LIMITS OF CONSTRAINED MINIMUM PROBLEMS IN VARIABLE DOMAINS

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Contents

Annotation

We consider a sequence of functionals $J_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ of the form

$$J_s(v) = \int_{\Omega_s} f_s(x, \nabla v)dx + G_s(v), \quad v \in W^{1,p}(\Omega_s),$$

where $\{\Omega_s\}$ is a sequence of domains in $\mathbb{R}^n$ contained in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ and $p > 1$.

We assume that the functions $f_s\colon \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ satisfy a convexity condition and the inequality

$$c_1|\xi|^p - \mu_s(x) \leqslant f_s(x, \xi) \leqslant c_2|\xi|^p + \mu_s(x)$$

for almost every $x \in \Omega_s$ and every $\xi \in \mathbb{R}^n$, where $c_1, c_2 > 0$ and $\mu_s \in L^1(\Omega_s)$ is nonnegative.

The functionals $G_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ are assumed to be weakly lower semicontinuous and coercive with respect to L^p-norms of functions in $W^{1,p}(\Omega_s)$.

Annotation

Along with the functionals J_s, we consider sequences $V_s \subset W^{1,p}(\Omega_s)$ of the following forms:

$$V_s = \{v \in W^{1,p}(\Omega_s) : \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\},$$

$$V_s = \{v \in W^{1,p}(\Omega_s) : h_s(v) \leqslant 0 \text{ a.e. in } \Omega_s\},$$

$$V_s = \{v \in W^{1,p}(\Omega_s) : M_s(v) \leqslant 0 \text{ a.e. in } \Omega_s\},$$

where $\varphi, \psi \colon \Omega \to \overline{\mathbb{R}}$ are measurable functions, $h_s \colon \mathbb{R} \to \mathbb{R}$, and M_s is a mapping from $W^{1,p}(\Omega_s)$ to the set of all functions on Ω_s.

We describe conditions for the convergence of minimizers and minimum values of the functionals J_s on the sets V_s.

1. Preliminaries. Definitions

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$, and let $p > 1$.

Let $\{\Omega_s\}$ be a sequence of domains in $\mathbb{R}^n$ contained in Ω.

Definition 1. We say that the sequence of domains Ω_s exhausts the domain Ω if, for every increasing sequence $\{m_j\} \subset \mathbb{N}$, we have

$$\operatorname{meas}\Big(\Omega \setminus \bigcup_{j=1}^{\infty} \Omega_{m_j}\Big) = 0.$$

We note that the condition that the sequence of domains Ω_s exhausts the domain Ω is essentially used for the limit passage in variational problems with irregular unilateral and bilateral constraints in variable domains
(see, e.g., **AK, Nonlinear Anal. 2016** and **AK, Ann. Mat. Pura Appl. 2022**).

1. Preliminaries. Definitions

We note that the sequence of domains Ω_s exhausts the domain Ω if and only if the following condition is satisfied:

$$v \in L^1(\Omega), \ \liminf_{s \to \infty} \int_{\Omega_s} |v| dx = 0 \implies v = 0 \ \text{a.e. in} \ \Omega.$$

1. Preliminaries. Definitions

For every $s \in \mathbb{N}$, let $q_s : W^{1,p}(\Omega) \to W^{1,p}(\Omega_s)$ be the mapping such that, for every function $v \in W^{1,p}(\Omega)$, $q_s v = v|_{\Omega_s}$.

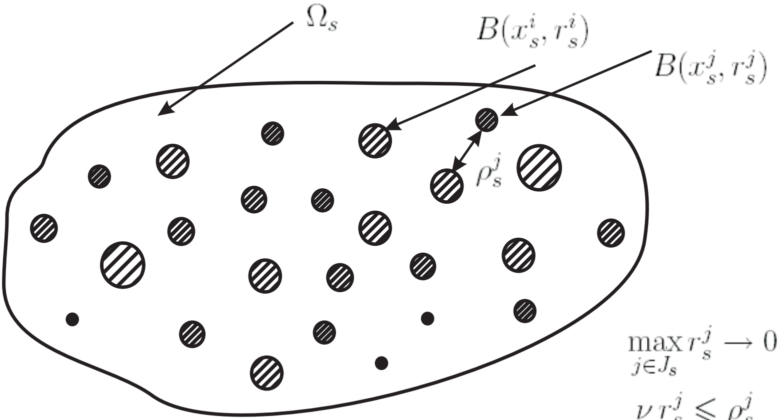
Definition 2. The sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$ if there exists a sequence of linear continuous operators $l_s : W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that:

