Navier-Stokes equations in algebraic approach.

Victor Zharinov

Steklov Mathematical Institute, MIAN

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Navier-Stokes equations

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where
$\mathbf{u} = (u^1, u^2, u^3)$ is the velocity field,
t is the time variable, $\mathbf{x} = (x^1, x^2, x^3)$ is the space variable,
the dot "$\cdot$" stands for the scalar product,
$\nabla = (\nabla_1, \nabla_2, \nabla_3) = (\partial_{x^1}, \partial_{x^2}, \partial_{x^3})$ is the gradient,
the parameter $\nu > 0$ is the viscosity of the flow,
$\Delta = \nabla_1^2 + \nabla_2^2 + \nabla_3^2$ is the Laplacian,
p is the pressure.

Here we study the Navier-Stokes equations from the algebra-geometrical point of view. The system (1)-(2) is not formally integrable, to get the equivalent formally integrable system one needs to add the trivial differential prolongations

$$\nabla \cdot \mathbf{u}_t = 0, \quad \nabla(\nabla \cdot \mathbf{u}) = 0, \tag{3}$$

and the non-trivial differential prolongation (*hidden integrability condition*)

$$\Delta p + \nabla((\mathbf{u} \cdot \nabla)\mathbf{u}) = \Delta p + \nabla \mathbf{u} \cdot \nabla \mathbf{u} = 0 \tag{4}$$

(we have used the equation (2)).

Remark

The equation (4) is a Poisson equation for the pressure p with the density $\rho = -\nabla \mathbf{u} \cdot \nabla \mathbf{u}$, it can be considered as the *inner constraint* for the Navier-Stokes equations.

The base space

$$\mathbf{B} = \mathrm{T} \times \mathrm{X} \times \mathbb{R}^{\mathrm{M}}_{\mathbb{I}} \times \mathbb{R}_{\mathbb{I}},$$

where

- $\mathrm{M} = \overline{1, m}, \ , m = 2, 3, \dots,$
  $\mathbb{I} = \{\mathrm{i} = (i^\mu) \mid i^\mu \in \mathbb{Z}_+, \ \mu \in \mathrm{M}\} = \mathbb{Z}^{\mathrm{M}}_+;$
  $\partial_{\mathrm{x}^{\mathrm{i}}} = (\partial_{x^1})^{i^1} \circ \dots \circ (\partial_{x^m})^{i^m}, \ \mathrm{i} = (i^1, \dots, i^m) \in \mathbb{I};$

- $\mathrm{T} = \{t \in \mathbb{R}\} = \mathbb{R}$ is the time variable;
  $\mathrm{X} = \{\mathrm{x} = (x^\mu) \mid x^\mu \in \mathbb{R}, \ \mu \in \mathrm{M}\} = \mathbb{R}^{\mathrm{M}}$ is the space variable;

- $\mathbb{R}^{\mathrm{M}}_{\mathbb{I}} = \{\mathbf{u} = (u^\mu_{\mathrm{i}}) \mid u^\mu_{\mathrm{i}} \in \mathbb{R}, \ \mu \in \mathrm{M}, \ \mathrm{i} \in \mathbb{I}\}$ is the velocity and its partial space derivatives, $u^\mu_{\mathrm{i}} = \partial_{x^{\mathrm{i}}} u^\mu$;
  $\mathbb{R}_{\mathbb{I}} = \{\mathbf{p} = (p_{\mathrm{i}}) \mid p_i \in \mathbb{R}, \ \mathrm{i} \in \mathbb{I}\}$ is the pressure and its partial space variables, $p_{\mathrm{i}} = \partial_{\mathrm{x}^{\mathrm{i}}} p$.

The base algebra

The base algebra is the unital commutative associative algebra

$$\mathcal{A}(\mathbf{B}) = \mathcal{C}^{\infty}_{\text{fin}}(\mathbf{B})$$

of all smooth real functions on the base space **B** *of a finite order*, i.e., depending on a finite number of the variables $x^\mu, u^\mu_{\mathrm{i}}, p_{\mathrm{i}}$, $\mu \in \mathrm{M}$, $\mathrm{i} \in \mathbb{I}$.

In more detail, the integer $r \in \mathbb{Z}_+$ is called the **u**-*order* of a function $f(x, \mathbf{u}, \mathbf{p}) \in \mathcal{A}(\mathbf{B})$, we write $\mathrm{ord}_{\mathbf{u}} f = r$, if the partial derivative $\partial_{u^\mu_{\mathrm{i}}} f \neq 0$ for some variable u^μ_{i}, $|\mathrm{i}| = r$, while partial derivatives $\partial_{u^\mu_{\mathrm{i}}} f = 0$ for all $|\mathrm{i}| > r$. In the same way, the **p**-order is defined.

