Experimental prover for tope logic

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Type theory with shapes

Context: type theory for synthetic ∞-categories

Type theory for synthetic ∞-categories (Riehl and Shulman 2017) is an extension over an (intentional) Martin-Löf Type Theory with two important features:

1. *tope logic*, which is used essentially to specify commutative diagram schemas.
2. *extension types*, which rely heavily on judgemental equality and the tope logic;

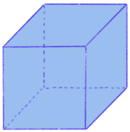
We will refer to Riehl and Shulman's Type Theory as RSTT.

RZK is an experimental proof assistant based on RSTT:
`https://fizruk.github.io/rzk/`

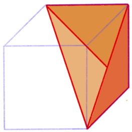
RSTT: bird's-eye view

A 3-layer type theory:

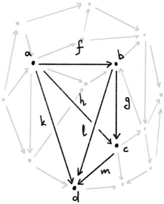
1. *cubes* provide spaces where points come from;
2. *topes* provide restrictions of those spaces;
3. *types* and *terms* are indexed by points in cubes, restricted by topes.



$(t_1, t_2, t_3) : \mathbb{2}^3$



$(t_3 \equiv 0 \wedge t_2 \leq t_1) \vee$
$(t_3 \leq t_2 \wedge t_1 \equiv 1) \vee$
$(t_3 \leq t_2 \wedge t_2 \equiv t_1)$



$a, b, c, d : A$
f, g, h, k, l, m
$g \circ f = h$
$m \circ g = l$
$m \circ h = k$

Tope logic

Cube layer

The **cube layer** is a simple intuitionistic type theory with finite products, i.e. we have variables, unit cube

$$\frac{(t : I) \in \Xi}{\Xi \vdash t : I} \qquad \frac{}{\mathbf{1}\ \text{cube}} \qquad \frac{}{\Xi \vdash \star : \mathbf{1}}$$

as well as products of cubes

$$\frac{I\ \text{cube} \qquad J\ \text{cube}}{I \times J\ \text{cube}} \qquad \frac{\Xi \vdash s : I \qquad \Xi \vdash t : J}{\Xi \vdash \langle s, t \rangle : I \times J} \qquad \frac{\Xi \vdash t : I \times J}{\Xi \vdash \pi_1(t) : I} \qquad \frac{\Xi \vdash t : I \times J}{\Xi \vdash \pi_2(t) : I}$$

For now, nothing else exists in this layer, but we will extend it later.

Tope layer (propositions and constants)

The **tope layer** is an intuitionistic logic over the cube layer. Specifically, the tope layer consists of

1. propositions

$$\frac{\phi \in \Phi}{\Xi \mid \Phi \vdash \phi}$$

2. top (true) and bottom (false) topes

$$\frac{}{\Xi \vdash \top \ \mathsf{tope}} \qquad \frac{}{\Xi \mid \Phi \vdash \top} \qquad \frac{}{\Xi \vdash \bot \ \mathsf{tope}} \qquad \frac{\Xi \mid \Phi \vdash \bot}{\Xi \mid \Phi \vdash \psi}$$

Tope layer (intersection)

The **tope layer** is an intuitionistic logic over the cube layer. Specifically, the tope layer consists of

3. tope intersection (logical AND)

$$\frac{\Xi \vdash \phi \ \text{tope} \qquad \Xi \vdash \psi \ \text{tope}}{\Xi \vdash (\phi \land \psi) \ \text{tope}} \qquad \frac{\Xi \mid \Phi \vdash \phi \qquad \Xi \mid \Phi \vdash \psi}{\Xi \mid \Phi \vdash \phi \land \psi}$$

$$\frac{\Xi \mid \Phi \vdash \phi \land \psi}{\Xi \mid \Phi \vdash \phi} \qquad \frac{\Xi \mid \Phi \vdash \phi \land \psi}{\Xi \mid \Phi \vdash \psi}$$

Tope layer (union)

The **tope layer** is an intuitionistic logic over the cube layer. Specifically, the tope layer consists of

4. tope union (logical OR)

$$\frac{\Xi \vdash \phi \text{ tope} \qquad \Xi \vdash \psi \text{ tope}}{\Xi \vdash (\phi \vee \psi) \text{ tope}} \qquad \frac{\Xi \mid \Phi \vdash \phi}{\Xi \mid \Phi \vdash \phi \vee \psi} \qquad \frac{\Xi \mid \Phi \vdash \psi}{\Xi \mid \Phi \vdash \phi \vee \psi}$$

$$\frac{\Xi \mid \Phi, \phi \vdash \chi \qquad \Xi \mid \Phi, \psi \vdash \chi \qquad \Xi \mid \Phi \vdash \phi \vee \psi}{\Xi \mid \Phi \vdash \chi}$$