(a) the sequence of norms $\|l_s\|$ is bounded;

(b) $s \in \mathbb{N}$, $v \in W^{1,p}(\Omega_s) \Longrightarrow q_s(l_s v) = v$ a.e. in Ω_s.

This definition goes back to the work by **E.Ya. Khruslov (Math. USSR-Sb. 1979)**.

1. Preliminaries



$$\max_{j \in J_s} r_s^j \longrightarrow 0$$

$$\nu \, r_s^j \leqslant \rho_s^j$$

$$\Omega_s = \Omega \setminus \bigcup_{j \in J_s} B(x_s^j, r_s^j)$$

Definition 3. For every $s \in \mathbb{N}$, let $I_s \colon W^{1,p}(\Omega_s) \to \mathbb{R}$, and let $I \colon W^{1,p}(\Omega) \to \mathbb{R}$. We say that the sequence $\{I_s\}$ Γ-converges to the functional I if:

(a) for every $v \in W^{1,p}(\Omega)$, there exists a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $I_s(w_s) \to I(v)$;

(b) for every $v \in W^{1,p}(\Omega)$ and every sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$, we have $\liminf\limits_{s\to\infty} I_s(v_s) \geqslant I(v)$.

1. Preliminaries

This definition is similar to the definition of Γ-convergence of functionals with the same domain studied, e.g., by

E. De Giorgi, T. Franzoni (1975),

G. Dal Maso (1981, 1993),

V.V. Zhikov (1983); V.V. Zhikov, S.E. Pastukhova (2014).

The Γ-convergence of functionals $I_s : W^{m,p}(\Omega_s) \to \mathbb{R}$ with taking into account the structure of domains Ω_s was studied, e.g., by:

L. Pankratov (2002),

B. Amaziane, M. Goncharenko, L. Pankratov (2005),

A.A. Kovalevsky (1992, 1996).

1. Preliminaries. Variational property of Γ-convergence

Theorem. *Assume that:*

(a) *the embedding of* $W^{1,p}(\Omega)$ *into* $L^p(\Omega)$ *is compact;*

(b) *the sequence of spaces* $W^{1,p}(\Omega_s)$ *is strongly connected with the space* $W^{1,p}(\Omega)$.

For every $s \in \mathbb{N}$, *let* $I_s \colon W^{1,p}(\Omega_s) \to \mathbb{R}$, *and let* $I \colon W^{1,p}(\Omega) \to \mathbb{R}$.

Assume that the sequence $\{I_s\}$ Γ*-converges to the functional* I.

For every $s \in \mathbb{N}$, *let* u_s *be a minimizer of* I_s *on* $W^{1,p}(\Omega_s)$.

Assume that $\{\|u_s\|_{W^{1,p}(\Omega_s)}\}$ *is bounded.*

Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in W^{1,p}(\Omega)$ *such that* u *minimizes* I *on* $W^{1,p}(\Omega)$ *and*

$$\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0, \qquad I_{s_j}(u_{s_j}) \to I(u).$$

1. Preliminaries. Variational property of Γ-convergence

Proof. In view of condition (b), $\{l_s u_s\}$ is bounded in $W^{1,p}(\Omega)$. Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in W^{1,p}(\Omega)$ such that $l_{s_j} u_{s_j} \to u$ weakly in $W^{1,p}(\Omega)$. Hence, by condition (a),

$$\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0.$$

Then, by the Γ-convergence of $\{I_s\}$ to I, we have

$$\liminf_{j \to \infty} I_{s_j}(u_{s_j}) \geqslant I(u). \tag{1}$$

Now, let $v \in W^{1,p}(\Omega)$. In view of the Γ-convergence of $\{I_s\}$ to I, we take a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $I_s(w_s) \to I(v)$. Then

$$\limsup_{j \to \infty} I_{s_j}(u_{s_j}) \leqslant \limsup_{j \to \infty} I_{s_j}(w_{s_j}) = I(v).$$