Here and below,

$$\mathbb{I} \ni \mathrm{i} = (i^1, \dots, i^m) \quad \Longrightarrow \quad |\mathrm{i}| = i^1 + \cdots + i^m \in \mathbb{Z}_+.$$

Derivations

- $\mathfrak{D}(\mathbf{B}) = \left\{\zeta = \zeta^\mu \partial_{x^\mu} + \zeta^\mu_{\mathrm{i}} \partial_{u^\mu_{\mathrm{i}}} + \zeta_{\mathrm{i}} \partial_{p_{\mathrm{i}}} \;\middle|\; \zeta^\mu, \zeta^\mu_{\mathrm{i}}, \zeta_{\mathrm{i}} \in \mathcal{A}(\mathbf{B})\right\}$;
- $\mathfrak{D}(\mathbf{B}) = \mathfrak{D}_{\mathrm{V}}(\mathbf{B}) \oplus_{\mathcal{A}(\mathbf{B})} \mathfrak{D}_{\mathrm{H}}(\mathbf{B})$,
  $\mathfrak{D}_{\mathrm{V}}(\mathbf{B}) = \{\zeta \in \mathfrak{D}(\mathbf{B}) \mid \zeta|_{\mathcal{C}^\infty(\mathrm{X})} = 0\} = \{\zeta = \zeta^\mu_{\mathrm{i}} \partial_{u^\mu_{\mathrm{i}}} + \zeta_{\mathrm{i}} \partial_{p_{\mathrm{i}}}\}$;
  $\mathfrak{D}_{\mathrm{H}}(\mathbf{B}) = \{\zeta = \zeta^\mu D_\mu \mid \zeta^\mu \in \mathcal{A}(\mathbf{B})\}$,
  $D_\mu = \partial_{x^\mu} + u^\lambda_{\mathrm{i}+(\mu)} \partial_{u^\lambda_{\mathrm{i}}} + p_{\mathrm{i}+(\mu)} \partial_{p_{\mathrm{i}}}, \;\; D_\mu|_{\mathcal{C}^\infty(\mathrm{X})} = \partial_{x^\mu}, \;\; [D_\lambda, D_\mu] = 0.$

Here and below,

- the summation over repeated upper and lower indices in the prescribed limits is assumed,
- $\mathrm{i} + (\mu) = (i^1, \ldots, i^\mu + 1, \ldots, i^m), \quad \mathrm{i} \in \mathbb{I}, \;\; \mu \in \mathrm{M}.$

The pair $(\mathcal{A}(\mathbf{B}), \mathfrak{D}_{\mathrm{H}}(\mathbf{B}))$ is called the *differential algebra* associated with the base space **B**.

Symmetries

The Lie algebra

$$\mathrm{Sym}(\mathcal{A}(\mathbf{B}), \mathfrak{D}_{\mathrm{H}}(\mathbf{B})) = \left\{\zeta = \mathrm{ev}_{\mathrm{f}} \in \mathfrak{D}_{\mathrm{V}}(\mathbf{B}) \;\middle|\; [D_\mu, \mathrm{ev}_{\mathrm{f}}] = 0,\ \mu \in \mathrm{M}\right\}$$

is the *Lie algebra of symmetries* of the differential algebra $(\mathcal{A}(\mathbf{B}), \mathfrak{D}_{\mathrm{H}}(\mathbf{B}))$, where

- $\mathrm{f} = (f^\mu, f) \in \mathcal{A}^{\mathrm{M}}(\mathbf{B}) \times \mathcal{A}(\mathbf{B})$, $\quad f^\mu = \zeta_0^\mu$, $f = \zeta_0$,
- $\mathrm{ev}_{\mathrm{f}} = D_{\mathrm{i}} f^\mu \cdot \partial_{u_{\mathrm{i}}^\mu} + D_{\mathrm{i}} f \cdot \partial_{p_{\mathrm{i}}}$, $\quad D_{\mathrm{i}} f^\mu = \zeta_{\mathrm{i}}^\mu$, $D_{\mathrm{i}} f = \zeta_{\mathrm{i}}$.