Tope layer (equality)

The **tope layer** is an intuitionistic logic over the cube layer. Specifically, the tope layer consists of

5. equality tope

$$\frac{\Xi \vdash t : I \qquad \Xi \vdash s : I}{\Xi \vdash (t \equiv s) \text{ tope}}$$

$$\frac{\Xi \vdash s : I}{\Xi \mid \Phi \vdash (s \equiv s)} \qquad \frac{\Xi \mid \Phi \vdash (s \equiv t)}{\Xi \mid \Phi \vdash (t \equiv s)} \qquad \frac{\Xi \mid \Phi \vdash (s \equiv t) \qquad \Xi \mid \Phi \vdash (t \equiv u)}{\Xi \mid \Phi \vdash (s \equiv u)}$$

$$\frac{\Xi \mid \Phi \vdash (s \equiv t) \qquad \Xi, x : I \vdash \psi \text{ tope} \qquad \Xi \mid \Phi \vdash \psi[s/x]}{\Xi \mid \Phi \vdash \psi[t/x]}$$

Tope layer (equality for products)

The **tope layer** is an intuitionistic logic over the cube layer. Specifically, the tope layer consists of

6. equality topes over products

$$\frac{\Xi \vdash t : \mathbf{1}}{\Xi \mid \Phi \vdash t \equiv \star}$$

$$\frac{\Xi \vdash s : I \qquad \Xi \vdash t : I}{\Xi \mid \Phi \vdash \pi_1(\langle s, t \rangle) \equiv s} \qquad \frac{\Xi \vdash s : I \qquad \Xi \vdash t : I}{\Xi \mid \Phi \vdash \pi_2(\langle s, t \rangle) \equiv t}$$

$$\frac{\Xi \vdash t : I \times J}{\Xi \mid \Phi \vdash t \equiv \langle \pi_1(t), \pi_2(t) \rangle}$$

Shapes and shape inclusion

A **shape** is a cube together with a tope in the corresponding singleton context:

$$\frac{I\,\mathsf{cube} \qquad t : I \vdash \phi\ \mathsf{tope}}{\{t : I \mid \phi\}\ \mathsf{shape}}$$

In RSTT, *extension types along cofibrations* are central to the theory.
The cofibrations are presented by shape inclusion

$$\{t : I \mid \phi\} \subseteq \{t : I \mid \psi\}$$

which means

$$t : I \mid \phi \vdash \psi$$

Tope logic extensions

Simplicial topes

Riehl and Shulman 2017 extend the basic tope logic to the logic of a directed interval in order to construct simplicial types required for synthetic ∞-categories.

First, a new cube for the directed interval is added:

$$\mathbb{2} \text{ cube} \qquad 0 : \mathbb{2} \qquad 1 : \mathbb{2}$$

Then, an inequality tope $(\leq)$ is defined:

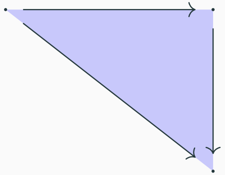
$$x : \mathbb{2}, y : \mathbb{2} \vdash (x \leq y) \text{ tope}$$

Simplicial topes (inequality axioms)

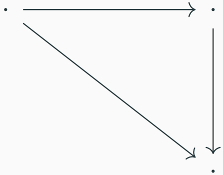
The following tope axioms are defined for the directed interval $\mathbb{2}$:

$$
\begin{aligned}
x : \mathbb{2} \mid \cdot &\vdash (x \leq x) \\
x : \mathbb{2}, y : \mathbb{2}, z : \mathbb{2} \mid (x \leq y), (y \leq z) &\vdash (x \leq z) \\
x : \mathbb{2}, y : \mathbb{2} \mid (x \leq y), (y \leq x) &\vdash (x \equiv y) \\
x : \mathbb{2}, y : \mathbb{2} \mid \cdot &\vdash (x \leq y) \vee (y \leq x) \\
x : \mathbb{2} \mid \cdot &\vdash (0 \leq x) \\
x : \mathbb{2} \mid \cdot &\vdash (x \leq 1) \\
\cdot \mid (0 \equiv 1) &\vdash \bot
\end{aligned}
$$

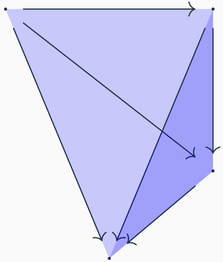
Examples of simplicial shapes



(a) 2-dim simplex, notated Δ^2: $\{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid s \leq t\}$