This and (1) imply that the function u minimizes I on $W^{1,p}(\Omega)$. □

1. Preliminaries. Functionals F_s

Let $c_1, c_2 > 0$, and, for every $s \in \mathbb{N}$, let $\mu_s \in L^1(\Omega_s)$ and $\mu_s \geqslant 0$ in Ω_s. Assume that $\{\|\mu_s\|_{L^1(\Omega_s)}\}$ is bounded.

For every $s \in \mathbb{N}$, let $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be a function such that:

for every $\xi \in \mathbb{R}^n$, the function $f_s(\cdot, \xi)$ is measurable on Ω_s;

for a.e. $x \in \Omega_s$, the function $f_s(x, \cdot)$ is convex on $\mathbb{R}^n$;

for a.e. $x \in \Omega_s$ and every $\xi \in \mathbb{R}^n$,

$$c_1|\xi|^p - \mu_s(x) \leqslant f_s(x, \xi) \leqslant c_2|\xi|^p + \mu_s(x).$$

We have $\quad s \in \mathbb{N},\ v \in W^{1,p}(\Omega_s) \Longrightarrow f_s(x, \nabla v) \in L^1(\Omega_s).$

For every $s \in \mathbb{N}$, let $F_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ be the functional such that

$$\forall v \in W^{1,p}(\Omega_s), \qquad F_s(v) = \int_{\Omega_s} f_s(x, \nabla v)dx.$$

By the above assumptions, for every $s \in \mathbb{N}$, the functional F_s is weakly lower semicontinuous.

1. Preliminaries. Functionals G_s

Next, let $c_3, c_4 > 0$, and, for every $s \in \mathbb{N}$, let $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ be a weakly lower semicontinuous functional.

We assume that, for every $s \in \mathbb{N}$ and every $v \in W^{1,p}(\Omega_s)$,

$$G_s(v) \geqslant c_3 \|v\|^p_{L^p(\Omega_s)} - c_4 .$$

Obviously, for every $s \in \mathbb{N}$, the functional $F_s + G_s$ is weakly lower semicontinuous and coercive.

Thus, if $s \in \mathbb{N}$ and U_s is a sequentially weakly closed set in $W^{1,p}(\Omega_s)$, then there exists a minimizer of $F_s + G_s$ on U_s.

1. Preliminaries. General assumptions

We assume that the following conditions are satisfied:

(C_1) the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact;

(C_2) the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$;

(C_3) the sequence of domains Ω_s exhausts the domain Ω;

(C_4) for every sequence of measurable sets $H_s \subset \Omega_s$ such that meas $H_s \to 0$, we have $\int_{H_s} \mu_s \, dx \to 0$;

(C_5) $\{F_s\}$ Γ-converges to a functional $F\colon W^{1,p}(\Omega) \to \mathbb{R}$;

(C_6) for every $v \in W^{1,p}(\Omega)$ and every sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$, we have $G_s(v_s) \to G(v)$, where $G\colon W^{1,p}(\Omega) \to \mathbb{R}$ is a strictly convex functional.

2. Minimum problems with bilateral constraints

For every measurable functions $\varphi, \psi \colon \Omega \to \overline{\mathbb{R}}$ and $s \in \mathbb{N}$, we define

$$V(\varphi, \psi) = \{v \in W^{1,p}(\Omega) \colon \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega\},$$

$$V_s(\varphi, \psi) = \{v \in W^{1,p}(\Omega_s) \colon \varphi \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}.$$

Theorem 1. *Let $\varphi, \psi \colon \Omega \to \overline{\mathbb{R}}$ be measurable functions, and assume that the following condition is satisfied:*

(A) *there exist functions $\bar{\varphi}, \bar{\psi} \in W^{1,p}(\Omega)$ such that $\varphi \leqslant \bar{\varphi} < \bar{\psi} \leqslant \psi$ a.e. in Ω.*

For every $s \in \mathbb{N}$, let u_s be a function in $V_s(\varphi, \psi)$ minimizing $F_s + G_s$ on $V_s(\varphi, \psi)$.