Here and below,

$$D_{\mathrm{i}} = (D_1)^{i^1} \circ \ldots \circ (D_m)^{i^m}, \quad \mathrm{i} = (i^1, \ldots, i^m) \in \mathbb{I}.$$

Horizontal differential complex

Horizontal differential forms are

$$\Omega_{\mathrm{H}}^q(\mathbf{B}) = \begin{cases} 0, & q < 0, q > m; \\ \mathcal{A}(\mathbf{B}), & q = 0; \\ \mathrm{Hom}_{\mathcal{A}(\mathbf{B})}(\wedge^q \mathfrak{D}_{\mathrm{H}}(\mathbf{B}); \mathcal{A}(\mathbf{B})), & 1 \leq q \leq m. \end{cases}$$

$$\mathrm{Hom}_{\mathcal{A}(\mathbf{B})}(\wedge^q \mathfrak{D}_{\mathrm{H}}(\mathbf{B}); \mathcal{A}(\mathbf{B})) \\ = \{\omega_{\mathrm{H}}^q = \omega_{\mu_1 \dots \mu_q} \cdot dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \mid \omega_{\mu_1 \dots \mu_q} \in \mathcal{A}(\mathbf{B}), + \text{s-s}\},$$

where the abbreviation "+s-s" states that components $\omega_{\mu_1 \dots \mu_q}$ are *skew-symmetric in indices* $\mu_1, \dots, \mu_q \in \mathrm{M}$.

Horizontal differentials $d_{\mathrm{H}}^q : \Omega_{\mathrm{H}}^q(\mathbf{B}) \to \Omega_{\mathrm{H}}^{q+1}(bB)$, $d_{\mathrm{H}}^{q+1} \circ d_{\mathrm{H}}^q = 0$, $q \in \mathbb{Z}$,

$$d_{\mathrm{H}}^q = d_{\mathrm{H}}\big|_{\Omega_{\mathrm{H}}^q(\mathbf{B})} : \Omega_{\mathrm{H}}^q(\mathbf{B}) \to \Omega_{\mathrm{H}}^{q+1}(\mathbf{B}),$$

$$\omega_{\mu_1 \dots \mu_q} \cdot dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \mapsto D_{[\mu_0} \omega_{\mu_1 \dots \mu_q]} \cdot dx^{\mu_0} \wedge \dots \wedge dx^{\mu_q},$$

the brackets $[\dots]$ denote the skew-symmetrization in indices $\mu_0, \dots, \mu_q \in \mathrm{M}$.

Horizontal cohomologies

$$H^q_{\mathrm{H}}(\mathbf{B}) = \operatorname{Ker} d^q_{\mathrm{H}} / \operatorname{Im} d^{q-1}_{\mathrm{H}}, \quad q \in \mathbb{Z}.$$

Theorem (The main theorem of the formal calculus of variations)

The linear spaces

$$H^q_{\mathrm{H}}(\mathbf{B}) = \begin{cases} 0, & q < 0, q > m; \\ \mathbb{R}, & q = 0; \\ 0, & 1 \leq q \leq m-1; \\ \mathcal{H}(\mathbf{B}), & q = m; \end{cases}$$

The Helmholtz linear space

$$\mathcal{H}(\mathbf{B}) = \left\{ \chi = (\chi_\mu, \chi) \in \mathcal{A}_{\mathrm{M}}(\mathbf{B}) \times \mathcal{A}(\mathbf{B}) \;\middle|\; \chi_* = \chi^* \right\}.$$

The linear mappings

$$\chi_*, \chi^* : \mathcal{A}^{\mathrm{M}}(\mathbf{B}) \times \mathcal{A}(\mathbf{B}) \to \mathcal{A}_{\mathrm{M}}(\mathbf{B}) \times \mathcal{A}(\mathbf{B})$$

act by the rules:

$$\begin{aligned}
&\mathcal{A}^{\mathrm{M}}(\mathbf{B}) \times \mathcal{A}(\mathbf{B}) \ni \mathrm{f} = (f^\mu, f) \mapsto \chi_*\mathrm{f} = \mathrm{g} = (g_\mu, g) \in \mathcal{A}_{\mathrm{M}}(\mathbf{B}) \times \mathcal{A}(\mathbf{B}),\\
&g_\mu = \partial_{u^\nu_{\mathrm{i}}}\chi_\mu \cdot D_{\mathrm{i}} f^\nu + \partial_{p_{\mathrm{i}}}\chi_\mu \cdot D_{\mathrm{i}} f, \quad g = \partial_{u^\nu_{\mathrm{i}}}\chi \cdot D_{\mathrm{i}} f^\nu + \partial_{p_{\mathrm{i}}}\chi \cdot D_{\mathrm{i}} f;\\
&\mathcal{A}^{\mathrm{M}}(\mathbf{B}) \times \mathcal{A}(\mathbf{B}) \ni \mathrm{f} = (f^\mu, f) \mapsto \chi^*\mathrm{f} = \mathrm{g} = (g_\mu, g) \in \mathcal{A}_{\mathrm{M}}(\mathbf{B}) \times \mathcal{A}(\mathbf{B}),\\
&g_\mu = (-D)_{\mathrm{i}}\big(f^\nu \cdot \partial_{u^\mu_{\mathrm{i}}}\chi_\nu + f \cdot \partial_{u^\mu_{\mathrm{i}}}\chi\big), \quad g = (-D)_{\mathrm{i}}\big(f^\nu \cdot \partial_{p_{\mathrm{i}}}\chi_\nu + f \cdot \partial_{p_{\mathrm{i}}}\chi\big).
\end{aligned}$$

