Quantum Structure of Black Holes

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Based on the joint works with I. Volovich

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III International Conference

“$M \cap \Phi$, Dynamical Systems, ∞-Dimensional Analysis” dedicated to the 100th anniversary of **V.S. Vladimirov** the 100th anniversary of **L.D. Kudryavtsev** and the 85th anniversary of **O.G. Smolyanov**

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Outlook

- Introduction
  - Four Laws of Thermodynamics vs Four Laws of Black Hole mechanics
  - Third Law of Thermodynamics and its violation by BHs
- Microscopic origin of the Bekenstein-Hawking entropy via Bose gas models
  - entropy of **non-local** Bose gas models with the zeta function regularizations
- BH entropy via random thin shell model
  - entropy of random thin shell models

Four Laws of Thermodynamics vs Four Laws of Black Hole mechanics

- There is a remarkable analogy between the laws of thermodynamics and the laws of black hole mechanics

Thermodynamics

- 0. $E, T, S, V, P, ...$
- 1. $dE = TdS - PdV$
- 2. $\delta S \geq 0$
- 3. $S \to 0$ if $T \to 0$

Black Hole mechanics
(Bardeen, Carter, Hawking,73'; Bekenstein 73')

- 0. surface gravity $\kappa = \frac{1}{M}$, $Q, a, ...$
- 1. $dM = \frac{1}{8\pi M} d\,\frac{\mathcal{A}}{4} + ...$
- 2. $\delta \mathcal{A} \geq 0$
- 3. States with $\kappa = 0$ are unattainable

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- A missing link in this area is a precise statistical mechanical interpretation of entropies for all varieties of black holes.
- We can try to find a statistical mechanics model with the same dependence of entropy on other thermodynamic variables as a particular black hole has
- However, there is a problem with the third law of thermodynamics

Third Law of Thermodynamics

- **In the Planck formulation** : Entropy $S \to 0$ as $T \to 0$ $(\beta = \frac{1}{T} \to \infty)$

- In the Nernst formulation

$$\delta S(T,x) \equiv S(T,x) - S(T,x') \to 0 \quad as \quad T \to 0 \tag{1}$$

or

$$\lim_{T \to 0} S(T,x) - \text{universal constant}$$

- Unattainability of $T = 0$

REFS: W.Israel, 1986; R.Wald, 1997;
F. Belgiorno and M. Martellini, 2004;
C. Kehle and R. Unger, 2211.1574.

Violation of Third Law in BH Thermodynamics

- **Schwarzschild black hole**
  - Hawking temperature $T = \frac{1}{8\pi M}$
  - Bekenstein-Hawking entropy $S = \frac{1}{16\pi T^2} \to \infty \quad as \quad T \to 0$

Violation in Planck formulation

- Reissner-Nordstrom black hole
  - Hawking temperature $T = \frac{\sqrt{M^2-Q^2}}{2\pi\left(\sqrt{M^2-Q^2}+M\right)^2} \to 0$ for $M \to Q$ or $M \to \infty$
  - BH entropy $S = \pi\left(\sqrt{M^2-Q^2}+M\right)^2 \to \pi Q^2$ for $T \to 0$ **depends on** Q
- Kerr

Violation in Nernst formulation

Physical systems with violation of the Third Law *

- Lattice models.
  The question of whether the third law is satisfied can be decided completely in terms of ground-state degeneracies
  M. Aizenman, El. Lieb 80'
- Ice models.
  V. F. Petrenko and R. W. Whitworth, 99', *Physics of Ice*
- Strange metals.
  J. Zaanen et al. 15', *Holographic duality in condensed matter physics.*

Few Refs. on microscopic origin of BH entropy *

- **The problem of the microscopic origin of the Bekenstein-Hawking entropy of a black hole has attracted a lot of attention over the past 30 years**

