Topological semantics of the predicate modal calculus QGL extended with non-well-founded proofs

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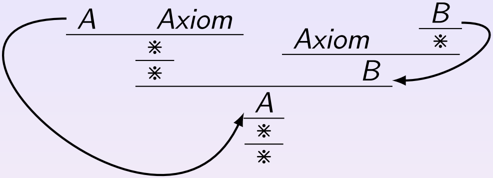
SCAN
2023

Cyclic and non-well-founded proofs



Proof systems allowing non-well-founded reasoning can be defined for

- modal μ-calculus
- action logic
- Peano arithmetic
- GL, Grz, K^+, etc.

Gödel-Löb provability logic GL

Axioms

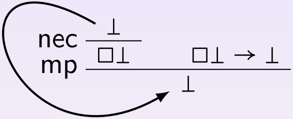
- axioms of classical propositional logic
- $\Box(A \to B) \to (\Box A \to \Box B)$
- $\Box A \to \Box\Box A$
- $\Box(\Box A \to A) \to \Box A$ (Löb's axiom)

Inference rules

$$\text{mp}\ \frac{A \qquad A \to B}{B} \qquad \text{nec}\ \frac{A}{\Box A}$$

The logic GL is sound and complete w.r.t. its provability interpretation, where $\Box A$ corresponds to «A is provable in Peano arithmetic».

Example of a cyclic derivation



Theorem

GL = K4 + cyclic derivations

A predicate modal calculus QGL

We fix a first-order signature without function symbols and constants.

Axioms

- axioms of GL
- $\forall x\, A(x) \to A(y)$
- $\forall x\, (A \to B) \to (A \to \forall x\, B)$, where $x \notin FV(A)$

Inference rules

$$\text{mp}\ \frac{A \qquad A \to B}{B} \qquad \text{nec}\ \frac{A}{\Box A} \qquad \text{gen}\ \frac{A}{\forall x\, A}$$

Theorems (Montagna 1987)

The calculus QGL is not arithmetically complete. Besides, it is not complete with respect to its Kripke semantics.

Recall that GL is weakly complete for its relational interpretation. Strong completeness is achieved if one considers topological (or neighbourhood) semantics of the given system.

At the same time, QGL is not complete for its relational interpretation.

In this talk, we consider non-well-founded proofs in QGL and show that QGL extended with these proofs is topologically complete.

We conjecture that QGL with ordinary proofs is topologically incomplete.

Non-well-founded derivations in QGL

Definition

An *∞-derivation* in QGL is a (possibly infinite) tree whose nodes are marked by predicate modal formulas and that is constructed according to the rules (mp), (gen) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec).

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots \quad \Box\forall x_2\, P_2(x_2) \qquad \Box\forall x_2\, P_2(x_2) \to P_1(x_1)}{P_1(x_1)}\,\text{mp}}{\forall x_1\, P_1(x_1)}\,\text{gen}}{\Box\forall x_1\, P_1(x_1)}\,\text{nec} \qquad \Box\forall x_1\, P_1(x_1) \to P_0(x_0)}{P_0(x_0)}\,\text{mp}
$$

Topological completeness

Definition

An *∞-proof* is an ∞-derivation, where all leaves are marked by axioms of QGL. We set $\mathrm{QGL}_\infty \vdash A$ if there is an ∞-proof with the root marked by A.

Theorem (topological completeness)

For any formula A, $\mathrm{QGL}_\infty \vdash A$ if and only if A is valid in every predicate topological frame.

Strong completeness

Definition

We put $\Gamma \vdash A$ if there is an ∞-derivation δ with the root marked by A such that, for each leaf a of δ that is not marked by an axiom, a is marked by a formula from Γ, and there are no applications of the rules (gen) and (nec) on the path from the root of δ to the leaf a.

$$
\text{mp}\ \dfrac{\text{nec}\ \dfrac{\text{gen}\ \dfrac{\text{mp}\ \dfrac{\begin{matrix}\vdots\\ \Box\forall x_2\, P_2(x_2)\end{matrix} \qquad \Box\forall x_2\, P_2(x_2) \to P_1(x_1)}{P_1(x_1)}}{\forall x_1\, P_1(x_1)}}{\Box\forall x_1\, P_1(x_1)} \qquad \Box\forall x_1\, P_1(x_1) \to P_0(x_0)}{P_0(x_0)}
$$

Theorem (strong completeness)

We have $\Gamma \vdash A$ if and only if A is a semantic consequence of Γ over the class of predicate topological QGL-frames.