The isomorphism $\delta = (\delta_{u^\mu}, \delta_p) : H^m_{\mathrm{H}}(\mathbf{B}) \simeq \mathcal{H}(\mathbf{B})$ of linear spaces is defined by the variational derivatives:

$$\begin{aligned}
&\Omega^m_{\mathrm{H}}(\mathbf{B}) \ni \omega^m_{\mathrm{H}} = \omega \cdot d^m x \mapsto \chi = \delta\omega = (\delta_{u^\mu}\omega, \delta_p\omega),\\
&\delta_{u^\mu}\omega = (-D)_{\mathrm{i}}\partial_{u^\mu_{\mathrm{i}}}\omega, \quad \delta_p\omega = (-D)_{\mathrm{i}}\partial_{p_{\mathrm{i}}}\omega.
\end{aligned}$$

Constraints

The continuity equation (2), $\mathrm{CE} = \{\partial_{x^\mu} u^\mu = 0\}$, has the algebraic counterpart

$$\mathbf{CE} = \left\{(x, \mathbf{u}, \mathbf{p}) \in \mathbf{B} \;\middle|\; \mathrm{CE}_{\mathrm{i}} = u^\mu_{\mathrm{i}+(\mu)} = 0,\ \mathrm{i} \in \mathbb{I}\right\}.$$

The integrability condition (4), $\Delta p + \nabla \mathbf{u} \cdot \nabla \mathbf{u} = 0$, has the algebraic counterpart

$$\mathbf{PE} = \left\{(x, \mathbf{u}, \mathbf{p}) \in \mathbf{B} \;\middle|\; \mathrm{PE}_{\mathrm{i}} = \Delta p_{\mathrm{i}} + D_{\mathrm{i}}(u^\lambda_{(\mu)} u^\mu_{(\lambda)}) = 0,\ \mathrm{i} \in \mathbb{I}\right\}.$$

where

- $(\mu) = 0 + (\mu) = (0, \dots, 0, 1, 0, \dots, 0)$, 1 stands in μth place;
- $\Delta = \delta^{\lambda\mu} D_\lambda \circ D_\mu = \delta^{\lambda\mu} D_{(\lambda)+(\mu)} = \sum_\mu D_{2(\mu)}$,
- $D_{\mathrm{i}}(u^\lambda_{(\mu)} u^\mu_{(\lambda)}) = \sum_{\mathrm{k}+\mathrm{l}=\mathrm{i}} \binom{\mathrm{i}}{\mathrm{k}} u^\lambda_{\mathrm{k}+(\mu)} u^\mu_{\mathrm{l}+(\lambda)}$.

The subspace

$$\mathbf{CPE} = \mathbf{CE} \cap \mathbf{PE} = \mathrm{T} \times \mathrm{X} \times \mathbb{R}^1_{\mathbb{I}_0} \times \mathbb{R}^{\mathrm{N}}_{\mathbb{I}} \times \mathbb{R}_{\mathbb{I}_1}$$

has the global coordinates $(t, \mathrm{x}, \mathbf{u}, \mathbf{p}) = \{t, x^\mu, u^1_{\mathrm{i}_0}, u^\alpha_{\mathrm{i}}, p_{\mathrm{i}_1}\}$, the indices
$\mu \in \mathrm{M} = \overline{1, m}$,
$\mathrm{i}_0 \in \mathbb{I}_0 = \{\mathrm{i} \in \mathbb{I} \mid i^1 = 0\}$,
$\alpha \in \mathrm{N} = \overline{2, m}$, $\quad \mathrm{i} \in \mathbb{I}$,
$\mathrm{i}_1 \in \mathbb{I}_1 = \{\mathrm{i} \in \mathbb{I} \mid i^1 = 0, 1\}$.