  - **Wheeler** considered of the BH interior as "bag of gold" (**Almheiri et al 20**)
  - **Strominger and Vafa, 96'** $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2$, $f(r) = \left(1 - (\frac{r_0}{r})^2\right)^2$, $r_0 = (\frac{8Q_H Q_F^2}{\pi^2})^{1/6}$, $S_{BH} = 2\pi\sqrt{\frac{Q_H Q_F^2}{2}}$
    D-0 branes interpretation: $d(n,c) \sim \exp(2\pi\sqrt{\frac{nc}{6}})$, $c = 6(\frac{1}{2}Q_F^2 + 1)$, $\quad n = Q_H$
    $$S_{stat} = \ln d(Q_F, Q_H) \sim 2\pi\sqrt{Q_H(\frac{1}{2}Q_F^2 + 1)}$$
  - **'t Hooft 84'** proposed to relate BH entropy with the entropy of thermally excited quantum fields in the vicinity of the horizon.
  - Recent searches **Balasubramanian et al 22'** for internal geometries that provide the entropy of BH.
  - Matrix models corresponding to BH in spacetime with topology $AdS_2 \times S^8$, **Maldacena'23**

To summarize Introduction

- **Schwarzschild BHs violate 3-d law of thermodynamics.**
- **Schwarzschild BH entropies in D-dim $S \to \infty$ rather than zero when $T \to 0$.**
- **We search for quantum statistical models with such exotic thermodynamic behaviour.**

Free energy of non-local Bose gas (NLBG).

- d-dim Bose gas

$$F_{BG}(d,\varepsilon) = \frac{\Omega_{d-1}}{\beta}\int_0^\infty \ln\left(1-e^{-\lambda\,\beta\;\varepsilon(k)}\right)\,k^{d-1}dk$$

  - standard (local case) $\varepsilon(k)=k^2$
  - d-dim **α-non-local** Bose gas $\varepsilon = k^\alpha$, $F_{BG}(d,\alpha) = F_{BG}(d,\varepsilon)\Big|_{\varepsilon=k^\alpha}$
  - d-dim **$\mathcal{F}$-non-local Bose gas**, $\mathcal{F}(k)$ -an analytical function. $\mathcal{F}(k)=\exp(ck^2)$. **V.S.Vladimirov, see B.Dragovich's talk**.

- Explicit form

$$F_{BG}(d,\alpha) = -\frac{2\pi^{d/2}}{d\Gamma(d/2)}\left(\frac{1}{\beta}\right)^{\frac{d}{\alpha}+1}\left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}}\Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right).$$

- Free energy of D-dim Schwarzschild BH $F_{BH}(D,\beta)$ [**see next slides**]

- $F_{BH}(D,\beta) = F_{BG}(d,\beta)$

D=4 Schwarzschild BH vs Bose Gas. 1/3

- Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

- Hawking temperature and Bekenstein-Hawking entropy

$$T = \frac{1}{8\pi M}, \qquad S = 4\pi M^2 = \frac{\beta^2}{16\pi}$$

- Free energy

$$F = \frac{\beta}{16\pi}$$

D=4 Schwarzschild BH vs Bose Gas. 2/3

- Equalizing: $F_{BG}(\beta) = F_{BH}(\beta)$

$$-\frac{\pi^{d/2}}{\beta^{\frac{d}{2}+1}\lambda^{\frac{d}{2}}}\,\zeta\left(\frac{d}{2}+1\right) = \frac{\beta}{16\pi} \quad (*)$$

- To fulfill (*) we have to assume

$$d = -4, \qquad \lambda^2 = -\frac{\pi}{16\,\zeta(-1)}.$$

- Taking into account that $\zeta(-1) = -1/12$, we get

$$\lambda = \sqrt{\frac{3\pi}{4}},$$

D=4 Schwarzschild BH vs Bose Gas. 3/3

- Therefore, we obtain that the thermodynamics of the 4-dim Schwarzschild BH is equivalent to the thermodynamics of the Bose gas in $d = -4$ spatial dimensions.
- We understand the thermodynamics of the Bose gas in negative spatial dimensions in the sense of the analytical continuation of the right hand site of

$$F_{BG} = -\frac{\pi^{d/2}}{\beta^{\frac{d}{2}+1}\lambda^{\frac{d}{2}}}\zeta\left(\frac{d}{2}+1\right).$$

D>4 Schwarzschild BH vs Bose Gas 1/2

- D-dimensional Schwarzschild black hole, $D \geq 4$,

$$ds^2 = -\left(1 - \frac{{r_h}^{D-3}}{r^{D-3}}\right) dt^2 + \frac{dr^2}{1 - \frac{{r_h}^{D-3}}{r^{D-3}}} + r^2 d\omega_{D-2}^2,$$

