Algorithmic complexity of monadic multimodal predicate logics with equality over finite Kripke frames

Irina Agadzhanian
HSE University, Moscow, Russia

Mikhail Rybakov
IITP RAS, Moscow, Russia
HSE University, Moscow, Russia
Tver State University, Tver, Russia

Dmitry Shkatov
University of the Witwatersrand, Johannesburg, South Africa

Motivation

$$x = y \to \Box_k (x = y)$$

$$x \neq y \to \Box_k (x \neq y)$$

Motivation

Decidable	Undecidable
The monadic fragment with equality of the $\mathbf{QCl}^{=}$	Monadic modal and superintuitionistic logics (as a rule)

What about finite frames?

Kripke trick

Every modal logic validated by **S5** *frames is undecidable with two monadic predicate letters: write* $\Diamond(P_1(x) \wedge P_2(y))$ *for* $R(x, y)$ *to obtain an embedding of an undecidable fragment of* **QCL**.

Previous researches

- **S. Maslov, G. Mints, and V. Orevkov** had shown that the fragment of the intuitionistic first-order logic **QInt** with a single monadic predicate letter is undecidable;
- **D. Gabbay and V. Shehtman** later had shown it for the modal and superintuitionistic logics of constant domains;

Previous research

- **R. Kontchakov, A. Kurucz, and M. Zakharyaschev** had shown that **QInt** and every modal logic validated by **S5** frames is undecidable with two individual variables (the proof uses two binary predicate letters and an unrestricted supply of unary letters).
- **M. Rybakov, D. Shkatov** had shown that **QInt**, as well as a number of related logics, including those containing the constant domain axiom, are undecidable in languages with two individual variables and a single monadic predicate letter.

Questions

1. Let $\mathcal{C}$ be a recursively enumerable class of finite Kripke n-frames. What is the algorithmic complexity of the monadic fragments with equality of the logics $L^{=}(\mathcal{C})$ and $L_c^{=}(\mathcal{C})$?
2. Let L be one of the logics **K**, **T**, **D**, **K4**, **K4.3**, **S4**, **S4.3**, **GL**,**GL.3**, **Grz**, **Grz.3**, **KB**, **KTB**, **K5**, **K45**, **S5**. What is the algorithmic complexity of the monadic fragments of the logics $\mathbf{Q}L_n^{wfin}$, $\mathbf{Q}L_n.\mathbf{bf}^{wfin}$ and the monadic fragments with equality of the logics $\mathbf{Q}^{=}L_n^{wfin}$, $\mathbf{Q}_n^{=}.\mathbf{bf}^{wfin}$?
3. Are the monadic fragments of logics $\mathbf{QAlt}_n$ decidable?

Language

Definition

The language $\mathcal{L}_n$ is obtained by adding unary modal connectives $\Box_1, \ldots, \Box_n$ to the classical predicate language $\mathcal{L}_0$.

The language $\mathcal{L}_n^=$ is obtained by adding a designated binary predicate letter $=$ to $\mathcal{L}_n$.

Definition

A monadic $\mathcal{L}_n$-formula is a formula that contains only monadic predicate letters.

A monadic $\mathcal{L}_n^=$-formula is a formula that contains only monadic predicate letters and $=$.

Notations

1. $\mathbf{Q}L$ is the minimal logic that contains **QCl** and the n-modal propositional logic L;
2. $\mathbf{Q}^{=}L$ is the minimal extension of $\mathbf{Q}L$ that contains the classical equality axioms;
3. $L.\mathbf{bf}$ is the minimal extension of an n-modal predicate logic L that contains the **Barcan formula** $\mathbf{bf}_k = \forall x \Box_k P(x) \to \Box_k \forall x P(x)$ for each $k \in \{1, \ldots, n\}$.

Kripke semantics

Definition

A Kripke n-modal frame is a tuple $F = \langle W, R_1, \ldots, R_n \rangle$, where W is a non-empty set and $R_1, \ldots, R_n$ are binary accessibility relations on W.

Kripke semantics

Definition

An augmented Kripke frame is a tuple $\mathfrak{F} = \langle F, D \rangle$, where F is a Kripke n-modal frame and D is a family $(D_w)_{w \in W}$ of non-empty domains that satisfy the expanding domains condition $\forall k \in \{1, \ldots, n\}\ \forall w,v \in W\ (wR_k v \Rightarrow D_w \subseteq D_v)$

Definition

A Kripke model is a tuple $\mathfrak{M} = \langle \mathfrak{F}, I \rangle$, where $\mathfrak{F}$ is an augmented n-frame and I is a family $(I_w)w \in W$ of interpretations of predicate letters $(I_w(P) \subseteq D_w^m$ for every m-ary letter $P)$.