Then there exists a unique function $u \in V(\varphi, \psi)$ minimizing $F + G$ on $V(\varphi, \psi)$ and the following relations hold:

$$\|u_s - q_s u\|_{L^p(\Omega_s)} \to 0, \qquad (F_s + G_s)(u_s) \to (F + G)(u).$$

2. Minimum problems with bilateral constraints

Theorem 2. *Assume that* $\|\mu_s\|_{L^1(\Omega_s)} \to 0$.

Let $\varphi, \psi \colon \Omega \to \overline{\mathbb{R}}$ *be measurable functions, and let*

$\varphi \leqslant 0$ *a.e. in* Ω *and* $\psi \geqslant 0$ *a.e. in* Ω.

For every $s \in \mathbb{N}$, *let* u_s *be a function in* $V_s(\varphi, \psi)$ *minimizing* $F_s + G_s$ *on* $V_s(\varphi, \psi)$.

Then there exists a unique function $u \in V(\varphi, \psi)$ *minimizing* $F + G$ on $V(\varphi, \psi)$ *and the following relations hold*:

$$\|u_s - q_s u\|_{L^p(\Omega_s)} \to 0, \qquad (F_s + G_s)(u_s) \to (F + G)(u).$$

Theorems 1 and 2 were proved in

Kovalevsky, A.A.: Convergence of solutions of variational problems with measurable bilateral constraints in variable domains. *Ann. Mat. Pura Appl.* **201**(2), 835–859 (2022).

3. Minimum problems with implicit constraints

For every function $h : \mathbb{R} \to \mathbb{R}$, we define

$$\Phi(h) = \{t \in \mathbb{R} : h(t) \leqslant 0\},$$

$$U(h) = \{v \in W^{1,p}(\Omega) : h(v) \leqslant 0 \text{ a.e. in } \Omega\}.$$

For every $s \in \mathbb{N}$ and every function $h : \mathbb{R} \to \mathbb{R}$, we define

$$U_s(h) = \{v \in W^{1,p}(\Omega_s) : h(v) \leqslant 0 \text{ a.e. in } \Omega_s\}.$$

We note that if $s \in \mathbb{N}$, $h \colon \mathbb{R} \to \mathbb{R}$, and the set $\Phi(h)$ is nonempty and closed, then the set $U_s(h)$ is nonempty and sequentially weakly closed in $W^{1,p}(\Omega_s)$.

Then, by the properties of the functionals $F_s + G_s$, we conclude: if $s \in \mathbb{N}$, $h \colon \mathbb{R} \to \mathbb{R}$, and the set $\Phi(h)$ is nonempty and closed, then there exists $u_s \in U_s(h)$ minimizing $F_s + G_s$ on $U_s(h)$.

3. Minimum problems with implicit constraints

Theorem 3. *For every $s \in \mathbb{N}$, let $h_s \colon \mathbb{R} \to \mathbb{R}$ be a function such that the set $\Phi(h_s)$ is nonempty and closed. Let $h \colon \mathbb{R} \to \mathbb{R}$ and let the set $\Phi(h)$ be nonempty and closed. Assume that:*

(a_1) *if $t \in \Phi(h)$, then there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$ and $t \in [t_1, t_2] \subset \Phi(h)$;*

(a_2) *if $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, $(t_1, t_2) \subset \Phi(h)$, $0 < \sigma < (t_2 - t_1)/2$, then $[t_1 + \sigma, t_2 - \sigma] \subset \Phi(h_s)$ for sufficiently large $s \in \mathbb{N}$;*

(a_3) *if $t_s \to t$ in $\mathbb{R}$, $\{\tilde{s}_j\}$ is an increasing sequence in $\mathbb{N}$, and, for any $j \in \mathbb{N}$, we have $t_{\tilde{s}_j} \in \Phi(h_{\tilde{s}_j})$, then $t \in \Phi(h)$.*

For every $s \in \mathbb{N}$, let $u_s \in U_s(h_s)$ minimize $F_s + G_s$ on $U_s(h_s)$.

