Mathematical Physics, Dynamical Systems and Infinite Dimensional Analysis - 2023 (MPDSIDA)

Dedicated to Oleg G. Smolyanov

EFFORTS & FLOWS IN QUANTUM SYSTEMS

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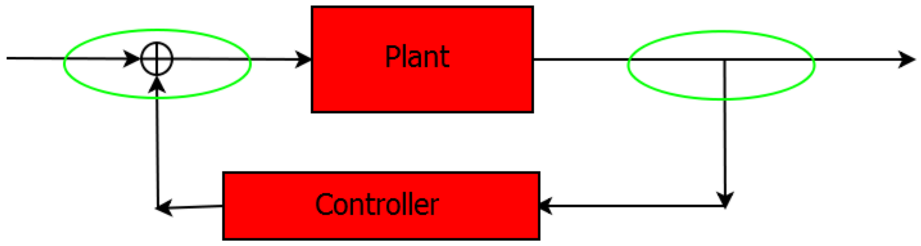
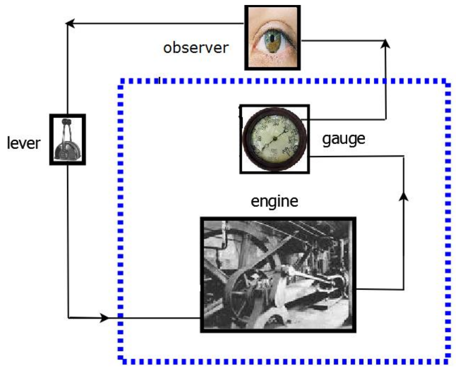
Networks and Feedback Control



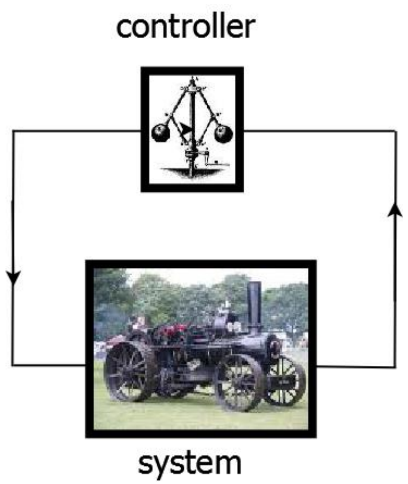
cannot happen in the quantum setting!!!

must use unitary junctions (e.g., beamsplitters)

- Measurement Based Feedback Control



- Coherent Feedback Control



Efforts & Flows

Joint work with P. Rouchon, N. Amini, B. Maschke, A. v.d. Schaft

- These are power-conjugate variables (e.g., current & voltage)
- Basic idea

$$\text{(Power)}\ \frac{dE}{dt} = f.e\ (\text{flow} \times \text{effort}).$$

- Central to classical control techniques such as **port-Hamiltonian systems**
- Efforts and flows can form a **Dirac structure**, in which case one may combine Dirac structures (interconnection!)
- Developed classically by van der Schaft, Maschke, etc.

Port Hamiltonian Systems

- Dynamical system

$$\dot{x} = (J - R)\nabla H + G\,u(t),$$

$$y = G^\top \nabla H + D\,u(t).$$

- Coefficients

$$J(x)^\top = -J(x), \quad R(x)^\top = R(x).$$

- Flows & efforts

$$f = -x, e = \nabla H \qquad f_R = \nabla H, e_R = -R\nabla H.$$

$$\begin{bmatrix} f \\ f_r \\ y \end{bmatrix} = \begin{bmatrix} -J & -I & -G \\ I & 0 & 0 \\ G^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ e_r \\ u \end{bmatrix},$$

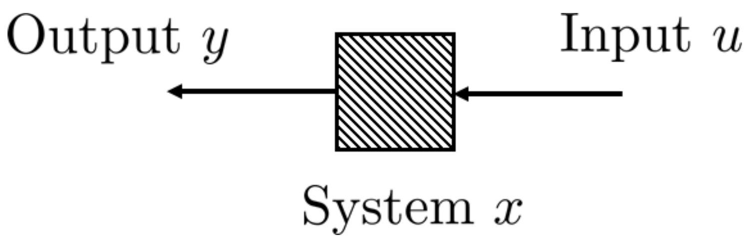
Power balance (Dirac Structure)

$$e^\top f + e_R^\top f_R + u^\top y = 0.$$

Linear Systems

A model is said to be **linear** if its state dynamics and input-output equations are then assumed to take the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$



The linear system is by the **model matrix**:

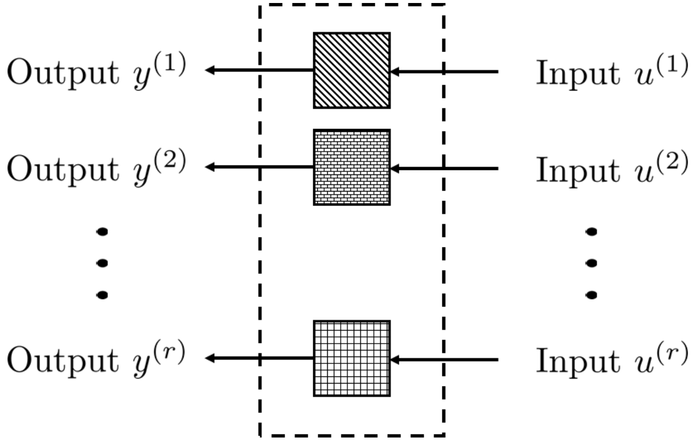
$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X} \oplus \mathcal{U} \mapsto \mathcal{X} \oplus \mathcal{Y},$$

written in block partition form.