The algebra $\mathcal{A}(\mathbf{CPE}) = \mathcal{C}^\infty_{\mathrm{fin}}(\mathrm{T} \times \mathrm{X} \times \mathbb{R}^1_{\mathbb{I}_0} \times \mathbb{R}^{\mathrm{N}}_{\mathbb{I}} \times \mathbb{R}_{\mathbb{I}_1})$.

Derivations

- $\mathfrak{D}_{\mathrm{V}}(\mathbf{CPE}) = \{\zeta = \zeta^1_{\mathrm{i}_0} \partial_{u^1_{\mathrm{i}_0}} + \zeta^\alpha_{\mathrm{i}} \partial_{u^\alpha_{\mathrm{i}}} + \zeta_{\mathrm{i}_1} \partial_{p_{\mathrm{i}_1}} \mid \zeta^1_{\mathrm{i}_0}, \zeta^\alpha_{\mathrm{i}}, \zeta_{\mathrm{i}_1} \in \mathcal{A}(\mathbf{CPE})\}$;
- $\mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}) = \{\zeta = \zeta^\mu D_\mu \mid \zeta^\mu \in \mathcal{A}(\mathbf{CPE})\}$;
- $D_\mu = \partial_{x^\mu} + u^1_{\mathrm{i}_0+(\mu)} \partial_{u^1_{\mathrm{i}_0}} + u^\alpha_{\mathrm{i}+(\mu)} \partial_{u^\alpha_{\mathrm{i}}} + p_{\mathrm{i}_1+(\mu)} \partial_{p_{\mathrm{i}_1}}, \quad \mu \in \mathrm{M}$,
  where $u^1_{\mathrm{i}_0+(1)} + u^\alpha_{\mathrm{i}_0+(\alpha)} = 0$, $\Delta p_{\mathrm{i}} + D_{\mathrm{i}}(u^\lambda_{(\mu)} u^\mu_{(\lambda)}) = 0$.

Differential algebra in the space **CPE**

The pair $(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}))$ is called the *differential algebra* associated with the constrained space **CPE**;

The Lie algebra

$$\mathrm{Sym}(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE})) = \{\mathrm{ev}_{\mathrm{f}} \in \mathfrak{D}_{\mathrm{V}}(\mathbf{B}) \mid [D_\mu, \mathrm{ev}_{\mathrm{f}}] = 0,\ \mu \in \mathrm{M}\}$$

is the *Lie algebra of symmetries* of the differential algebra $(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}))$, where

- $\mathrm{ev}_{\mathrm{f}} = D_{\mathrm{i}_0} f^1 \cdot \partial_{u^1_{\mathrm{i}_0}} + D_{\mathrm{i}} f^\alpha \cdot \partial_{u^\alpha_{\mathrm{i}}} + D_{\mathrm{i}_1} f \cdot \partial_{p_{\mathrm{i}_1}}$,
- $\mathrm{f} = (f^\mu, f) \in \mathcal{A}(\mathbf{CPE})^{\mathrm{M}} \times \mathcal{A}(\mathbf{CPE})$,
- $D_\mu f^\mu = 0, \quad \Delta f + \mathrm{ev}_{\mathrm{f}}\left(u^\lambda_{(\mu)} u^\mu_{(\lambda)}\right) = 0$,

Horizontal differential complex in the space **CPE**

Horizontal differential forms are

$$\Omega^q_{\mathrm{H}}(\mathbf{CPE}) = \begin{cases} 0, & q < 0, q > m; \\ \mathcal{A}(\mathbf{CPE}), & q = 0; \\ \mathrm{Hom}_{\mathcal{A}(\mathbf{CPE})}(\wedge^q \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}); \mathcal{A}(\mathbf{CPE})), & 1 \le q \le m; \end{cases}$$

$$\mathrm{Hom}_{\mathcal{A}(\mathbf{CPE})}(\wedge^q \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}); \mathcal{A}(\mathbf{CPE})) = \{\omega^q_{\mathrm{H}} = \omega_{\mu_1 \dots \mu_q} \cdot dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \mid \omega_{\mu_1 \dots \mu_q} \in \mathcal{A}(\mathbf{CPE})\}.$$