Kripke semantics with equality

Definition

An augmented n-frame with equality is a tuple $\langle \mathfrak{F}, D, \equiv \rangle$, where $\langle \mathfrak{F}, D \rangle$ is an augmented n-frame and $\equiv$ is a family $(\equiv_w)_{w \in W}$ of equivalence relations that satisfies the heredity condition: $\forall k \in \{1, \ldots, n\}\ \forall w, v \in W\ (wR_k v \Rightarrow \equiv_w \subseteq \equiv_v)$

Definition

A Kripke model with equality is a tuple $\langle \mathfrak{F}, I \rangle$, where $\mathfrak{F}$ is an augmented n-frame with equality and I is a family $(I_w)_{w \in W}$ of interpretations of predicate letters such that $\equiv_w$ is a congruence on the classical model $M_w = \langle D_w, I_w \rangle$.

Truth relation

Definition

If a,$b \in D_w$ and $\overline{c}$ is a list of elements in D_w of suitable length, then:

$$\begin{array}{lcl}
\mathfrak{M}, w \vDash a = b & \iff & a \equiv_w b \\
\mathfrak{M}, w \vDash P(\overline{c}) & \iff & \overline{c} \in I_w(P) \\
\mathfrak{M}, w \vDash \forall x \varphi(x,\overline{c}) & \iff & \mathfrak{M}, w \vDash \varphi(d,\overline{c}), \text{ for every } d \in D_w \\
\mathfrak{M}, w \vDash \Box_k \varphi(\overline{c}) & \iff & \mathfrak{M}, w \vDash \varphi(\overline{c}), \text{ for every } v \in R_k(w)
\end{array}$$

Notations

If $\mathcal{C}$ is a class of Kripke n-frames, then:

1. $L(\mathcal{C})$ is the set of $\mathcal{L}_n$-formulas valid on $\mathcal{C}$;
2. $L_c(\mathcal{C})$ is the set of $\mathcal{L}_n$-formulas valid on locally constant augmented n-frames over a Kripke frame from $\mathcal{C}$;
3. $L^=(\mathcal{C})$ is the set of $\mathcal{L}_n^=$-formulas valid on $\mathcal{C}$;
4. $L_c^=(\mathcal{C})$ is the set of $\mathcal{L}_n^=$-formulas valid on locally constant augmented n-frames with equality over a Kripke frame from $\mathcal{C}$.

Question 1

Question:

Let $\mathcal{C}$ be a recursively enumerable class of finite Kripke n-frames. What is the algorithmic complexity of the monadic fragments with equality of the logics $L^{=}(\mathcal{C})$ and $L_c^{=}(\mathcal{C})$?

Lemma

If a monadic n-modal formula with equality containing n monadic predicate letters and m individual variables is refuted on an augmented frame with equality $\mathfrak{F} = \langle W, R_1, \ldots, R_n, D, \equiv\rangle$ with finite W, then it is refuted on an augmented frame with equality $\mathfrak{F}' = \langle W, R_1, \ldots, R_n, D', \equiv'\rangle$ with $|D'^{+}| \leqslant |W| \cdot m \cdot 2^{|W|(n+1)}$.

Question 1

Theorem

Let $\mathfrak{F}$ be a finite Kripke frame. Then, the monadic fragment with equality of the logics $L^{=}(\mathfrak{F})$ and $L_c^{=}(\mathfrak{F})$ are both decidable.

1. $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$
2. φ be a monadic n-modal formula with equality containing n predicate letters
3. $\varphi \in L^{=}(\mathfrak{F}) \iff$ exists $\mathfrak{F}' = \langle \mathfrak{F}, D \rangle$ such that $\mathfrak{F}' \vDash \varphi$ $(|D^{+}| \leqslant |W| \cdot m \cdot 2^{|W|(n+1)})$

Question 1

Theorem

Let $\mathcal{C}$ be a recursively enumerable class of finite Kripke n-frames. Then the monadic fragments with equality of the logics $L^{=}(\mathcal{C})$ and $L^{=}_{c}(\mathcal{C})$ are both in Π^0_1.

1. the complements of the monadic fragments of $L^{=}(\mathcal{C})$ and $L^{=}_{c}(\mathcal{C})$ are both belong to Σ^0_1
2. $\varphi_1, \varphi_2, \ldots; \mathfrak{F}_1, \mathfrak{F}_2, \ldots$
3. for every $k : \ k \geqslant 1$ it is decidable whether $\mathfrak{F}_i \models \varphi_j$ for every $i,j \in \{1, \ldots, k\}$

Question 2

Question:

Let L be one of the logics **K**, **T**, **D**, **K4**, **K4.3**, **S4**, **S4.3**, **GL**,**GL.3**, **Grz**, **Grz.3**, **KB**, **KTB**, **K5**, **K45**, **S5**. *What is the algorithmic complexity of the monadic fragments of the logics* $\mathbf{Q}L_n^{wfin}$, $\mathbf{Q}L_n.\mathbf{bf}^{wfin}$ *and the monadic fragments with equality of the logics* $\mathbf{Q}^{=}L_n^{wfin}$, $\mathbf{Q}_n^{=}.\mathbf{bf}^{wfin}$ *?*

Question 2

Definition

A **fusion** of 1-modal propositional logics $L_1, \ldots, L_n$ is the logic $L_1 * \ldots * L_n = \mathbf{K}_n \oplus (L_1 \cup L'_2 \cup \ldots \cup L'_n)$, where L'_i is obtained from L_i by replacing every occurence of $\Box_1$ with $\Box_i$.

If $L \in [\mathbf{QK}^{wfin}, \mathbf{QGL.3.bf}^{wfin}] \cup [\mathbf{QK}^{wfin}, \mathbf{QGrz.3.bf}^{wfin}] \cup [\mathbf{QK}^{wfin}, \mathbf{QS5}^{wfin}]$, then the monadic fragment of L is Π^0_1-hard.

Question 2

Corollary

Let $L = L_1 * \cdots * L_n$*, where* $L_1, \ldots, L_n$ *are normal* 1*-modal propositional logics, such that*

- $L_i \subseteq \mathbf{S5}$ *or* $L_i \subseteq \mathbf{Grz.3}$ *or* $L_i \subseteq \mathbf{GL.3}$*, for some* $i \in \{1, \ldots, n\}$*;*
- *the class of finite Kripke frames validating* L *is recursively enumerable.*

Then, the monadic fragments of the logics $\mathbf{Q}L^{wfin}$, $\mathbf{Q}L.\mathbf{bf}^{wfin}$ *and the monadic fragments with equality of the logics* $\mathbf{Q}^{=}L^{wfin}$, $\mathbf{Q}^{=}L.\mathbf{bf}^{wfin}$ *are all* Π^0_1*-complete.*

Question 2

Theorem

Let L be one of the logics **K**, **T**, **D**, **K4**, **K4.3**, **S4**, **S4.3**, **GL**,**GL.3**, **Grz**, **Grz.3**, **KB**, **KTB**, **K5**, **K45**, **S5**. *Then the monadic fragments of the logics* $\mathbf{Q}L_n^{wfin}$, $\mathbf{Q}L_n.\mathbf{bf}^{wfin}$ *and the monadic fragments with equality of the logics* $\mathbf{Q}^{=}L_n^{wfin}$, $\mathbf{Q}^{=}L_n.\mathbf{bf}^{wfin}$ *are* Π_1^0*-complete.*

Question 3

Question:

Are the monadic fragments of logics $\mathbf{QAlt}_n$ *decidable?*

Question 3

Theorem

Let $\mathcal{C}$ be a decidable class of Kripke 1-frames closed under the operation of taking subframes and satisfying the condition that there exists $n \in \mathbb{N}$ such that $|R(w)| \leqslant n$ whenever $\langle W, R\rangle \in \mathcal{C}$ and $w \in W$. Then the monadic fragments of the logics $L(\mathcal{C}), L_c(\mathcal{C})$ and the monadic fragments with equality of the logics $L^=(\mathcal{C}), L_c^=(\mathcal{C})$ are decidable.

1. $\varphi \notin L(\mathcal{C}) \Rightarrow$ exists $\mathfrak{F} = \langle W, R\rangle,\ w_0 \in W$ such that $\mathfrak{F}, w_0 \not\models \varphi$
2. $\mathfrak{F}_\varphi$ such that $\{R^n(w_0) :\ n \leqslant \mathrm{md}\varphi\} \Rightarrow \mathfrak{F}_\varphi$ contains

$$|W'| \leqslant \frac{n^{\mathrm{md}\varphi+1} - 1}{n-1}$$

Question 3

Theorem

Let $\mathcal{C}$ be a decidable class of Kripke 1-frames closed under the operation of taking subframes and satisfying the condition that there exists $m \in \mathbb{N}$ such that $|R(w)| \leqslant m$ whenever $\langle W, R\rangle \in \mathcal{C}$ and $w \in W$. Then the monadic fragments of the logics $L(\mathcal{C}), L_c(\mathcal{C})$ and the monadic fragments with equality of the logics $L^{=}(\mathcal{C}), L_c^{=}(\mathcal{C})$ are decidable.

Theorem

The monadic fragments of logics $\mathbf{QAlt}_n$, $\mathbf{QAlt}_n.\mathbf{bf}$, $\mathbf{Q^{=}Alt}_n$, $\mathbf{Q^{=}Alt}_n.\mathbf{bf}$ *are all decidable.*

Thank you!