Linear Systems

Given a pair of models $M_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$ on $(\mathcal{Y}_i, \mathcal{X}, \mathcal{U}_i)$, for $i = 1, 2$, their **superimposition** is the model on $(\mathcal{Y}_1 \oplus \mathcal{Y}_2, \mathcal{X}, \mathcal{U}_1 \oplus \mathcal{U}_2)$ given by

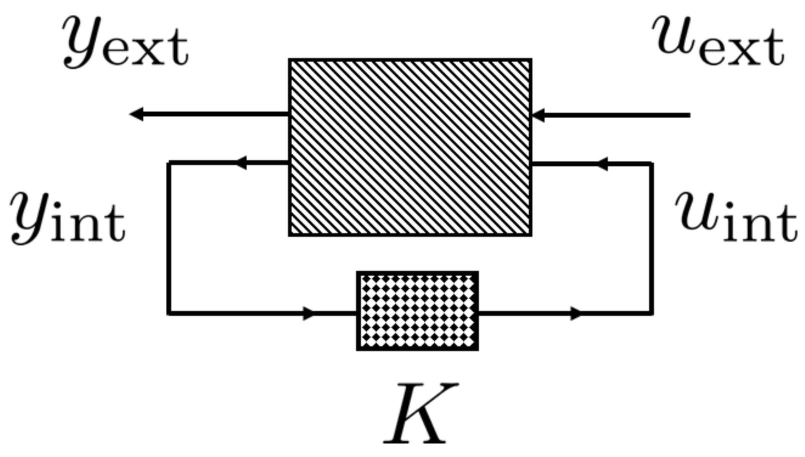
$$M_1 \boxplus M_2 = \begin{bmatrix} A_1 + A_2 & B_1 & B_2 \\ C_1 & D_1 & 0 \\ C_2 & 0 & D_2 \end{bmatrix}.$$



Linear Systems

We begin with an open-loop model on $(\mathcal{Y}_{\text{ext}} \oplus \mathcal{Y}_{\text{int}}, \mathcal{X}, \mathcal{U}_{\text{ext}} \oplus \mathcal{U}_{\text{int}})$ with

$$M = \begin{bmatrix} A & B_{\text{e}} & B_{\text{i}} \\ C_{\text{e}} & D_{\text{ee}} & D_{\text{ei}} \\ C_{\text{i}} & D_{\text{ie}} & D_{\text{ii}} \end{bmatrix}$$



To close the internal loop, we set $y_{\text{i}} = K\,u_{\text{i}}$,

This leads to the **feedback reduction**

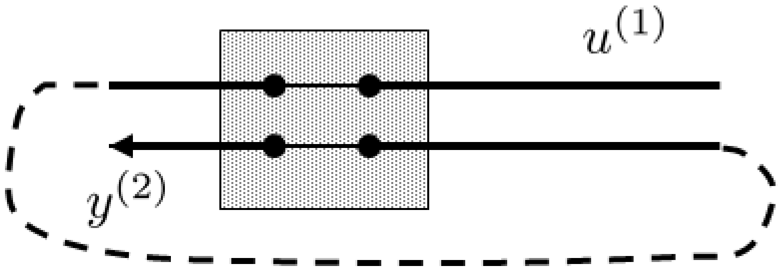
$$M_{\text{fb}}(K) = \begin{bmatrix} A & B_{\text{e}} \\ C_{\text{e}} & D_{\text{ee}} \end{bmatrix} + \begin{bmatrix} B_{\text{i}} \\ D_{\text{ei}} \end{bmatrix} (I - KD_{\text{ii}})^{-1} K \begin{bmatrix} C_{\text{i}} & D_{\text{ie}} \end{bmatrix}$$

Well-posedness requires that $(I - KD_{\text{ii}})$ is invertible.

