Ramified continua as global attractors of C^1-smooth self-maps of a cylinder close to skew products

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L.S. Efremova,
Nizhni Novgorod State University,
Moscow Institute of Physics and Technology, Russia

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Continua in dynamical problems

In the current time the theory of dynamical systems on complicated continua, in particular, on ramified continua, is intensively developing. There are dynamical systems (in particular, ordinary differential equations) that admit continua with a complicated topological structure as their attractors. For example, Plykin attractor is an indecomposable continuum. Indecomposable continua also arise in the study of inverse limits of maps with homoclinic tangencies. Dendrites appear not only in considerations of inverse limits of maps with attractors of Hénon and Lozi types, but also in investigations of limit sets of some Kleinian groups on hyperbolic 3-manifolds.

Let a discrete dynamical system with the phase space M be given by a map $F : M \to M$. Let $B \subseteq M$ be *an absorbing domain* in M, i.e., the inclusion $F(\overline{B}) \subset B$ holds, where $\overline{(\cdot)}$ is the closure of a set.

Main definitions and results

The maximal attractor A_{max} in the absorbing domain B is said to be the set

$$A_{max} = \bigcap_{n=1}^{+\infty} F^n(B).$$

An invariant set A is said to be *an attractor* of F if there exists an absorbing domain, for which A is *the maximal attractor*.

A connected compact Hausdorff space A is called *a continuum*.

Let A be a continuum, z be a point in A. We say that z is a point of the order $n \in \mathbb{N}$ if z is a unique common endpoint of every two of exactly n arcs contained in A.

By *a ramified continuum* we mean a continuum admitting points of an order $n \geqslant 3$. Such points are called *ramification points*.

Maps under consideration, I

Let a self-map F of the phase space M (equipped with x and y coordinates) be presented in the form

$$F(x, y) = (f(x) + \mu(x, y),\ g_x(y)),\ \text{where } g_x(y) = g(x, y). \quad (1)$$

If M is the plane, then we obtain Hénon map for

$$f(x) + \mu(x, y) = 1 - ax^2 + y,\ \ g_x(y) = by;$$

and Lozi map for

$$f(x) + \mu(x, y) = 1 - a|x| + y,\ \ g_x(y) = by.$$

If M is a cylinder (or a torus), then we obtain Belykh map for

$$f(x) + \mu(x, y) = x + h(x) + y,\ \ g_x(y) = \lambda(h(x) + y).$$

Maps under consideration, II

Let M be a compact two-dimensional cylinder $M = S^1 \times I_2$. Here S^1 is a circle, I_2 is a compact interval of the real line. We consider C^1-smooth maps (1) close (in the C^1-norm) to skew products on the cylinder M and so that a function μ satisfies zero boundary conditions.

A map $\varphi \in C^1(S^1)$ is said to be Ω-*stable* (in the C^1-norm) if for every $\delta > 0$ there exists $\varepsilon > 0$ such that for a map $\psi \in B^1_{1,\varepsilon}(\varphi)$ one can find δ-close in the C^0-norm to the identity map homeomorphism $h : \Omega(\varphi) \to \Omega(\psi)$ satisfying the equality

$$h \circ \varphi_{|\Omega(\varphi)} = \psi_{|\Omega(\psi)} \circ h.$$

If the Ω-stable map $\varphi \in C^1(S^1)$ satisfies the equality $\Omega(\varphi) = S^1$, than the map φ is said to be C^1-*structurally stable*.

Some definitions of one-dimensional dynamics

Let $S^1 = \{x \in \mathbb{C} : |x| = 1\} = \{e^{2\pi it} : t \in \mathbb{R}^1\}$. Then for every continuous map $\varphi : S^1 \to S^1$ there is a continuous map $\widehat{\varphi} : \mathbb{R}^1 \to \mathbb{R}^1$ satisfying

$$\varphi \circ exp = exp \circ \widehat{\varphi},$$

where $exp\, t = e^{2\pi it}$ for every $t \in \mathbb{R}^1$.
The above continuous map $\widehat{\varphi}$ is said to be *a lifting* of the circle map φ.