The differential $d^q_{\mathrm{H}} : \Omega^q_{\mathrm{H}}(\mathbf{CPE}) \to \Omega^{q+1}_{\mathrm{H}}(\mathbf{CPE})$, $d^{q+1}_{\mathrm{H}} \circ d^q_{\mathrm{H}} = 0$, where

$$\omega_{\mu_1 \dots \mu_q} \cdot dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \mapsto D_{[\mu_0} \omega_{\mu_1 \dots \mu_q]} \cdot dx^{\mu_0} \wedge \dots \wedge dx^{\mu_q}.$$

$$H^q_{\mathrm{H}}(\mathbf{CPE}) = \mathrm{Ker}\, d^q_{\mathrm{H}} / \mathrm{Im}\, d^{q-1}_{\mathrm{H}}, \quad q \in \mathbb{Z}.$$

Horizontal cohomologies in the space **CPE**

Theorem

The linear spaces of the cohomologies of the differential algebra $(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}))$ *are*

$$H^q_{\mathrm{H}}(\mathbf{CPE}) = \begin{cases} 0, & q < 0, 1 \leq q \leq m-2, q > m; \\ \mathbb{R}, & q = 0; \\ \operatorname{Ker} D_1^{m-1} \simeq \mathcal{S} \cap \mathcal{H}, & q = m-1; \\ H^{m-1}(\Theta) \big/ \operatorname{Im} D_1^{m-1}, & q = m; \end{cases}$$

- $\mathcal{S} = \mathrm{Sol}(D_1 + \mathrm{f}^*)$ *is the linear space of solutions* $\chi = (\chi_1^0, \chi_\alpha^{i1}, \chi^0, \chi^1)$ *of the linear system* $D_1\chi + \mathrm{f}^*\chi = 0$*;*
- $\mathcal{H} = \{\chi = (\chi_1^0, \chi_\alpha^{i1}, \chi^0, \chi^1) \mid \chi_* = \chi^*\}$ *is the Helmholtz space of the differential algebra* $(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}))$*.*

Evolution in the space **CPE**

The evolution in the space

$$\mathbf{CPE} = \mathrm{T} \times \mathrm{X} \times \left(\mathbb{R}^1_{\mathbb{I}_0} \times \mathbb{R}^{\mathrm{N}}_{\mathbb{I}}\right) \times \mathbb{R}_{\mathbb{I}_1} = \left\{t, \mathrm{x} = (x^\mu), \mathbf{u} = (u^1_{\mathrm{i}_0}, u^\alpha_{\mathrm{i}}), \mathbf{p} = (p_{\mathrm{i}_1})\right\}$$

$(\mathrm{M} = 2, 3)$ is governed by an evolution derivation

$$D_t = \partial_t + \mathrm{ev}_{\mathrm{E}},$$

where

- $\mathrm{ev}_{\mathrm{E}} = D_{\mathrm{i}_0} E^1 \cdot \partial_{u^1_{\mathrm{i}_0}} + D_{\mathrm{i}} E^\alpha \cdot \partial_{u^\alpha_{\mathrm{i}}} + D_{\mathrm{i}_1} E \cdot \partial_{p_{\mathrm{i}_1}} \in \mathrm{Sym}(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}))$;
- $\mathrm{E} = (E^\mu, E) \in \mathcal{A}^{\mathrm{M}}(\mathbf{CPE}) \times \mathcal{A}(\mathbf{CPE})$;
- $D_\mu E^\mu = 0, \quad \Delta E + 2\left(u^\lambda_{(\mu)} D_\lambda E^\mu\right) = 0.$

Evolutionary differential algebra in the space **CPE**

There is defined the differential algebra $(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{E}}(\mathbf{CPE}))$, where

- $\mathfrak{D}(\mathbf{CPE}) = \mathfrak{D}_{\mathrm{V}}(\mathbf{CPE}) \oplus_{\mathcal{A}(\mathbf{CPE})} \mathfrak{D}_{\mathrm{E}}(\mathbf{CPE})$;
- $\mathfrak{D}_{\mathrm{V}}(\mathbf{CPE}) = \{\zeta = \zeta^1_{\mathrm{i}_0}\partial_{u^1_{\mathrm{i}_0}} + \zeta^\alpha_{\mathrm{i}}\partial_{u^\alpha_{\mathrm{i}}} + \zeta_{\mathrm{i}_1}\partial_{p_{\mathrm{i}_1}} \mid \zeta^1_{\mathrm{i}_0}, \zeta^\alpha_{\mathrm{i}}, \zeta_{\mathrm{i}_1} \in \mathcal{A}(\mathbf{CPE})\}$;
- $\mathfrak{D}_{\mathrm{E}}(\mathbf{CPE})$ has the $\mathcal{A}(\mathbf{CPE})$-basis $\{D_t, D_\mu \mid \mu \in \mathrm{M}\}$, the time derivation $D_t = \partial_t + \mathsf{ev}_{\mathrm{E}}$, $[D_t, D_\mu] = 0$, $\mu \in \mathrm{M}$, so

$$\mathfrak{D}_{\mathrm{E}}(\mathbf{CPE}) = \left\{\zeta = \zeta^t D_t + \zeta^\mu D_\mu \mid \zeta^t, \zeta^\mu \in \mathcal{A}(\mathbf{CPE})\right\}.$$