Linear Systems

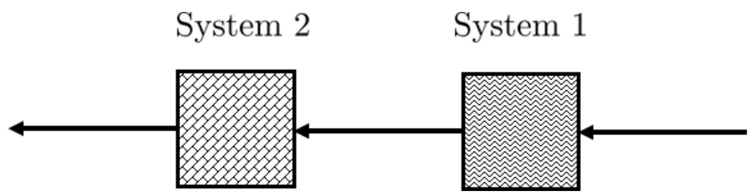
Open loop superimposition formula

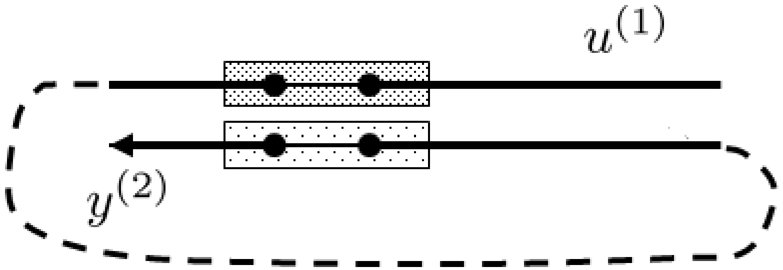
$$M_1 \boxplus M_2 = \begin{bmatrix} A_1 + A_2 & B_1 & B_2 \\ C_1 & D_1 & 0 \\ C_2 & 0 & D_2 \end{bmatrix},$$



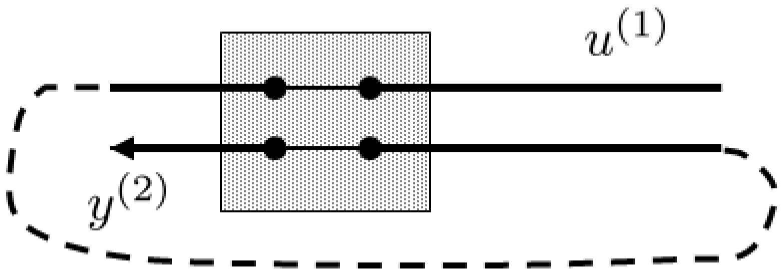
we do not assume a decomposition of the state space into two components!

Note: a cascade is a special case.





Linear Systems



Closed Loop feedback connection $u_2 = y_1$

$$\begin{aligned} M_2 \lhd M_1 &\triangleq (M_1 \boxplus M_2)_{\mathrm{fb}} \\ &= \begin{bmatrix} A_1 + A_2 & B_1 \\ C_2 & 0 \end{bmatrix} + \begin{bmatrix} B_2 \\ D_2 \end{bmatrix} (I - 0)^{-1} \begin{bmatrix} C_1 & D_1 \end{bmatrix} \\ &= \begin{bmatrix} A_1 + A_2 + B_2 C_1 & B_1 + B_2 D_1 \\ C_2 + D_2 C_1 & D_2 D_1 \end{bmatrix}. \end{aligned}$$

We shall refer to $M_2 \lhd M_1$ as the **series product** of models M_2 and M_1.

Link to the (extended) Heisenberg group!

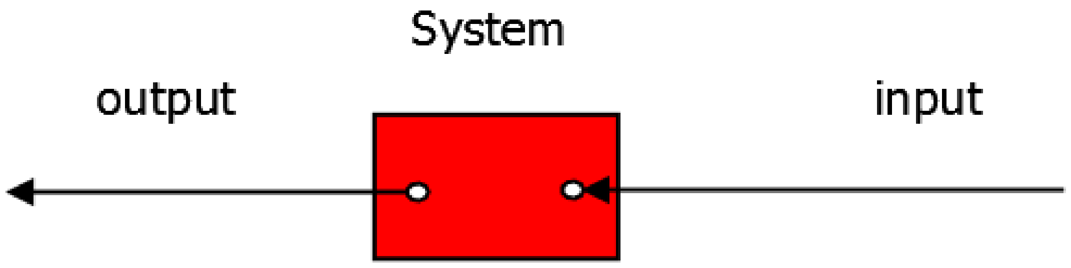
If $\mathbf{V}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \triangleq \begin{bmatrix} I & B & A \\ 0 & D & C \\ 0 & 0 & I \end{bmatrix}$, then $\mathbf{V}(M_2 \lhd M_1) \equiv \mathbf{V}(M_2)\,\mathbf{V}(M_1)$.

Quantum Input-Output Systems

Hudson, Parthasarathy (1984)

V.P. Belavkin (1979+)

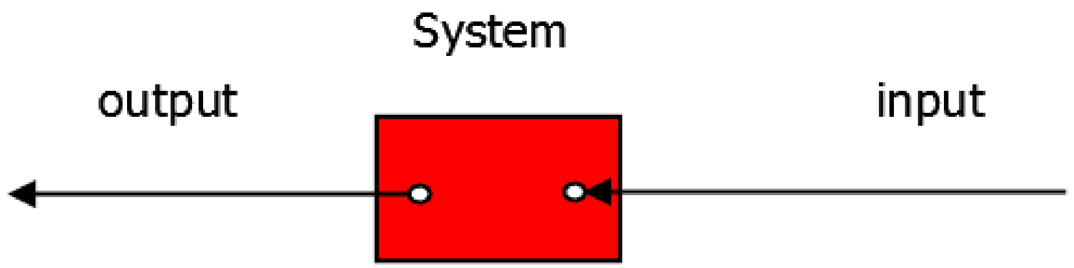
Gardiner, Collett (1985)