The degree of a map φ is said to be an integer number

$$deg\, \varphi = \frac{1}{n}(\widehat{\varphi}(t+n) - \widehat{\varphi}(t)),$$

that does not depend on $t \in \mathbb{R}^1$; moreover, it is defined for an integer number n for any lifting $\widehat{\varphi} : \mathbb{R}^1 \to \mathbb{R}^1$ of the circle map φ.

The main result of one-dimensional dynamics

Theorem. *In the space $C^1(S^1)$ there is an open everywhere dense set* $\mathbf{L}$ *of maps that equals the union of two subsets* $\mathbf{L_1}$ *and* $\mathbf{L_2}$*, where* $\mathbf{L_1}$ *is the set of Ω-stable maps with a completely disconnected nonwandering set, and* $\mathbf{L_2}$ *is the set of structurally stable maps such that every $\varphi \in \mathbf{L_2}$ is topologically conjugate with an expanding map of the same degree $deg\,\varphi$, where $|deg\,\varphi| > 1$.*

We interpret a circle S^1 as the unit interval $[0,\,1]^*$ with identified points 0 and 1. Let $I_2 = [a,\,b]$.
We consider the class $C^1_L(M)$ of C^1-smooth maps (1), where $f \in \mathbf{L}$, and μ satisfies the condition: (i_μ) $\mu(0,\,y) = \mu(1,\,y) = \mu(x,\,a) = = \mu(x,\,b) = 0$ for $x \in [0,\,1]^*$, $y \in [a,\,b]$. Then $T^1_L(M) \subset C^1_L(M)$, where $T^1_L(M)$ is the space of C^1-smooth skew products ($\mu(x,\,y) \equiv 0$ in (1)) with quotient from the set L.

Let the base of the C^1-topology in $C^1_L(M)$ be given by the family of ε-balls $B^1_\varepsilon(F)$ for every $F \in C^1_L(M)$ and $\varepsilon > 0$.

The condition of smallness of the function μ

We also need the class $C^1(M,\, S^1)$ of C^1-smooth maps of the cylinder M into the circle S^1 endowed with the standard C^1-norm $||\cdot||_{C^1(M,S^1)}$. Then for every $y \in I_2$ we have:

$$||\mu||_{C^1(S^1)} \leqslant ||\mu||_{C^1(M,S^1)}.$$

Let $\delta > 0$. Since $f \in \mathbf{L}$, then we find $\varepsilon > 0$, using the definition of the Ω-stability (or the structural stability). The "condition of smallness"in the C^1-norm means that the following inequality is valid:

$(ii_{\mu,\varepsilon})$ $||\mu||_{C^1(M,S^1)} < \varepsilon.$

Definition of C^r-local one-dimensional lamination ($r \geqslant 0$)

Let L_α be a C^r-curve, $L_\alpha \subset M$, L_α be C^r-regularly embedded to M.

Let A be a subset of M satisfying $A = \bigcup_\alpha L_\alpha$. Here α belongs to an index set; C^r-curves $\{L_\alpha\}_\alpha$ are pairwise disjoint. The family of curves $\{L_\alpha\}_\alpha$ is said to be *one-dimensional C^r- local lamination (without singularities)*, if for every point $z \in A$, $z = z(x, y)$, there exist a neighborhood $U(z) \subset M$ and a C^r-diffeomorphism for $r \geqslant 1$ or homeomorphism for $r = 0$ $\chi : U(z) \to \mathbb{R}^2$ (here $\mathbb{R}^2$ is the plane) such that every connected component of the intersection $U(z) \bigcap L_\alpha$ (if it is not empty) is mapping by means of χ into a straight line such that

$$\chi_{|U(z) \bigcap L_\alpha} : U(z) \bigcap L_\alpha \to \chi(U(z) \bigcap L_\alpha)$$

is a C^r-diffeomorphism for $r \geqslant 1$ or homeomorphism for $r = 0$ on the image.