(b) Boundary of Δ^2, notated $\partial\Delta^2$: $\{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid s \equiv \mathbf{0} \vee s \equiv t \vee t \equiv \mathbf{1}\}$



(c) A 3-dim horn, notated Λ^3_2: $\{\langle t_1, t_2, t_3\rangle : \mathbb{2}^3 \mid (t_3 \equiv \mathbf{0} \wedge t_2 \leq t_1) \vee (t_3 \leq t_2 \wedge t_2 \equiv t_1) \vee (t_3 \leq t_2 \wedge t_1 \equiv \mathbf{1})\}$

Experimental prover(s)

Standalone prover

An experimental standalone prover for the tope logic (with an online playground) is available at https://github.com/fizruk/simple-topes and features:

1. user-defined cubes and topes (with axioms)
2. relies on a naïve sequent-based proof search with some heuristics
3. only contains the tope logic (not full RSTT)
4. rendering for 2D and 3D shapes (LaTeX output)

See playground at https://fizruk.github.io/simple-topes/.

Standalone prover (cube $\mathbb{2}$)

The cube $\mathbb{2}$ is defined in a straightforward way:

```
-- | The strict interval cube (see RS17 Section 3.1).
cube 2 with
  point 0 -- ^ 0 point (left point).
  point 1 -- ^ 1 point (right point).
```

Standalone prover (inequality tope)

1. User-defined axioms are formulated in the form of sequent calculus inference rules.
2. The order of the rules affects the proof search.
3. All user-defined rules as considered by the cut rule, except
   3.1 rules where right hand side is a tope (propositional) variable or
   3.2 rules where right hand side is already a trivial consequence of current premises in $\mathbf{LJ}^{\simeq}$.

```
-- | Inequality tope for the strict interval cube.
-- Here axioms are formulated in sequent calculus form.
-- NOTE: the order of the rules affects the proof search!
tope ≤(2, 2) with
  rule "(≤) distinct" where
    ─────────────
    · | 0 ≡ 1 ⊢ ⊥

  rule "(≤) antisymmetry" where
    ───────────────────────────────────────
    t : 2, s : 2 | ≤(t, s), ≤(s, t) ⊢ s ≡ t

  rule "(≤) transitivity" where
    ─────────────────────────────────────────────────
    t : 2, s : 2, u : 2 | ≤(t, s), ≤(s, u) ⊢ ≤(t, u)

  rule "(≤) excluded middle" where
    ────────────────────────────────────
    t : 2, s : 2 | · ⊢ ≤(t, s) ∨ ≤(s, t)

  rule "(≤) one" where
    ───────────────────
    t : 2 | · ⊢ ≤(t, 1)

  rule "(≤) zero" where
    ───────────────────
    t : 2 | · ⊢ ≤(0, t)

  rule "(≤) reflexivity" where
    ───────────────────
    t : 2 | · ⊢ ≤(t, t)
```

Rzk proof assistant

Rzk proof assistant has a built-in prover used by the typechecker:

1. only built-in cubes and topes (directed interval and inequality tope)
2. similar naïve implementation of the prover
3. implements a version of an existential quantifier, which is used when
4. rendering 1D, 2D, and 3D simplicial diagrams (SVG output)

See documentation and playground at `https://fizruk.github.io/rzk/`.

Interestingly, in practice we do not get large contexts or formulas (so far) with at most 6 dimensions required to be taken care of by the prover. So a naïve prover seems sufficient for now.

However, *shape types* complicate things, bringing entire RSTT back to the tope layer.

Future work and open problems

Even though we have working provers for the tope logic, with satisfactory performance in practice, we would like to improve:

1. Identify and implement an efficient prover for the tope logic:
   1.1 solving many *small* problems fast,
   1.2 formalise the algorithm for simplicial/cubical theories,
   1.3 prove soundness, completeness, and, hopefully, termination.
2. Support user-defined cubes and topes in Rzk.
3. Test the prover on different shapes and shape types, by formalising cubical theory and the work of Buchholtz and Weinberger 2023.

Thank you!

Buchholtz, Ulrik and Jonathan Weinberger (2023). "Synthetic fibered $(\infty, 1)$-category theory". In: *Higher Structures* 7 (1), pp. 74–165. arXiv: 2105.01724 [math.CT].

Riehl, Emily and Michael Shulman (2017). "A type theory for synthetic ∞-categories". In: *Higher Structures* 1 (1). arXiv: 1705.07442 [math.CT].