Field quanta of type *k* annihilated at the system at time *t* $b_{\mathrm{in},k}(t)$

$$[b_{\mathrm{in},j}(t), b_{\mathrm{in},k}(s)^*] = \delta_{jk}\,\delta(t-s).$$

Quantum Ito Table



- Fundamental Processes

$$B^*_{\mathrm{in},k}(t) = \int_0^t b_{\mathrm{in},k}(s)^* ds, \qquad B_{\mathrm{in},k}(t) = \int_0^t b_{\mathrm{in},k}(s) ds, \quad \Lambda_{\mathrm{in},jk}(t) \equiv \int_0^t b_j(s)^* b_k(s) ds$$

- Table

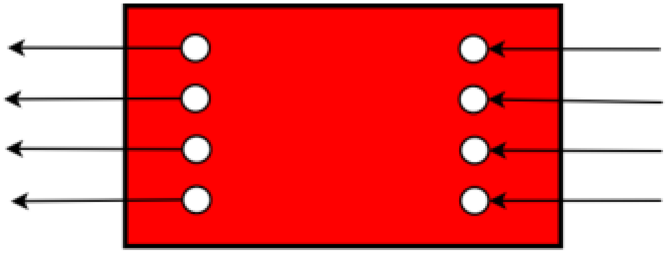
$$dB_j dB^*_k = \delta_{jk} dt$$

$$\begin{aligned} d\Lambda_{jl} dB^*_k &= \delta_{lk} dB^*_j \\ dB_j d\Lambda_{kl} &= \delta_{jk} dB_l \\ d\Lambda_{jl} d\Lambda_{ki} &= \delta_{lk} d\Lambda_{ji} \end{aligned}$$

- Product Rule

$$d(XY) = dX(t)\,Y(t) + X(t)\,dY(t) + dX(t)\,dY(t).$$

Quantum Stochastic Models



- General (*S* ,*L* , *H*) case (Hudson & Parthasarathy)

$$dU(t) = \Big\{(S_{jk} - \delta_{jk} I) \otimes d\Lambda_{jk}(t) + L_j \otimes dB_j^*(t) - L_j^* S_{jk} \otimes dB_k(t) - (\frac{1}{2} L_k^* L_k + iH) \otimes dt \Big\} U(t)$$

$$H^* = H \qquad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix} \qquad S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \qquad S^{-1} = S^*$$

Hamiltonian

Coupling/ Collapse Dissipators

Scattering Operators

Heisenberg-Langevin Dynamics

i.e., a Hudson-Evans Flow!

$$j_t(X) = U(t)^* \{ X \otimes I \} U(t)$$

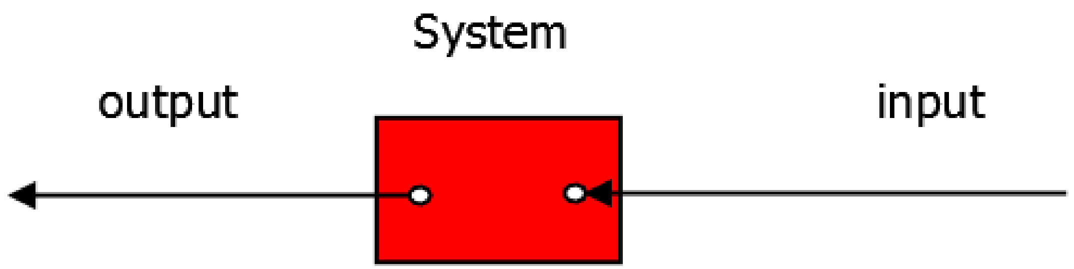
Heisenberg Equations of Motion

$$\begin{aligned} dj_t(X) \quad = \quad & j_t(S_{lj}^* X S_{lk} - \delta_{jk} X) \otimes d\Lambda_{\mathrm{in},jk}(t) + j_t(S_{lj}^*[L_l, X]) \otimes dB_{\mathrm{in},j}^*(t) \\ & + j_t([X, L_l^*] S_{lk}) \otimes dB_{\mathrm{in},k(t)} + j_t(\mathscr{L} X) \otimes dt. \end{aligned}$$

Lindblad Generator

$$\mathscr{L} X = \frac{1}{2} L_k^* [X, L_k] + \frac{1}{2} [L_k^*, X] L_k - i[X, H]$$

Quantum Output Process



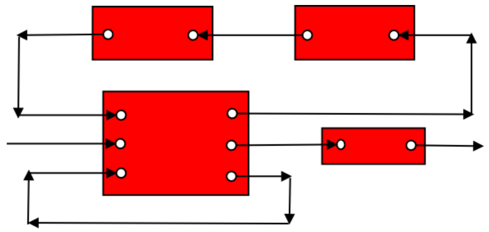
$$B_{\mathrm{out},k}(t) = U(t)^* \{ I \otimes B_{\mathrm{in},k}(t) \} U(t)$$

Input-Output Relations

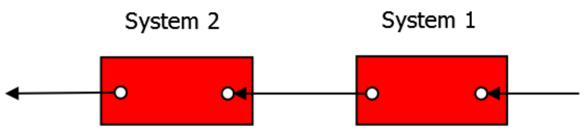
$$dB_{\mathrm{out},j}(t) = j_t(S_{jk}) \otimes dB_{\mathrm{in},k}(t) + j_t(L_j) \otimes dt$$

Quantum Networks

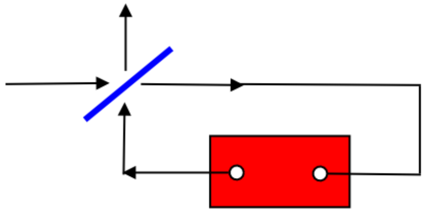
- How to connect models?