Existence of a C^1-smooth local lamination

Theorem 1. *Let $\Phi \in T^1_L(M)$ be a map of the form*

$$\Phi(x, y) = (f(x), g_x(y)). \tag{2}$$

Let $\delta > 0$. Then there is an ε-neighborhood $B^1_\varepsilon(\Phi)$ of the map Φ in the space $C^1_L(M)$ such that every map $F \in B^1_\varepsilon(\Phi)$ obtained from Φ by means of the C^1-smooth perturbation $\mu = \mu(x, y)$, where μ satisfies the condition $(ii_{\mu,\varepsilon})$, has an invariant C^1-smooth local lamination $L^1_{loc}(F)$, which is a lamination for $f \in \mathbf{L_1}$, and a foliation for $f \in \mathbf{L_2}$. Fibres of this local lamination start from the points of the set $\Omega(f) \times \{a_2\}$ and are pairwise disjoint graphs of C^1-smooth functions $x = x(y)$ on the interval I_2. Moreover, every curvilinear fibre is ε'-close in the C^1-norm (for some $\varepsilon' > 0$, $\varepsilon' = \varepsilon'(\delta)$) to the vertical closed interval that starts from the same initial point of the set $\Omega(f) \times \{a_2\}$ just as the curvilinear fibre.

The definition of geometric integrability

Definition 1. *A map $F : M \to M$ is said to be geometrically integrable on a nonempty F-invariant set $A(F) \subseteq M$ if there exist a self-map ψ of an arc $J \subseteq M_1$ and ψ-invariant set $B(\psi) \subseteq J$ such that the restriction $F_{|A(F)}$ is semiconjugate with the restriction $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$, i.e., the following equality holds:*

$$H \circ F_{|A(F)} = \psi_{|B(\psi)} \circ H.$$

The map $\psi_{|B(\psi)}$ is said to be the quotient of $F_{|A(F)}$.

Remark 1. In the framework of our approach the concept of geometric integrability is introduced for some multifunctions (see L.S. Efremova, The Trace Map and Integrability of the Multifunctions, J. Phys.: Conf. Ser., **990** (2018), 012003).

The geometric criterion of integrability on an invariant set

We use further first pr_1 and second pr_2 natural projections.
Theorem 2. Let F be a self-map of M, $A(F)$ be a closed F-invariant subset of M satisfying

$$pr_2(A(F)) = I_2. \qquad (3)$$

Let $J \subseteq S^1$ be an arc, ψ be a self-map of J, $B(\psi)$ be a closed ψ-invariant subset of J.
Then $F_{|A(F)}$ is the geometrically integrable map with the quotient $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$ such that for every $y \in M_2$ the map H is an injection on x, if and only if $A(F)$ is the support of a continuous invariant lamination for $A(F) \neq M$ (of a continuous invariant foliation for $A(F) = M$) with fibres $\{\gamma_{x'}\}_{x' \in B(\psi)}$ that are pairwise disjoint graphs of continuous functions $x = x_{x'}(y)$ for every $y \in I_2$. Moreover, the inclusion $F(\gamma_{x'}) \subseteq \gamma_{\psi(x')}$ holds.

On the geometric integrability of maps under consideration

Define a curvilinear projection H on the support of the local lamination $L^1_{loc}(F)$. Let (x, y) be a point of the support of $L^1_{loc}(F)$. Then there is a fibre $\gamma_{x'}$ such that $(x, y) \in \gamma_{x'}$, where $x' \in \Omega(f)$. We set

$$H(x, y) = x'.$$

Theorem 3. Let $\Phi \in T^1_L(M)$ be a map of the form (2). Let $\delta > 0$. Then there is an ε-neighborhood $B^1_\varepsilon(\Phi)$ of the map Φ in the space $C^1_L(M)$ such that every map $F \in B^1_\varepsilon(\Phi)$ obtained from Φ by means of the C^1-smooth perturbation $\mu = \mu(x, y)$, where μ satisfies the condition $(ii_{\mu,\,\varepsilon})$, is geometrically integrable on the support of the local lamination $L^1_{loc}(F)$ with the quotient $f_{|\Omega(f)}$ by means of a C^1-smooth curvilinear projection H such that for every $y \in I_2$ the map H is an injection on x satisfying the inequality

$$\frac{\partial}{\partial x} H(x, y) \neq 0$$

.

