyeq b \Rightarrow b \models p) \text{ (for } a \in W).$$

Main results

Theorem 3

Logic HC *is the logic of the class of all audit set models with* $W = T_2$ *— the infinite binary tree.*

Theorem 4

Logic HC *is the logic of the class of all audit set models with* $W = T_2$ *in which* Aud *and* $T_2 \setminus \mathrm{Aud}$ *both are confinal.*

Theorem 5

Logic HC *is complete with respect to topological frame* $(\mathbb{A}, \tau_{\leqslant}, \mathbb{Q})$*, where* $\mathbb{A}$ *is the set of algebraic numbers,* $\tau_{\leqslant}$ *is the standart order topology and* $\mathbb{Q}$ *is the set of rational numbers.*

Thanks for your attention!