Then there exist an increas. seq. $\{s_j\} \subset \mathbb{N}$ and *$u \in U(h)$ such that:*

(a) *u minimizes $F + G$ on $U(h)$;*

(b) $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$; (c) $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.

3. Minimum problems with implicit constraints

Theorem 3 was proved in

Kovalevsky, A.A.: On the convergence of solutions of variational problems with variable implicit pointwise constraints in variable domains. *Ann. Mat. Pura Appl.* **198**(4), 1087–1119 (2019).

We note that

1. Conditions (a_1)–(a_3) of Theorem 3 imply that

   $\Phi(h_s) \to \Phi(h)$ in the sense of Kuratowski.

2. In general, conditions (a_1)–(a_3) of Theorem 3 cannot be replaced by the requirement that $\Phi(h_s) \to \Phi(h)$ in the sense of Kuratowski without violating the conclusion of this theorem.

3. If $h\colon \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = \sin t$

   and, for every $s \in \mathbb{N}$, $h_s\colon \mathbb{R} \to \mathbb{R}$ is defined by

   $$h_s(t) = \sin(t - t^2/s),$$

   then conditions (a_1)–(a_3) of Theorems 3 are satisfied.

In this case, $U(h) = \{v \in W^{1,p}(\Omega) : \sin v \leqslant 0 \text{ a.e. in } \Omega\}$,

$$U_s(h_s) = \{v \in W^{1,p}(\Omega_s) : \sin(v - v^2/s) \leqslant 0 \text{ a.e. in } \Omega_s\}.$$

4. Minimum problems with functional constraints

For every $s \in \mathbb{N}$, we denote by $\mathcal{F}(\Omega_s)$ the set of all functions $v \colon \Omega_s \to \mathbb{R}$.

For every $s \in \mathbb{N}$, let $M_s \colon W^{1,p}(\Omega_s) \to \mathcal{F}(\Omega_s)$.

We assume that:

(A_1) there exists a sequence $\psi_s \in W^{1,p}(\Omega_s)$ such that $\sup\limits_{s \in \mathbb{N}} \|\psi_s\|_{W^{1,p}(\Omega_s)} < +\infty$ and, for every $s \in \mathbb{N}$, $M_s(\psi_s) \leqslant 0$ a.e. in Ω_s;

(A_2) if $s \in \mathbb{N}$ and $v_m \to v$ strongly in $W^{1,p}(\Omega_s)$, then there exists an increasing sequence $\{m_j\} \subset \mathbb{N}$ such that $M_s(v_{m_j}) \to M_s(v)$ a.e. in Ω_s;

(A_3) if $s \in \mathbb{N}$, $v, w \in W^{1,p}(\Omega_s)$, and $\tau \in [0,1]$, then

$$M_s((1-\tau)v + \tau w) \leqslant (1-\tau)M_s(v) + \tau M_s(w) \text{ a.e. in } \Omega_s.$$

4. Minimum problems with functional constraints

For every $s \in \mathbb{N}$, we define

$$V_s = \{v \in W^{1,p}(\Omega_s) \colon M_s(v) \leqslant 0 \text{ a.e. in } \Omega_s\}.$$

It follows from conditions (A_1)–(A_3) that, for every $s \in \mathbb{N}$, the set V_s is nonempty, closed, and convex.

We denote by $\mathcal{F}(\Omega)$ the set of all functions $v \colon \Omega \to \mathbb{R}$.