The analytic criterion of integrability on an invariant set

Theorem 4. *Let $F : M \to M$, $A(F)$ be a nonempty closed F-invariant subset of M satisfying the (3). Let $J \subseteq S^1$ be an arc, ψ be a self-map of J, $B(\psi)$ be a closed ψ-invariant subset of J. Then $F_{|A(F)}$ is the geometrically integrable map with the quotient $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$ such that for every $y \in I_2$ the map H is an injection on x, if and only if there is a homeomorphism $\widetilde{H}$ that maps the set $A(F)$ on the set $B(\psi) \times I_2$ and reduces the restriction $F_{|A(F)}$ to the skew product $\Psi_{|B(\psi)\times I_2}$ satisfying the equality*

$$\Psi_{|B(\psi)\times I_2}(u, v) = (\psi_{|B(\psi)}(u), g_{x'}(v)), \;\; x' = pr_1 \circ \widetilde{H}^{-1}(u, v).$$

Here $\widetilde{H}^{-1} : B(\psi) \times I_2 \to A(F)$ is the inverse homeomorphism for $\widetilde{H}$,

$$\text{and } \; \widetilde{H}(x, y) = (H(x, y), y), \text{ for all } (x, y) \in A(F).$$

C^1-smooth conjugacy of maps (1) to skew products

Theorem 5. *Let* $\Phi \in T^1_L(M)$ *be a map of the form (2). Let* $\delta > 0$*. Then there is an* ε*-neighborhood* $B^1_\varepsilon(\Phi)$ *of the map* Φ *in the space* $C^1_L(M)$ *such that every map* $F \in B^1_\varepsilon(\Phi)$ *obtained from* Φ *by means of the* C^1*-smooth perturbation* $\mu = \mu(x,\, y)$*, where* μ *satisfies the condition* $(ii_{\mu,\,\varepsilon})$*, possesses the following property: the restriction* $F_{|L^1_{loc}(F)}$ *is* C^1*-smoothly conjugate under the* C^1*-diffeomorphism* $\widetilde{H} : L^1_{loc}(F) \to \Omega(f) \times I_2$ *to the skew product* $\Psi_{|\Omega(f)\times I_2}$ *satisfying:*

$$\Psi_{|\Omega(f)\times I_2}(u, v) = (f_{|\Omega(f)}(u), g_{x'}(v)), \;\; x' = pr_1 \circ \widetilde{H}^{-1}(u, v).$$

Here $\widetilde{H}^{-1} : \Omega(f) \times I_2 \to L^1_{loc}(F)$ *is the inverse* C^1*-diffeomorphism for* $\widetilde{H}$*,*

$$\textit{and} \;\; \widetilde{H}(x,\, y) = (H(x,\, y),\, y), \;\; \textit{for all} \;\; (x,\, y) \in L^1_{loc}(F).$$

Example of maps with one-dimensional ramified attractors, I

Let C^1-smooth maps $F_k : M \to M$ ($k > 1$, $k \in \mathbf{N}$, $I_2 = [0, 1]$) be so that

$$F_k(x, y) = (f_k(x) + \mu(x, y), g_x(y)), \quad f_k(x) = kx \,(mod\, 1).$$

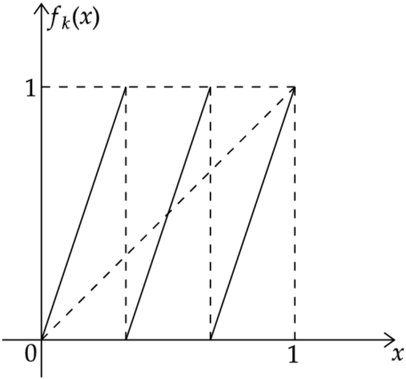


Рис.: The graph of f_k for $k = 3$.