- Hawking temperature $T = 1/\beta = \frac{D-3}{4\pi r_h}$
  r_h is the radius of the horizon.
- The entropy and the free energy are

$$S = \frac{\Omega_{D-2}}{4}\left(\frac{D-3}{4\pi}\,\frac{1}{T}\right)^{D-2}; \; F = \frac{(D-3)^{D-3}\beta^{D-3}\,\Omega_{D-2}}{4(4\pi)^{D-2}}$$

$S \to \infty$, when $T \to 0$ — a violation of the 3-d law

- Equalizing: $F_{BG}(\beta) = F_{BH}(\beta)$ series of solutions

I.Volovich's talk

D>4 Schwarzschild BH vs Bose Gas. 2/2
4 series of solutions

oframenumbering4 series of solutions

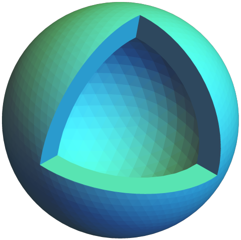
D	d	α
$D=4k+1, \quad k=1,2,3...$	$d=(4k-1)\lvert\alpha\rvert$	$\alpha=-q,\ q=1,2,3$
$D=4k+1, \quad k=1,2,3...$	$d=-(4k-1)\alpha$	$\frac{4r}{4k-1}<\alpha<\frac{2(2r+1)}{4k-1}, \quad r=0,1,2,...$
$D=4k+3, \quad k=1,2,3...$	$d=-(4k+1)\alpha$	$\frac{2(2r+1)}{4k+1}<\alpha<\frac{4(r+1)}{4k+1}, \quad r=0,1,2...$
$D=2k, \quad k=2,3,4...$	$d=-2(k-1)\alpha$	$\alpha=\frac{p}{k-1}, \quad p=1,2,..$

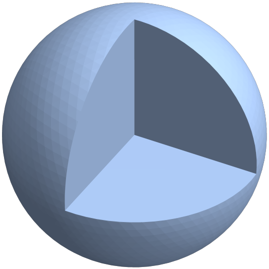
- Euclid $d=3$
  Kaluza-Klein $d=5$
  Superstrings $d=10$
  Here $d<0$

Thin Shells in $M \cap \Phi$

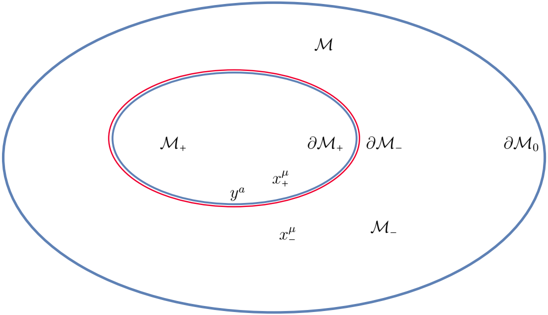
- A shell in D-dimensional space is D-dimensional body whose thickness is very small compared to its other dimensions
- A thin shell in D-dimensional space is a D-1 dimensional hyper surface

D=3





Thin Shells in Math.Phys.

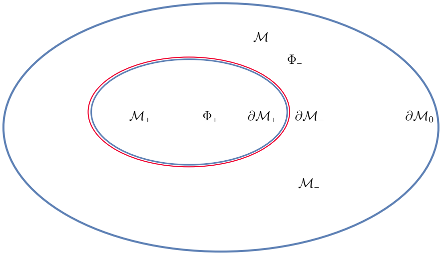


- The hypersurfaces $\partial\mathcal{M}_+$ and $\partial\mathcal{M}_-$ coincide, $\partial\mathcal{M}_+ = \partial\mathcal{M}_- = \Sigma$
- The normal vectors $\mathbf{n}^\pm$ to them are oppositely directed. The unit normal $\mathbf{n}^\pm$ are directed from $\mathcal{M}^\mp$ to $\mathcal{M}^\pm$.
- Σ divides the spacetime $\mathcal{M}$ into two regions $\mathcal{M}^+$ and $\mathcal{M}^-$
- $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$. The boundary $\mathcal{M}$ is $\partial\mathcal{M} = \partial\mathcal{M}_+ \cup \partial\mathcal{M}_- \cup \partial\mathcal{M}_0$.