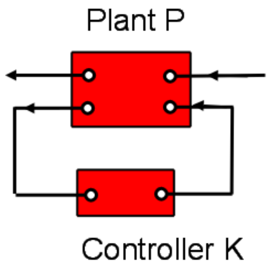
- Cascaded models



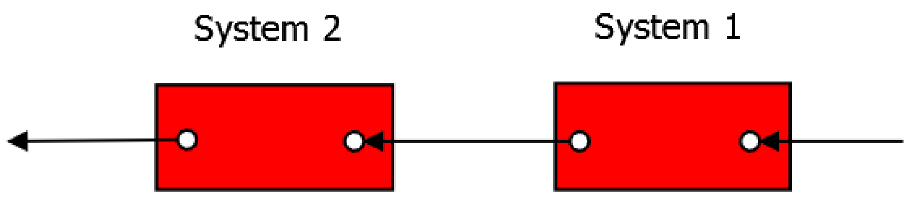
- Algebraic loops



- Feedback Control



The Series Product



The cascaded system in the **instantaneous feedforward** limit is equivalent to the single component

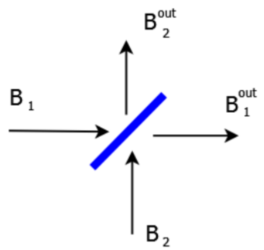
$$(S_2, L_2, H_2) \triangleleft (S_1, L_1, H_1) = \left(S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \mathrm{Im}\left\{ L_2^\dagger S_2 L_1 \right\} \right).$$

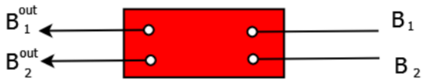
J. G., M.R. James, *The Series Product and Its Application to Quantum Feedforward and Feedback Networks* IEEE Transactions on Automatic Control, 2009.

This is the ***quantum*** extended Heisenberg group again!

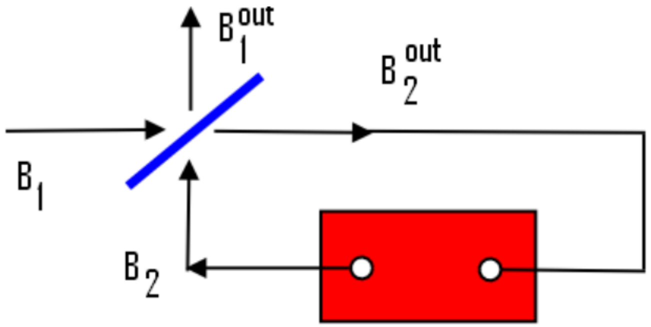
Beam-splitters

$$\begin{bmatrix} B_1^{\text{out}} \\ B_2^{\text{out}} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$





beamsplitter $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$,
and *in-loop component* $(S_0, L_0, 0)$:



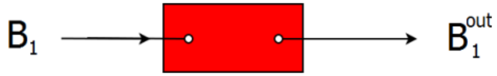
$$dB_2 = S_0 dB_2^{\text{out}} + L_0 dt = S_0(S_{21} dB_1 + S_{22} dB_2) + L_0 dt$$

$$\Rightarrow dB_1^{\text{out}} = S_{11} dB_1 + S_{12} dB_2 \equiv \hat{S}_0 dB_1 + \hat{L}_0 dt$$

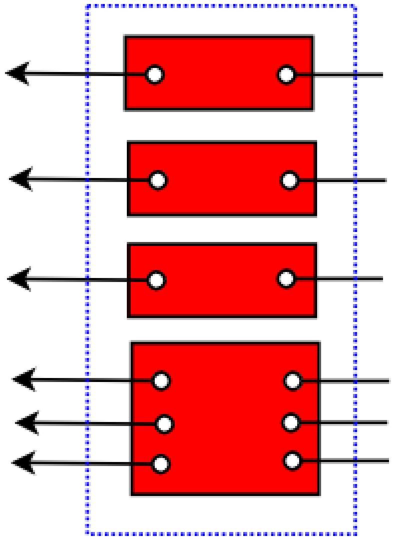
where

$$\hat{S}_0 = S_{11} + S_{12}(I - S_0 S_{22})^{-1} S_0 S_{21}, \quad \hat{L}_0 = S_{12}(I - S_{22})^{-1} S_0 L_0.$$