The Lie algebra of symmetries here is

$$\begin{aligned}\mathsf{Sym}(\mathcal{A}(\mathbf{CPE}),&\mathfrak{D}_{\mathrm{E}}(\mathbf{CPE}))\\ &= \left\{ \mathsf{ev}_{\mathrm{f}} \in \mathsf{Sym}(\mathcal{A}(\mathbf{CPE}, \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE})) \mid [D_t, \mathsf{ev}_{\mathrm{f}}] = 0 \right\},\end{aligned}$$

where the condition $[D_t, \mathsf{ev}_{\mathrm{f}}] = 0$ reduces to the equation $(D_t - \mathrm{E}_*)\mathrm{f} = 0$.

Evolutionary differential complex in the space **CPE**

We split

- $\Omega^q_{\mathrm{E}}(\mathbf{CPE}) = dt \wedge \Omega^{q-1}_{\mathrm{H}}(\mathbf{CPE}) \oplus_{\mathcal{A}(\mathbf{CPE})} \Omega^q_{\mathrm{H}}(\mathbf{CPE}), \quad q \in \mathbb{Z};$
- where

$$
\begin{aligned}
0 \to \Omega^{q-1}_{\mathrm{H}} &\to \Omega^q_{\mathrm{E}} \to \Omega^q_{\mathrm{H}} \to 0, \\
\omega^{q-1}_{\mathrm{H}} &\mapsto \omega^q_{\mathrm{E}} = (-1)^{q-1} dt \wedge \omega^{q-1}_{\mathrm{H}}, \\
&\qquad \omega^q_{\mathrm{E}} = dt \wedge \omega^{q-1}_{\mathrm{H}} + \omega^q_{\mathrm{H}} \mapsto \omega^q_{\mathrm{H}};
\end{aligned}
$$

- $d^q_{\mathrm{E}} = d^q_t + d^q_{\mathrm{H}} : \Omega^q_{\mathrm{E}} \to \Omega^{q+1}_{\mathrm{E}}, \quad d_t = dt \wedge D_t, \quad d_{\mathrm{H}} = dx^\mu \wedge D_\mu,$
  $\omega^q_{\mathrm{E}} = dt \wedge \omega^{q-1}_{\mathrm{H}} + \omega^q_{\mathrm{H}} \mapsto d_{\mathrm{E}}\omega^q_{\mathrm{E}} = dt \wedge (D_t\omega^q_{\mathrm{H}} - d_{\mathrm{H}}\omega^{q-1}_{\mathrm{H}}) + d_{\mathrm{H}}\omega^q_{\mathrm{H}};$
- $D^q_t : \Omega^q_{\mathrm{H}}(\mathbf{CPE}) \to \Omega^q_{\mathrm{H}}(\mathbf{CPE}), \quad q \in \mathbb{Z},$
  $D^q_t(\omega_{\mu_1\dots\mu_q} \cdot dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q}) = (D_t\omega_{\mu_1\dots\mu_q}) \cdot dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q}.$

Evolutionary cohomologies in the space **CPE**

$$H^q_{\mathrm{E}}(\mathbf{CPE}) = \operatorname{Ker} d^q_{\mathrm{E}} \big/ \operatorname{Im} d^{q-1}_{\mathrm{E}}, \quad q \in \mathbb{Z}.$$

Theorem

The linear spaces of cohomologies of the differential algebra $(\mathcal{A}(\mathbf{CPE}); \mathfrak{D}_{\mathrm{E}}(\mathbf{CPE}))$ *are*

$$H^q_{\mathrm{E}}(\mathbf{CPE}) = \begin{cases} 0, & q < 0, 1 \le q \le m-2, q > m+1; \\ \mathbb{R}, & q = 0; \\ \operatorname{Ker} D^{m-1}_t, & q = m-1; \\ H^m_{\mathrm{H}}(\mathbf{CPE}) \big/ \operatorname{Im} D^m_t, & q = m+1; \end{cases}$$

while in the case $q = m$ *one has* $H^m_{\mathrm{E}}(\mathbf{CPE}) \big/ \operatorname{Im} H^{m-1}_{\mathrm{H}}(\mathbf{CPE}) = \operatorname{Ker} D^m_t$.