Action Principle for Shell Equation

Elliptic case

В.С. Владимиров, О. А. Ладыженская,...



- $\mathcal{M}_\pm$ are given
- $\Phi_\pm$ are field on $\mathcal{M}_\pm$
- The action [energy] is

$$\begin{aligned} L &= \int_{\mathcal{M}_-} \mathcal{L}(\Phi_-, \partial\Phi_-) d^D x \\ &+ \int_{\mathcal{M}_+} \mathcal{L}(\Phi_+, \partial\Phi_+) d^D x \\ \mathcal{L}(\Phi, \partial\Phi) &= \frac{1}{2}(\partial_i \Phi)^2 + V(\Phi) \end{aligned}$$

- The principle of least action $\delta L = 0$. We assume $\delta\Phi_+\Big|_{\mathcal{M}_+} = \delta\Phi_-\Big|_{\mathcal{M}_-}$

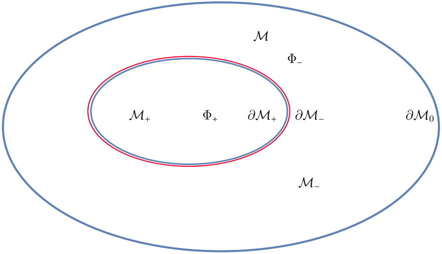
$$\delta L = \delta L_{bulk} + \delta L_{bnd}; \qquad \delta L_{bulk} = \sum_{a=\pm} \int_{\partial\mathcal{M}_a} (-\partial^2 \Phi_a + V')\delta\Phi_a \, d^D x$$

$$\delta L_{bnd} = \int_{\partial\mathcal{M}_0} \underbrace{(\partial_i\Phi_-, n_i)}_{=0\, Neum.bnd} \delta\Phi_- dS + \int_{\partial\mathcal{M}_-} \underbrace{(\partial_i\Phi_- - \partial_i\Phi_+, n_i)}_{=0\, junction\, cnd} \delta\Phi_- dS$$

- To get $[\partial_n\Phi] \neq 0$ we have to add $L = \int \Phi J_n dS_n$

Action Principle for Shell Equation

Hyperbolic case



- $\mathcal{M}_\pm$ are given, $\partial\mathcal{M}_\pm$ - **NON-NULL**
- $\Phi_\pm$ are field on $\mathcal{M}_\pm$
- The action [energy] is

$$L = \int_{\mathcal{M}_-} \mathcal{L}(\Phi_-, \partial\Phi_-) d^D x + \int_{\mathcal{M}_+} \mathcal{L}(\Phi_+, \partial\Phi_+) d^D x$$

$$\mathcal{L}(\Phi, \partial_\mu\Phi) = \frac{-\partial_0\Phi^2 + \partial_i\Phi^2}{2} + V(\Phi)$$

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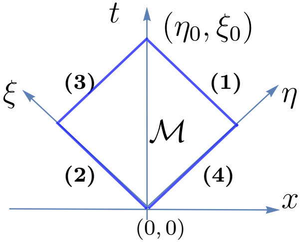
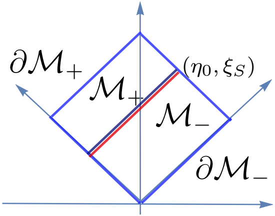
- To get $[\partial_\mathbf{n}\Phi] \neq 0$ we have to add $L = \int \Phi J_\mathbf{n} dS_\mathbf{n}$

Action Principle for Shell Equation

Hyperbolic case





- $\partial\mathcal{M}$ - **NULL**
- The action is $L_0 = \int_{\mathcal{M}} \partial_\xi\Phi\, \partial_\eta\Phi\, d\xi d\eta$

$$\delta L_{0,bnd} = \int_0^{\xi_0} d\xi\delta\Phi\partial_\xi\Phi\Big|_0^{\eta_0} + \int_0^{\eta_0} d\eta\delta\Phi\partial_\eta\Phi\Big|_0^{\xi_0}$$

$$1)\, \partial_\xi\phi(\eta_0, \xi) = 0;\ 2)\partial_\xi\phi(0, \xi) = 0;$$
$$3)\, \partial_\eta\phi(\eta, \xi_0) = 0;\ 4)\, \partial_\eta\phi(\eta, 0) = 0$$