Equivalent component $(\hat{S}_0, \hat{L}_0, \hat{H}_0)$:



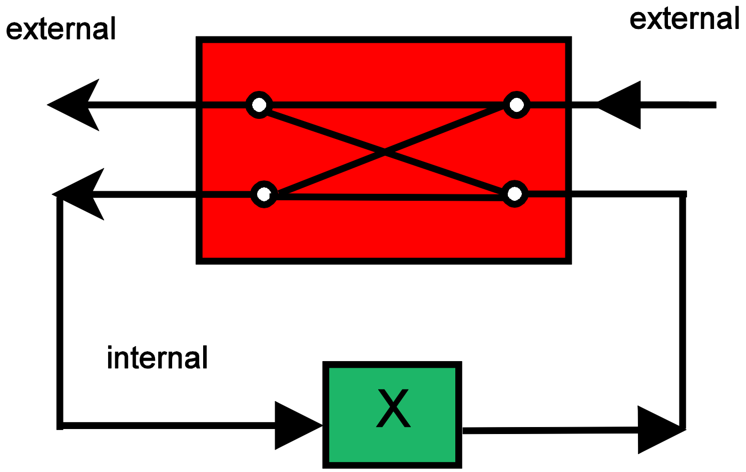
Network Rule # 1
Open loop systems in parallel



Models $(S_j, L_j, H_j)_{j=1}^{n}$ in parallel

$$\left(\begin{bmatrix} S_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_n \end{bmatrix}, \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, H_1 + \cdots + H_n \right).$$

Network Rule # 2
Feedback Reduction Formula



$$S = \begin{bmatrix} \mathsf{S}_{\mathtt{ii}} & \mathsf{S}_{\mathtt{ie}} \\ \mathsf{S}_{\mathtt{ei}} & \mathsf{S}_{\mathtt{ee}} \end{bmatrix}, \mathsf{L} = \begin{bmatrix} \mathsf{L}_{\mathtt{i}} \\ \mathsf{L}_{\mathtt{e}} \end{bmatrix}$$

The reduced model obtained by eliminating all the internal channels (instantaneous feedback) is determined by the operators $(\mathsf{S}^{\mathrm{fb}}, \mathsf{L}^{\mathrm{fb}}, \mathsf{H}^{\mathrm{fb}})$ given by

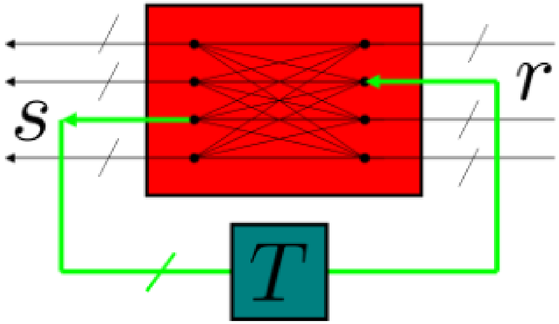
$$\begin{aligned} \mathsf{S}^{\mathrm{fb}} &= \mathsf{S}_{\mathtt{ee}} + \mathsf{S}_{\mathtt{ei}} X \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{S}_{\mathtt{ie}}, \\ \mathsf{L}^{\mathrm{fb}} &= \mathsf{L}_{\mathtt{e}} + \mathsf{S}_{\mathtt{ei}} X \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{L}_{\mathtt{i}}, \\ \mathsf{H}^{\mathrm{fb}} &= \mathsf{H} + \sum_{i=\mathtt{i},\mathtt{e}} \mathrm{Im} \mathsf{L}_j^{\dagger} X \mathsf{S}_{j\mathtt{i}} \left(1 - \mathsf{S}_{\mathtt{ii}} X\right)^{-1} \mathsf{L}_{\mathtt{i}}. \end{aligned}$$

J. G., M.R. James, *Quantum Feedback Networks: Hamiltonian Formulation*, Commun. Math. Phys., 1109-1132, Volume 287, Number 3 / May, 2009.

The rules are very similar to classical linear systems

The quantum **model matrix** for $G \sim (S, L, H)$:

$$\mathsf{V} = \begin{bmatrix} -\frac{1}{2}L^* L - iH & -L^* S \\ L & S \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sum_j L_j^* L_j - iH & -\sum_j L_j^* S_{j1} & \cdots & -\sum_j L_j^* S_{jm} \\ L_1 & S_{11} & \cdots & S_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ L_n & S_{n1} & \cdots & S_{nn} \end{bmatrix} = \begin{bmatrix} \mathsf{V}_{00} & \mathsf{V}_{01} & \cdots & \mathsf{V}_{0m} \\ \mathsf{V}_{10} & \mathsf{V}_{11} & \cdots & \mathsf{V}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{V}_{n0} & \mathsf{V}_{n1} & \cdots & \mathsf{V}_{nn} \end{bmatrix}.$$



The feedback reduction yields the model matrix

$$\left[\mathscr{F}_{(r,s)}(\mathsf{V}, T)\right]_{\alpha\beta} = \mathsf{V}_{\alpha\beta} + \mathsf{V}_{\alpha r} T \left(1 - \mathsf{V}_{rs} T\right)^{-1} \mathsf{V}_{s\beta}$$

for $\alpha \neq r$ and $\beta \neq s$.