Example of maps with one-dimensional ramified attractors, II

To construct maps $g_x(y)$ on $S^1 \times [0,\ 1]$ ($S^1 = [0,\ 1]^*$) for the cylinder map F_k, we use a C^1-smooth Urysohn function $y = h_k(x)$, where $x \in [0,\ 1]^*$ such that

$$h_k(0) = h_k(1) = \frac{3}{4},\ \ h_k(\frac{1}{k}\left[\frac{k}{2}\right]) = 0;$$
$$h_k'(0) = h_k'(1) = h_k'(\frac{1}{k}\left[\frac{k}{2}\right]) = 0.$$

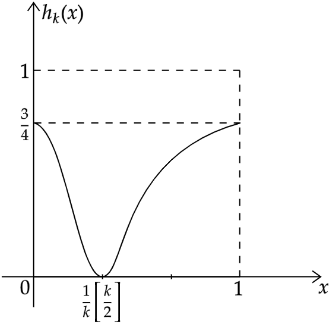


Рис.: The graph of Urysohn function h_k for $k = 3$.

Example of maps with one-dimensional attractors, III

Then for small enough δ the graph of the function $y = h_k(x)$ intersects every fibre $\gamma_{x'}$ in the unique point without tangency, and the equality $y = h_k(x_{x'}(y))$ is equivalent to $y = y(x')$, where $x' \in S^1$; moreover, the function $y = y(x')$ is C^1-smooth on S^1. We use two connected sets

$$D' = \{(x,\, y) \in \bigcup_{x' \in [0,1]^*} \gamma_{x'} \,:\, 0 \leqslant y \leqslant y(x')\};$$

$$D'' = \{(x,\, y) \in \bigcup_{x' \in [0,1]^*} \gamma_{x'} : y(x') < y \leqslant 1\}.$$

We set

$$g_x(y) = \begin{cases} y, & \text{if} \quad (x,\, y) \in \gamma_{x'} \bigcap D', \\ h_k(x) + \sin(y - h_k(x)), & \text{if} \quad (x,\, y) \in \gamma_{x'} \bigcap D''. \end{cases}$$

Example of maps with one-dimensional attractors, IV

By Theorem 5 maps F_k $(k > 1)$ C^1-smoothly conjugate to skew products $\widehat{\Phi}_k = (f_k, \widehat{g}_{x'})$ for all $x' \in S^1$.
Graphs of fibre maps for different $x' \neq 1/k[k/2]$ are presented in the picture.

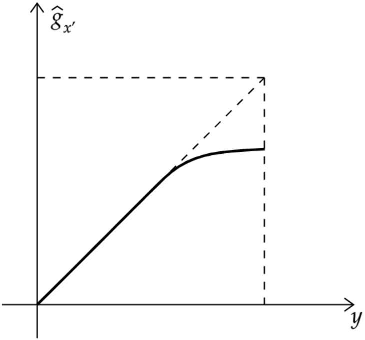


Рис.: The graph of $\widehat{g}_{x'}$ for $x' \neq 1/k\left[k/2\right]$.

Description of the nonwandering set of $\widehat{\Phi}_k$: Ω-function

We use the topological space 2^{I_2} of all closed subsets of the segment I_2 endowed with the exponential topology, i. e. the weakest topology in which the sets 2^A are open in 2^{I_2} for open sets A, and closed in 2^{I_2} for closed sets A. Here 2^A means the set of all closed subsets lying in $A \subset I_2$.

Definition 6. *The Ω-function* of a skew product $\Phi \in T^0(M)$ is defined to be the function $\Omega^\Phi : \Omega(\varphi) \to 2^{I_2}$ such that for any $x \in \Omega(\varphi)$ the equality holds:

$$\Omega^\Phi(x) = \big(\Omega(\Phi)\big)(x),$$

where $(\Omega(\Phi))(x)$ is the slice of the nonwandering set $\Omega(\Phi)$ by the fibre over x.