V.Sakbaev, I. Volovich, 1312.4302

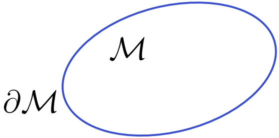
- $\mathcal{M}_\pm$ are given, $\partial\mathcal{M}_\pm$ - **NULL**
- $\Phi_\pm$ are field on $\mathcal{M}_\pm$; $\delta\Phi_+\Big|_{\mathcal{M}_+} = \delta\Phi_-\Big|_{\mathcal{M}_-}$
- The part of variation on the shell

$$\delta L_{0,shell} = \int_0^{\eta_0} d\eta\delta\Phi(\eta, \xi_s)(\underbrace{\partial_\eta\Phi_+ - \partial_\eta\Phi_-}_{=0\, Neumann\, bnd.})$$

$[\partial_{tangential}\Phi] = 0$ NO RESTRICTIONS on $[\partial_n\Phi]$

In agreement with: discontinuities can only propagate along characteristics

Action Principle in General Relativity



Gibbons, Hawking, Yost

- HE action $S_{\rm EH} = \int_{\mathcal{M}} R\sqrt{-g}\, d^4x$,

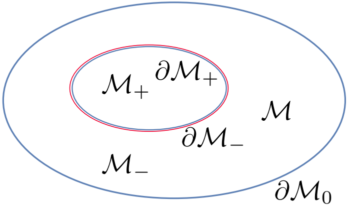
$\partial\mathcal{M}$ NOT NULL

$$\delta S_{\rm EH} = \int_{\mathcal{M}} G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\, d^4x + \int_{\partial M} (Kh_{ab} - K_{ab})\,\delta h^{ab}\sqrt{|h|} d^3y$$
$$+ \int_{\partial\mathcal{M}} \epsilon\, h^{\alpha\beta}\delta g_{\alpha\beta,\mu} n^{\mu} |h|^{1/2}\, d^3y \qquad g^{\alpha\beta} = \epsilon n^{\alpha} n^{\beta} + h^{\alpha\beta}$$

- GHY term $S_{GHY} = 2\int_{\partial M} K\sqrt{|h|} d^3y$, $\delta S_{GHY} = \int_{\partial\mathcal{M}} \epsilon h^{\alpha\beta}\delta g_{\alpha\beta,\mu} n^{\mu}|h|^{1/2}\, d^3y$

$$\delta(S_{\rm EH} + S_{GHY}) = \int_{\mathcal{M}} G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\, d^4x + \int_{\partial\mathcal{M}} (Kh_{ab} - K_{ab})\,\delta h^{ab}\sqrt{|h|} d^3y$$

Thin Shells in General Relativity



- The boundary NOT NULL

$$\partial\mathcal{M} = \partial\mathcal{M}_0 \cup \partial\mathcal{M}_- \cup \partial\mathcal{M}_-$$

- HE + GHY action
  $S_{\rm EH} + S_{GHY} = \int_{\mathcal{M}} R\sqrt{-g}\, d^4x$

  $+ \int_{\partial\mathcal{M}} K\, \sqrt{|h|} d^3y$

- $g_{\mu\nu}\Big|_{\partial\mathcal{M}_+} = g_{\mu\nu}\Big|_{\partial\mathcal{M}_-}$,

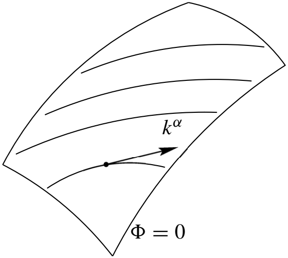
$$\delta(S_{\rm EH} + S_{GHY}) = \int_{\mathcal{M}} G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\, d^4x + \int_{\partial\mathcal{M}_0} (Kh_{ab} - K_{ab})\,\delta h^{ab}\sqrt{|h|} d^3y$$

$$+ \int_{\partial\mathcal{M}_+} (Kh_{ab} - K_{ab})\,\delta h^{ab}\sqrt{|h|} d^3y + \int_{\partial\mathcal{M}_+} (Kh_{ab} - K_{ab})\,\delta h^{ab}\sqrt{|h|} d^3y$$

$$\Rightarrow \quad [Kh_{ab} - K_{ab}] = 0 \qquad + \text{Matter on shell: } \underbrace{[Kh_{ab} - K_{ab}] = T_{ab}}_{\text{Israel's BC}}$$