Classical Hamiltonian Systems

- Closed Hamiltonian with external inputs (efforts)

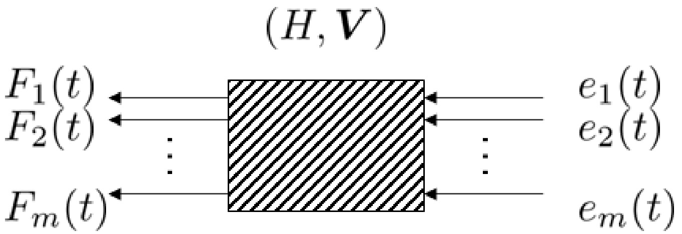
$$\Upsilon(t) = H + \sum_k V_k \, e_k(t)$$

- Power delivered to the system

$$\frac{d}{dt} j_t(H) = \sum_k F_k(t) \; e_k(t)$$

- Flow variables

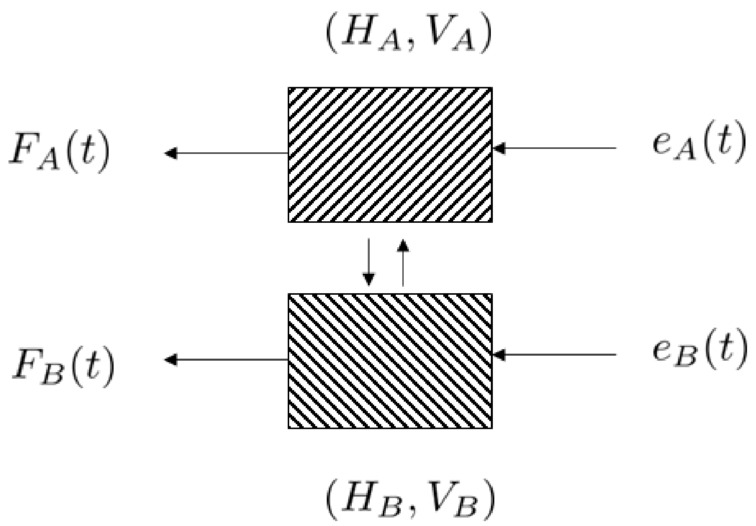
$$F_k(t) = j_t\left(\{H, V_k\}\right).$$



Coupled Hamiltonian Systems

- Total Hamiltonian

$$\Upsilon(t) = H_A + H_B + e_A(t)V_A + e_B(t)V_B + V_{AB}$$



- Power relations

$$\begin{aligned} \frac{d}{dt} j_t(H_A) &= F_A(t)\, e_A(t) + P_{B\to A}(t), \\ \frac{d}{dt} j_t(H_B) &= F_B(t)\, e_B(t) + P_{A\to B}(t), \end{aligned}$$

Classical Hamiltonian Systems with Noise

- Stratonovich form

$$dj_t(A) = j_t\big(\{A, H\}\big)\, dt + \sum_k j_t\big(\{A, V_k\}\big) \circ \Big(e_k(t)dt + dW_k(t)\Big).$$

- Ito form

$$dj_t(A) = j_t\big(\mathcal{L}(A)\big)\, dt + \sum_k j_t\big(\{A, V_k\}\big)\Big(e_k(t)dt + dW_k(t)\Big)$$

where we have the second-order differential operator

$$\mathcal{L}(A) \equiv \{A, H\} + \frac{1}{2}\sum_k \{\{A, V_k\}\, V_k\}.$$

Classical Hamiltonian Systems with Noise

- The power is now

$$dj_t(H) = j_t(G)\, dt + \sum_k F_k(t)\Big(e_k(t)dt + dW_k(t) \Big)$$

- Here there is a background power delivered even when no signal is present

$$G = \mathcal{L}(H) = \tfrac{1}{2} \sum_k \{\{H, V_k\}\, V_k\}$$

Symplectic Structure and Quantum Mechanics

- Holomorphic coordinates

The classical phase space $\mathbb{R}^{2n}$ may be equivalently modelled as $\mathbb{C}^n$ where we introduce complex vectors $\boldsymbol{\beta} = \frac{1}{\sqrt{2}}(\mathbf{q} + i\mathbf{p})$.

- Skew form = symplectic area

$$\beta \wedge \beta' \triangleq 2\mathrm{Im}\left(\beta^* \beta'\right) = \frac{1}{i}\left(\beta^* \beta' - \beta'^* \beta\right) = qp' - pq'.$$

For several dimensions, we just define $\boldsymbol{\beta} \wedge \boldsymbol{\beta}' = \sum_k \beta_k \wedge \beta'_k$.