Weak topological completeness

Topological semantics

A *predicate topological frame (for* QGL*)* is a tuple (X, τ, D), where (X, τ) is a scattered topological space and D is a non-empty domain.

A *valuation in D* is a function sending each n-ary predicate letter to an n-ary relation on D, and a *variable assignment* is a function from the set of variables $Var = \{x_0, x_1, x_2, \dots\}$ to the domain D.

A *predicate topological model* $\mathcal{M} = (X, \tau, D, \xi)$ is a predicate topological frame (X, τ, D) together with an indexed family of valuations $\xi = (\xi_w)_{w \in X}$ in D.

Topological semantics

The *truth of a formula A at a world w of a model $\mathcal{M} = (X, \tau, D, \xi)$ under a variable assignment h* is defined as

- $\mathcal{M}, w, h \nvDash \bot$,
- $\mathcal{M}, w, h \vDash P(x_1, \ldots, x_n) \iff (h(x_1), \ldots, h(x_n)) \in \xi_w(P)$,
- $\mathcal{M}, w, h \vDash A \to B \iff \mathcal{M}, w, h \nvDash A$ or $\mathcal{M}, w, h \vDash B$,
- $\mathcal{M}, w, h \vDash \Box A \iff \exists U \in \tau\, (w \in U$ and $\forall w' \in U \setminus \{w\}\ \mathcal{M}, w', h \vDash A)$,
- $\mathcal{M}, w, h \vDash \forall x\, A \iff \mathcal{M}, w, h' \vDash A$ for any varible assignment h' such that $h' \stackrel{x}{=} h$,

where $h' \stackrel{x}{=} h$ means that $h'(y) = h(y)$ for each $y \in Var \setminus \{x\}$.

A *formula A is true in $\mathcal{M}$* if A is true at all worlds of $\mathcal{M}$ under all variable assignments. In addition, *A is valid in a frame $\mathcal{F}$* if A is true in all models over $\mathcal{F}$.

Topological completeness via a sequent calculus

Lemma (soundness)

If $\mathsf{QGL}_\infty \vdash A$, then A is valid in every predicate topological frame.

The converse direction:

1. consider a sequent calculus for QGL_∞ with non-well-founded proofs;
2. combine sequent-based completeness proofs for the classical predicate calculus and for GL extended with non-well-founded proofs.

A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. For a finite multiset of formulas $\Gamma = B_1, \dots, B_n$, we set $\Box\Gamma := \Box B_1, \dots, \Box B_n$.

Non-well-founded sequent calculus

Initial sequents and inference rules of the sequent calculus S have the following form:

$$\Gamma, P(\vec{x}) \Rightarrow P(\vec{x}), \Delta, \qquad \Gamma, \bot \Rightarrow \Delta,$$

$$\to_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}, \qquad \to_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta},$$

$$\forall_{\mathsf{L}} \frac{\Gamma, A(y), \forall x\, A \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta}, \qquad \forall_{\mathsf{R}} \frac{\Gamma \Rightarrow A(y), \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}\,(y \notin FV(\Gamma \cup \Delta)),$$

$$\Box \frac{\Gamma, \Box\Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A, \Delta}.$$

Every infinite branch in a non-well-founded proof of this calculus must contain infinitely many applications of the rule $(\Box)$.

Example

$$
\forall_L \; \dfrac{\exists_L \; \dfrac{\to_L \; \dfrac{\dfrac{\text{Ax}}{P(x_0), \boxdot\forall y\, \exists z\, (\Box P(z) \to P(y)) \Rightarrow P(x_0)} \qquad \Box \; \dfrac{\vdots \atop \boxdot\forall y\, \exists z\, (\Box P(z) \to P(y)) \Rightarrow P(x_1)}{\boxdot\forall y\, \exists z\, (\Box P(z) \to P(y)) \Rightarrow \Box P(x_1), P(x_0)}}{\Box P(x_1) \to P(x_0), \boxdot\forall y\, \exists z\, (\Box P(z) \to P(y)) \Rightarrow P(x_0)}}{\exists z\, (\Box P(z) \to P(x_0)), \boxdot\forall y\, \exists z\, (\Box P(z) \to P(y)) \Rightarrow P(x_0)}}{\boxdot\forall y\, \exists z\, (\Box P(z) \to P(y)) \Rightarrow P(x_0)} \; ,
$$

where

$$
\boxdot\forall y\, \exists z\, (\Box P(z) \to P(y)) = \{\forall y\, \exists z\, (\Box P(z) \to P(y)), \Box\forall y\, \exists z\, (\Box P(z) \to P(y))\}.
$$

The proof of completeness

Observation
If A is valid in every predicate topological frame, then A is true in any predicate topological model with a countable domain under any bijective variable assignment.