Action principle for null boundaries in GR

- **Suitable coordinates**



Σ: $\Phi = 0$ - null, $\quad k_\alpha = \Phi_{,\alpha} \quad g^{\alpha\beta} k_\alpha k_\beta = 0$

Coordinate system on Σ: $y^\alpha = (\lambda, \theta^A)$,
$dx^\alpha = k^\alpha \lambda$, $e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda$.

An auxiliary null vector field N^α:
$N_\alpha k^\alpha = 1$, $N_\alpha e^\alpha_A = 0$

$$g^{\alpha\beta} = -k^\alpha N^\beta - N^\alpha k^\beta + \sigma^{AB} e^\alpha_A e^\beta_B,$$

$$(\delta S_{HE})_{null} = \int_{\partial \mathcal{M}} d^3y \ \left\{-2\delta\left[(\Theta + \kappa)\right] + \sqrt{\sigma}\left[\Theta_{AB} - (\Theta + \kappa)\sigma_{AB}\right]\delta\sigma^{AB}\right\}$$

where Θ_{AB} is the second fundamental form, $\Theta = \Theta^A_A$ is the expansion scalar, κ is the non-affinity coefficient on the null surface, i.e. $k^\alpha \nabla_\alpha k^\beta = \kappa k^\beta$

Null Thin Shells in General Relativity

- Junction condition on null surfaces without matter

$$[\Theta_{AB} - (\Theta + \kappa)\sigma_{AB}]_{NullShell} = 0$$

No restrictions on $\perp$ directions

- Junctions on null surfaces with matter

$$[\Theta_{AB} - (\Theta + \kappa)\sigma_{AB} - T_{AB}]_{NullShell} = 0$$

Gas of random quantum thin shells

- **A spherical symmetric thin shell $\Sigma = \mathbb{R} \times \mathbb{S}^2$ in spherical symmetric background divides the spacetime on**
  - the internal spacetime, $\mathcal{M}^-$ (with Schw. coord. $(t_-, r_-, \phi\theta)$), and
  - an external spacetime, $\mathcal{M}^+$(with Schw. coord. t_+, r_+, ϕ, θ)
- The shell can be describe by equations

$$r_\pm = r = R(\tau), \quad t_\pm = t(\tau).$$

- In term of intrinsic coord. of the shell (τ, θ, ϕ), the induced metric on Σ is

$$ds^2_\Sigma = d\tau^2 - R^2(\tau)d\Omega^2$$

Berezin, Kusmin,Tkachev,1988

Effective action for the shell

- **The effective action for the shell in the proper time**

$$S = \int d\tau \left[-m + f(R)\sqrt{1 + R_\tau^2} \right], \qquad f(R) = \frac{Gm^2}{2R}$$

- **Hamiltonian**

$$H = m - \sqrt{f^2 - P^2},$$

- **Wheeler-DeWitt equation**

$$\left[(-i\partial_\tau + m)^2 - \partial_R^2 - f^2 \right] \Psi(\tau, R) = 0.$$

Stationary solutions of WdW eq. Spectrum

Taking $\Psi(\tau, R)$ in the form

$$\Psi(\tau, R) = e^{-i\mathcal{E}\tau}\psi(R),$$

we get the stationary version WdW eq.

$$\psi''(R) - \left[(m-\mathcal{E})^2 - \frac{m^4}{4m_p^4 R^2}\right]\psi(R) = 0, \quad (*)$$

m is the shell mass, m_p is the Planck mass, $m_p = 1/\sqrt{G}$, G is the Newton gravitational constant. $\hbar = c = 1$.

