Blow-up of solutions of nonlocal parabolic equation under nonlocal boundary condition

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We consider the initial boundary value problem for nonlinear nonlocal parabolic equation

$$u_t = \Delta u + au^p \int_\Omega u^q(y,t)\,dy - bu^m,\ x \in \Omega,\ t > 0, \tag{1}$$

with nonlinear nonlocal boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} = \int_\Omega k(x,y,t)u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0, \tag{2}$$

and initial datum

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{3}$$

where a, b, p, q, m, l are positive numbers, Ω is a bounded domain in R^N for $N \geq 1$ with smooth boundary $\partial\Omega$, ν is unit outward normal on $\partial\Omega$.

We suppose that the functions $k(x,y,t)$ and $u_0(x)$ satisfy the following conditions:

$$k(x,y,t) \in C(\partial\Omega \times \overline{\Omega} \times [0,+\infty)),\ k(x,y,t) \geq 0;$$

$$u_0(x) \in C^1(\overline{\Omega}),\ u_0(x) \geq 0 \text{ in } \Omega,\ \frac{\partial u_0(x)}{\partial \nu} = \int_\Omega k(x,y,0)u_0^l(y)\,dy \text{ on } \partial\Omega.$$

Initial boundary value problem for parabolic equation (1) with nonlocal boundary condition

$$u(x,t) = \int_\Omega k(x,y,t)u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0$$

was considered in [1; 2].

Let $Q_T = \Omega \times (0,T),\ S_T = \partial\Omega \times (0,T)$, $\Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, $T > 0$.

Definition. We say that a nonnegative function $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a supersolution of (1)–(3) in Q_T if

$$u_t \geq \Delta u + au^p \int_\Omega u^q(y,t)\,dy - bu^m,\ (x,t) \in Q_T, \tag{4}$$

$$\frac{\partial u(x,t)}{\partial \nu} \geq \int_\Omega k(x,y,t)u^l(y,t)\,dy,\ (x,t) \in S_T, \tag{5}$$

$$u(x,0) \geq u_0(x),\ x \in \Omega, \tag{6}$$

and $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a subsolution of (1)–(3) in Q_T if $u \geq 0$ and it satisfies (4)–(6) in the reverse order. We say that $u(x,t)$ is a solution of problem (1)–(3) in Q_T if $u(x,t)$ is both a subsolution and a supersolution of (1)–(3) in Q_T.

Theorem 1. *Let* $\overline{u}$ *and* $\underline{u}$ *be a supersolution and a subsolution of problem (1)–(3) in* Q_T, *respectively. Suppose that* $\underline{u}(x,t) > 0$ *or* $\overline{u}(x,t) > 0$ *in* $Q_T \cup \Gamma_T$ *if* $\min(p,q,l) < 1$. *Then* $\overline{u}(x,t) \geq \underline{u}(x,t)$ *in* $Q_T \cup \Gamma_T$.

Theorem 2. *Let* $\max(p+q,l) \leq 1$ *or* $1 < \max(p+q,l) < m$. *Then every solution of (1)–(3) is global.*

To formulate finite time blow-up result we suppose that

$$\inf_\Omega \int_{\partial\Omega} k(x,y,0)\,dS_x > 0. \tag{7}$$

Theorem 3. *Let either* $r+p > \max(m,1)$ *or* $l > \max(m,1)$ *and (7) hold. Then solutions of (1)–(3) blow up in finite time if initial data are large enough.*

Remark. We improve comparison principle, global existence and blow-up results in [4].

The results of the talk have been published in [3].

References

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