Lemma
If a formula $\bigwedge\Gamma \to \bigvee\Delta$ is true in in any predicate topological model with a countable domain under any bijective variable assignment, then the sequent $\Gamma \Rightarrow \Delta$ is provable in S.

Lemma
If a sequent $\Gamma \Rightarrow \Delta$ is provable in S, then $\mathrm{QGL}_\infty \vdash \bigwedge\Gamma \to \bigvee\Delta$.

Corollary (completeness)
For any formula A, $\mathrm{QGL}_\infty \vdash A$ if and only if A is valid in every predicate topological frame.

Strong topological completeness

Shehtman's ultrabouquet construction

Ultrabouquet of topological spaces

For any $i \in I$, let $\mathcal{X}_i = (X_i, \tau_i)$ be a topological space and w_i be a closed point in it. Let $\mathcal{U}$ be a non-principal ultrafilter in I. The *ultrabouquet* $\bigvee_{\mathcal{U}}(\mathcal{X}_i, w_i)$ is a topological space obtained as a set from the disjoint union $\bigsqcup_{i \in I} X_i$ by identifying all points w_i. A set *U is open in* $\bigvee_{\mathcal{U}}(\mathcal{X}_i, w_i)$ if and only if

- the set $U \cap (X_i \setminus \{w_i\})$ is open in $\mathcal{X}_i$ for any $i \in I$,
- $\{i \in I \mid U \cap X_i \text{ is open in } \mathcal{X}_i\} \in \mathcal{U}$ whenever $w_* \in U$,

where w_* is the point of $\bigvee_{\mathcal{U}}(\mathcal{X}_i, w_i)$ obtained by identifying points w_i.

An ultrabouquet of scattered topological spaces is a scattered topological space.

The case of predicate models

Ultrabouquet of predicate topological models

For $i \in I$, let $\mathcal{M}_i = (X_i, \tau_i, D_i, \xi_i)$ be a predicate topological model, w_i be a closed point in it and h_i be a variable assignment in $\mathcal{M}_i$. Given a non-principal ultrafilter $\mathcal{U}$ in I, we define $\bigvee_{\mathcal{U}}(\mathcal{M}_i, w_i)$ as a tuple (X, τ, D, ξ), where $(X, \tau) = \bigvee_{\mathcal{U}}((X_i, \tau_i), w_i)$ and $D = \prod_{i \in I} D_i$. In addition, $(a_1, \dots, a_n) \in \xi_w(P)$ if and only if

- $(\pi_i(a_1), \dots, \pi_i(a_n)) \in (\xi_i)_w(P)$ for $w \in X_i \setminus \{w_i\}$,
- $\{i \in I \mid (\pi_i(a_1), \dots, \pi_i(a_n)) \in (\xi_i)_w(P)\} \in \mathcal{U}$ whenever $w = w_*$.

Besides, we define the variable assignment $h \colon Var \to D$ so that $\pi_i \circ h = h_i$ for any $i \in I$.

Topological compactness

Definition

We set $\Gamma \vDash A$ if for any predicate topological model $\mathcal{M} = (X, \tau, D, \xi)$, any world w of $\mathcal{M}$ and any variable assignment $h\colon Var \to D$

$$\forall B \in \Gamma\ \mathcal{M}, w, h \vDash B \Longrightarrow \mathcal{M}, w, h \vDash A.$$

Theorem

If $\Gamma \vDash A$, then there is a finite subset Γ_0 of Γ such that $\Gamma_0 \vDash A$.

Strong topological completeness

Theorem

For any set of formulas Γ and any formula A,

$$\Gamma \vdash A \Longleftrightarrow \Gamma \vDash A.$$

Proof of the right-to-left implication:

If $\Gamma \vDash A$, then $\Gamma_0 \vDash A$ for some finite subset Γ_0 of Γ. Therefore, $\bigwedge \Gamma_0 \to A$ is valid in every predicate topological frame and $\mathsf{QGL}_\infty \vdash \bigwedge \Gamma_0 \to A$. It follows that $\Gamma \vdash A$.

Thank you!