Let $M \colon W^{1,p}(\Omega) \to \mathcal{F}(\Omega)$. We define

$$V = \{v \in W^{1,p}(\Omega) \colon M(v) \leqslant 0 \text{ a.e. in } \Omega\}.$$

4. Minimum problems with functional constraints

Theorem 4. *Assume that the set V is convex and:*

(B$_1$) *if* $v \in W^{1,p}(\Omega)$, $v_s \in W^{1,p}(\Omega_s)$ *is a sequence such that* $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$ *and* $\sup_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, *and* $\{\bar{s}_k\}$ *is an increasing sequence in* $\mathbb{N}$, *then there exist an increasing sequence* $\{s_j\} \subset \{\bar{s}_k\}$ *and a sequence of nonnegative functions* $\beta_j : \Omega \to \mathbb{R}$ *such that* $\beta_j \to 0$ *a.e. in* Ω *and, for every* $j \in \mathbb{N}$, $M_{s_j}(v_{s_j}) \geqslant M(v) - \beta_j$ *a.e. in* Ω_{s_j};

(B$_2$) *if* $v \in W^{1,p}(\Omega)$ *and* $M(v) \leqslant 0$ *a.e. in* Ω, *then there exists a sequence* $w_s \in W^{1,p}(\Omega_s)$ *such that* $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$, $\limsup_{s\to\infty} F_s(w_s) \leqslant F(v)$, *and* $\forall s \in \mathbb{N}$, $M_s(w_s) \leqslant 0$ *a.e. in* Ω_s.

For any $s \in \mathbb{N}$, *let* u_s *be a function in* V_s *minimizing* $F_s + G_s$ *on* V_s.

Then $\exists!\, u \in V$ *minimizing* $F + G$ *on* V *and* $\|u_s - q_s u\|_{L^p(\Omega_s)} \to 0$, $(F_s + G_s)(u_s) \to (F + G)(u)$.

4. Minimum problems with functional constraints

Theorem 4 was proved in

Kovalevsky, A.A.: On the convergence of solutions of variational problems with pointwise functional constraints in variable domains. *J. Math. Sci. (N.Y.)* **254**(3), 375–396 (2021).

4. Minimum problems with functional constraints

Example 1. Let $\varphi \in W^{1,p}(\Omega)$, and let $\forall s \in \mathbb{N},\ \varphi_s \in W^{1,p}(\Omega_s)$.

We assume that

(a) $\|\varphi_s - q_s\varphi\|_{L^p(\Omega_s)} \to 0$; (b) $\sup_{s\in\mathbb{N}} \|\varphi_s\|_{W^{1,p}(\Omega_s)} < +\infty$;

(c) for every sequence of measurable sets $H_s \subset \Omega_s$ such that meas $H_s \to 0$, $\int_{H_s} |\nabla\varphi_s|^p\, dx \to 0$.

Now, for every $s \in \mathbb{N}$, let $M_s \colon W^{1,p}(\Omega_s) \to \mathcal{F}(\Omega_s)$ be such that

$$\forall v \in W^{1,p}(\Omega_s), \qquad M_s(v) = v - \varphi_s.$$

The mappings M_s satisfy conditions (A_1)–(A_3).

Next, let $M \colon W^{1,p}(\Omega) \to \mathcal{F}(\Omega)$ be such that

$$\forall v \in W^{1,p}(\Omega), \qquad M(v) = v - \varphi.$$

The mappings M_s and M satisfy conditions (B_1) and (B_2) of Th. 4.

4. Minimum problems with functional constraints

We note that, for the mappings M_s and M considered in this example, the sets V_s and V take the form

$$V_s = \{v \in W^{1,p}(\Omega_s) : v \leqslant \varphi_s \text{ a.e. in } \Omega_s\},$$

$$V = \{v \in W^{1,p}(\Omega) : v \leqslant \varphi \text{ a.e. in } \Omega\}.$$

4. Minimum problems with functional constraints

Example 2. Let $\Phi\colon W^{1,p}(\Omega) \to \mathbb{R}$, and, for every $s \in \mathbb{N}$, let $\Phi_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be a continuous convex functional.

We assume that:

$(*)$ for every $v \in W^{1,p}(\Omega)$ and every sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $\sup_{s\in\mathbb{N}} \|v_s\|_{W^{1,p}(\Omega_s)} < +\infty$, we have $\Phi_s(v_s) \to \Phi(v)$.

Let $\varphi\colon \Omega \to \mathbb{R}$, and, for every $s \in \mathbb{N}$, let $\varphi_s\colon \Omega_s \to \mathbb{R}$.