Power Variables for Closed Quantum Systems

- Time-dependent Hamiltonian

$$\Upsilon(t) = H + \boldsymbol{L} \wedge \boldsymbol{\beta}(t)$$

- Coupling to external inputs (efforts)

$$\boldsymbol{L} \wedge \boldsymbol{\beta}(t) \equiv \frac{1}{i}\left(\mathbf{L}^{*}\boldsymbol{\beta}(t) - \boldsymbol{\beta}(t)^{*}\mathbf{L}\right) = \frac{1}{i}\sum_{k}\left(L_{k}^{*}\beta_{k}(t) - \beta_{k}(t)^{*}L_{k}\right).$$

- Power

$$\frac{d}{dt}j_{t}(H) = -j_{t}([H, \mathbf{L}^{*}])\,\boldsymbol{\beta}(t) + \boldsymbol{\beta}(t)^{*}j_{t}([H, \mathbf{L}]) \equiv \boldsymbol{F}(t) \wedge \boldsymbol{\beta}(t)$$

- Flows

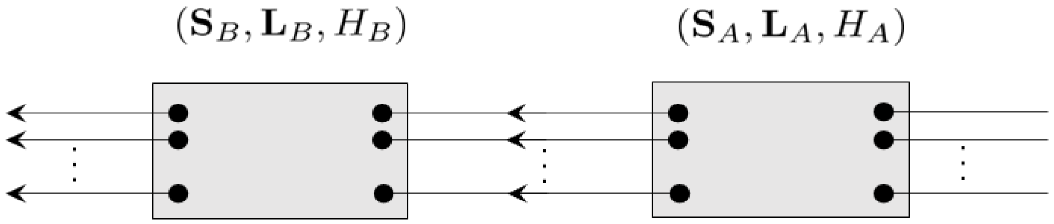
$$F_{k}(t) = \frac{1}{i}\,j_{t}\left([H, L_{k}]\right).$$

Open Quantum Systems?

- Efforts a quantum semimartingales!

$$dB(t) \to dE(t) = dB(t) + \beta(t)\,dt$$

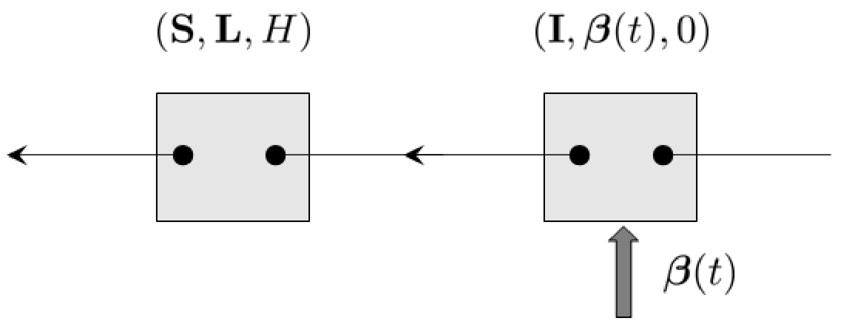
- Recall the Series Product



$$\begin{aligned}(\mathbf{S}, \mathbf{L}, H) &= (\mathbf{S}_B, \mathbf{L}_B, H_B) \triangleleft (\mathbf{S}_A, \mathbf{L}_A, H_A) \\ &\triangleq \left(\mathbf{S}_B \mathbf{S}_A, \mathbf{L}_B + \mathbf{S}_B \mathbf{L}_A, H_A + H_B + \frac{1}{2} \mathbf{L}_B \wedge (\mathbf{S}_B \mathbf{L}_A) \right).\end{aligned}$$

Adding in the signal to the noise

- Weyl Box



- Driven component

$$(\mathbf{S}, \mathbf{L}, H) \triangleleft (\mathbf{I}, \boldsymbol{\beta}(t), 0) = \left(\mathbf{S}, \mathbf{L} + \mathbf{S}\boldsymbol{\beta}(t), H + \tfrac{1}{2}\mathbf{L} \wedge \left(\mathbf{S}\boldsymbol{\beta}(t)\right) \right).$$

The Quantum Power Balance

- The power delivered

$$dj_t(H) = G(t)\,dt + \mathbf{F}(t) \wedge d\mathbf{E}(t) + \sum_{jk} j_t\Big(\sum_{l} S_{lj}^{*} H S_{lk} - \delta_{jk} H \Big) \otimes dN_{jk}(t),$$

- Vacuum power

$$G(t) = j_t\big(\mathcal{L}_{(\mathbf{L},H)}(H)\big) = \tfrac{1}{2} j_t\Big([\mathbf{L}^{*}, H]\mathbf{L} + \mathbf{L}^{*}[H, \mathbf{L}] \Big),$$

- Flow variables

$$\mathbf{F}(t) = \tfrac{1}{i} j_t\big(\mathbf{S}^{*}[H, \mathbf{L}]\big),$$

- Scattering terms

$$dN_{jk}(t) = d\Lambda_{jk}(t) + dB_j(t)^{*}\, \beta_k(t) + \beta_j(t)^{*} dB_k(t) + \beta_j(t)^{*} \beta_k(t)\, dt.$$

Спасибо!