$$D^q_t : H^q_{\mathrm{H}}(\mathbf{CPE}) \to H^q_{\mathrm{H}}(\mathbf{CPE}), \quad D^q_t[\omega^q_{\mathrm{H}}] = [D_t \omega^q_{\mathrm{H}}], \quad q \in \mathbb{Z}.$$

Navier-Stokes equations as evolution in the space **CPE**

We treat the Navier-Stokes system (1)-(2) as the evolution process governed by the equation (1) in the space **CPE**. The algebraic counterpart of the equation (1) is the symmetry

$$\mathrm{ev}_{\mathrm{E}} = D_{\mathrm{i}_0} E^1 \cdot \partial_{u^1_{\mathrm{i}_0}} + D_{\mathrm{i}} E^{\alpha} \cdot \partial_{u^{\alpha}_{\mathrm{i}}} + D_{\mathrm{i}_1} E \cdot \partial_{p_{\mathrm{i}_1}} \in \mathrm{Sym}(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE})),$$

where

- $\mathrm{E} = (E^{\mu}, E) \in \mathcal{A}(\mathbf{CPE})^{\mathrm{M}} \times \mathcal{A}(\mathbf{CPE})$;
- $E^{\mu} = -u^{\lambda} u^{\mu}_{(\lambda)} + \nu \Delta u^{\mu} - p_{(\mu)}$;
- $u^{\mu}_{\mathrm{i}+(\mu)} = 0, \quad \Delta u^{\mu} = \sum_{\lambda} u^{\mu}_{2(\lambda)}$;
- E to be defined from the condition $\mathrm{ev}_{\mathrm{E}} \in \mathrm{Sym}(\mathcal{A}(\mathbf{CPE}), \mathfrak{D}_{\mathrm{H}}(\mathbf{CPE}))$.

Here $D_{\mu} E^{\mu} = 0$, while the condition

$$\Delta E + \mathrm{ev}_{\mathrm{E}}(u^{\lambda}_{(\mu)} u^{\mu}_{(\lambda)}) = \Delta E + 2 u^{\lambda}_{(\mu)} D_{\lambda} E^{\mu} = 0 \tag{5}$$

is the *Poisson equation for the component* $E \in \mathcal{A}(\mathbf{CPE})$

Relevant equations:

- $D_\sigma f = 0,\ \ \sigma = t, \mu,\ \ \mu \in \mathrm{M},\ \ f \in \mathcal{A} = \mathcal{A}(\mathbf{B}), \mathcal{A}(\mathbf{CPE}),$
  $\Leftrightarrow \quad f = \mathrm{const} \in \mathbb{R} \subset \mathcal{A};$
- $D_\mu f^\mu = 0, \mathrm{f} = (f^\mu) \in \mathcal{A}(\mathbf{B})^{\mathrm{M}},$
  $\Leftrightarrow \quad f^\mu = D_\nu g^{\mu\nu},\ \ g^{\mu\nu} = -g^{\nu\mu} \in \mathcal{A}(\mathbf{B});$
- $D_\mu f^\mu = 0, \mathrm{f} = (f^\mu) \in \mathcal{A}(\mathbf{CPE})^{\mathrm{M}}$, ?;
- $\Delta f = 0,\ \ f \in \mathcal{A} = \mathcal{A}(\mathbf{B}), \mathcal{A}(\mathbf{CPE}), \quad \Leftrightarrow \quad$?;
- $\Delta f = g,\ \ f, g \in \mathcal{A} = \mathcal{A}(\mathbf{B}), \mathcal{A}(\mathbf{CPE}), \quad \Leftrightarrow \quad$?;
- $\Delta f + 2u^\lambda_{(\mu)} D_\lambda f^\mu = 0,\ \ \mathrm{f} = (f^\mu, f) \in \mathcal{A}(\mathbf{CPE})^{\mathrm{M}} \times \mathcal{A}(\mathbf{CPE}),$?;
- $(D_t - \mathrm{E}_*)\mathrm{f} = 0, \quad (D_t + \mathrm{E}^*)\chi = 0,$?.

Conclusion

It can be seen from the above constructions that the Navier-Stokes equations are subject to meaningful analysis within the framework of the algebraic approach to differential equations. The resulting equations for finding algebraic characteristics of Navier-Stokes equations, such as symmetries and cohomologies, are essentially complicated. One may hope to find their partial solutions at least, especially using analytical computational packets (Mathematica, for example).

THANK YOU

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