Let $\{\tau_s\} \subset [0, +\infty)$, $\tau_s \to 0$, and, for every $s \in \mathbb{N}$, let $\alpha_s\colon \Omega \to \mathbb{R}$ be a nonnegative function.

We assume that $\alpha_s \to 0$ a.e. in Ω and

$$\forall s \in \mathbb{N}, \qquad \varphi - \tau_s \leqslant \varphi_s \leqslant \varphi + \alpha_s \text{ a.e. in } \Omega_s. \tag{2}$$

4. Minimum problems with functional constraints

Let $h\colon \mathbb{R} \to \mathbb{R}$ be a convex function which is nondecreasing or nonincreasing, and let $c > 0$. We assume that

$$\forall s \in \mathbb{N}, \qquad \varphi_s \geqslant h(0) + \Phi_s(\theta_s) + c \ \text{ a.e. in } \Omega_s, \tag{3}$$

where θ_s is the zero function on Ω_s.

Now, for every $s \in \mathbb{N}$, let $M_s\colon W^{1,p}(\Omega_s) \to \mathcal{F}(\Omega_s)$ be such that

$$\forall v \in W^{1,p}(\Omega_s), \qquad M_s(v) = h(v) - \varphi_s + \Phi_s(v).$$

The mappings M_s satisfy conditions (A_1)–(A_3).

Next, let $M\colon W^{1,p}(\Omega) \to \mathcal{F}(\Omega)$ be such that

$$\forall v \in W^{1,p}(\Omega), \qquad M(v) = h(v) - \varphi + \Phi(v).$$

The mappings M_s and M satisfy conditions (B_1) and (B_2) of Th. 4.

4. Minimum problems with functional constraints

We note that, for the mappings M_s and M considered in this example, the sets V_s and V take the form

$$V_s = \{v \in W^{1,p}(\Omega_s) : h(v) + \Phi_s(v) \leqslant \varphi_s \text{ a.e. in } \Omega_s\},$$

$$V = \{v \in W^{1,p}(\Omega) : h(v) + \Phi(v) \leqslant \varphi \text{ a.e. in } \Omega\}$$

and it is admissible that $h = 0$ in $\mathbb{R}$ or $h(t) = t$ for every $t \in \mathbb{R}$ or $h(t) = -t$ for every $t \in \mathbb{R}$.

For instance, for every $t \in \mathbb{R}$, let $h(t) = t$.

In addition, for every $s \in \mathbb{N}$, let $\Phi_s \colon W^{1,p}(\Omega_s) \to \mathbb{R}$,

$$\Phi_s(v) = \int_{\Omega_s} |v|^p \, dx, \quad v \in W^{1,p}(\Omega_s).$$

Finally, let $c = 1$, let $\varphi \colon \Omega \to \mathbb{R}$, $\varphi \geqslant 1$ a.e. in Ω, and, for every $s \in \mathbb{N}$, let $\varphi_s = \varphi|_{\Omega_s}$, $\tau_s = 0$, and $\alpha_s = 0$ in Ω.

Then assumptions (2) and (3) are fulfilled.

4. Minimum problems with functional constraints

Next, we assume that:

($*'$) there exists a nonnegative bounded measurable function b on Ω such that, for every open cube $Q \subset \Omega$, we have

$$\operatorname{meas}(Q \cap \Omega_s) \to \int_Q b\,dx.$$

We define $\Phi\colon W^{1,p}(\Omega) \to \mathbb{R}$ by $\Phi(v) = \int_\Omega b|v|^p\,dx$.

We have: condition ($*'$) $\Longrightarrow$ condition ($*$).

Thus, all the above assumptions on Φ_s, φ_s, and h are fulfilled, and the sets V_s and V take the form

$$V_s = \Big\{v \in W^{1,p}(\Omega_s)\colon v + \int_{\Omega_s} |v|^p\,dx \leqslant \varphi \text{ a.e. in } \Omega_s\Big\},$$

$$V = \Big\{v \in W^{1,p}(\Omega)\colon v + \int_{\Omega} b|v|^p\,dx \leqslant \varphi \text{ a.e. in } \Omega\Big\}.$$

THANK YOU FOR YOUR ATTENTION !