Spectrum

The spectrum of equation (*) for $m > m_p$ is **Vaz, 2022**

$$\mathcal{E}_n(m) = m\left(1 - e^{-n\pi/\mathfrak{b}}\right), \qquad (**)$$
$$\mathfrak{b} = \frac{1}{2m_p^2}\sqrt{m^4 - m_p^4}, \quad m_p < m$$

n is a positive integer.

Free energy of bose gas of shells

- The free energy of bose gas of shells at temperature $T = 1/\beta$ and chemical potential μ

$$F_{gas-of-shells}(\beta,\mu,m) = \frac{1}{\beta}\sum_{n}\ln\left(1 - e^{\beta\,(\mu-\mathcal{E}_n(m))}\right)$$

here $\mathcal{E}_n(m)$, $n = 1, 2, 3, ...$ is the spectrum (**)

Free energy of RANDOM bose gas of shells, 1/4

At temperature $T = 1/\beta$

$$\mathcal{F}_{gas-of-shells}(\beta, \mu, m_p)$$

$$= \frac{1}{\beta} \int \sum_{n}^{N} \ln\left(1 - e^{\beta\,(\mu - \mathcal{E}_n(m))}\right) d\sigma(m)$$

$\mathcal{E}_n(m)$, $n = 1, 2, 3, ...$ is spectrum (**) for fixed **random parameter** m
.

d$\sigma = d\sigma(m)$ **is the probability measure**

Free energy of random bose gas of shells, 2/4

Now we specify the measure $\sigma = \sigma(m)$ and deal with $\mathcal{F}_{gas-of-shells}(\beta, \mu, m_p)$ given by

$$\mathcal{F}_{gas-of-shells}(\beta, \mu, m_p) = \frac{1}{\beta} \int_{m_p(1+\Delta)}^{2m_p} \sum_{n}^{N} \ln \left(1 - e^{\beta\,(\mu - \mathcal{E}_n(m))}\right) \frac{C\, dm}{(m - m_p)^3},$$

where $\Delta > 0$ is the regularization parameter and the constant C is derive from normalization

$$C^{-1} = \int_{m_p(1+\Delta)}^{2m_p} \frac{dm}{(m - m_p)^3}$$

and for small regularization parameter Δ

$$C = -m_p^2 (\frac{2\Delta^2}{\Delta^2 - 1}) \approx 2m_p^2 \Delta^2$$

Free energy of random bose gas of shells, 3/4

After the change of the variable $m - m_p = x m_p$ and taking $\mathcal{E}_n(m) \approx m = m_p(1+x)$ we get the representation

$$\mathcal{F}_{gas.shells}(\beta, \mu, m_p) = \frac{2N\Delta^2}{\beta} \int_{\Delta}^{1} \ln\left(1 - e^{\beta\,(\mu - m_p(1+x))}\right) \frac{dx}{x^3},$$

Taking $\mu = m_p$ we finally get

$$\mathcal{F}_{gas.shells}(\beta, m_p) \approx \frac{2N\Delta^2}{\beta} \int_{\Delta}^{1} \ln\left(1 - e^{-\beta\, m_p x}\right) \frac{dx}{x^3},$$

Free energy of random bose gas of shells, 4/4

Denote $N\Delta^2 = \lambda$ and consider $\Delta \to 0$ and $N \to \infty$,

$$N\beta C\, I(a) = N\beta 2m_p^2\, \Delta^2\, I(a) = 2m_p^2 \lambda\beta\, I(a)$$

Taking the renormalized value of I we get at $a \to 0$

$$\mathcal{F}_{ren,gas-of-shells}(\beta, m_p) \quad \approx \quad 2\,\lambda\, I_{ren}\, m_p^2 \beta$$

and the entropy is equal to

$$S = 2\,\lambda I_{ren}\, m_p^2 \beta^2$$

We set $2\,\lambda I_{ren} = \frac{1}{16\pi}$. This gives the BH entropy

$$S = \frac{1}{16\pi}\, m_p^2 \beta^2 = \frac{1}{16\pi G}\, \beta^2 = 4\pi G M^2$$

Conclusion

- Black holes violet the third law of thermodynamics
- Model of bose gas violeting the third law of thermodynamics is proposed
- Random quantum gas of thin shells reproducing the black hole entropy is proposed