The slice $(A')(x)$ of a set $A' \subseteq M$ by the vertical fibre over a point $x \in S^1$ is the following set:

$$\big(A'\big)(x) = \{y \in I_2 : (x, y) \in A'\}.$$

Properties of $\Omega(\widehat{\Phi}_k)$ in terms of the Ω-function, I

Theorem 7. *The Ω-function $\Omega^{\widehat{\Phi}_k} : [0, 1]^* \to 2^{[0, 1]}$ of a skew product $\widehat{\Phi}_k \in T^1_L(M)$ $(k \geqslant 2)$ possesses the following properties:*

(7.1) *it is upper semicontinuous;*

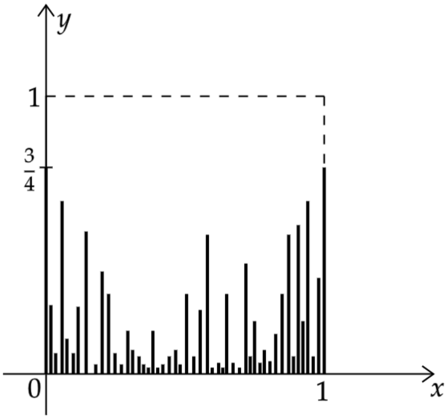
(7.2) *in each point $x' \in [0, 1]^*$ with f_k-trajectory, the closure of which does not contain $x'_* = 1/k\left[k/2\right]$, $\Omega^{\widehat{\Phi}_k}$ is discontinuous function, and $\Omega^{\widehat{\Phi}_k}(x') = [0, (c^k)(x')]$, where $c^k(x') > 0$;*

(7.3) *in each point $x' \in [0, 1]^*$ with f_k-trajectory, the closure of which contains $x'_* = 1/k\left[k/2\right]$, $\Omega^{\widehat{\Phi}_k}$ is continuous function, and $\Omega^{\widehat{\Phi}_k}(x') = \{0\}$;*

(7.4) *the graph of the Ω-function $\Omega^{\widehat{\Phi}_k}$ in M is one-dimensional topological continuum; its ramification points form an everywhere dense subset of $[0, 1]^* \times \{0\}$ of the cardinality equal to the continuum $\mathfrak{c}$, and each ramification point has the order 3 (see the following Figure);*

Properties of $\Omega(\widehat{\Phi}_k)$ in terms of the Ω-function, II

(7.5) the graph of the Ω-function $\Omega^{\widehat{\Phi}_k}$ in M is the global chaotic attractor of $\widehat{\Phi}_k$ that coincides with the closure of the set $Per(\widehat{\Phi}_k)$ and with the closed set $\bigcup_{(x',y)\in M} \omega_{\widehat{\Phi}_k}((x', y))$; moreover, the second eigenvalue of each $\widehat{\Phi}_k$-periodic point equals 1.



Definition 8. We say that a map $F : M \to M$ with a global chaotic attractor A possesses the property of *dense intermittency* (in the complement to the attractor) of attraction sets of different ω-limit sets, the union of which coincides with the global attractor A, if every point $(\overline{x}', \overline{y}) \in M \setminus A$ has a neighborhood $U((\overline{x}', \overline{y})) \subset M \setminus A$ satisfying the property:
in the intersection of $U((\overline{x}', \overline{y}))$ with an arbitrary circle $y = const$ (if it is not empty) between arbitrary two different points $(x'_1; y)$ and $(x'_2; y)$ of attraction sets of different ω-limit sets there is a point $(x'_3; y)$ that belongs to the attraction set of an ω-limit set different from each of previous two ω-limit sets.

Theorem 9. *Let $\Phi_k \in T^1_L(M)$ be a map of the form (2) with the quotient f_k and fibres maps described above. Let δ be a positive number, $\delta < \overline{\delta}$ (for some $\overline{\delta} > 0$). Then there is an ε-neighborhood $B^1_\varepsilon(\Phi_k)$ of the map Φ_k in the space $C^1_L(M)$ such that every map $F_k \in B^1_\varepsilon(\Phi_k)$ obtained from Φ_k by means of the C^1-smooth perturbation $\mu = \mu(x, y)$, where μ satisfies the condition $(ii_{\mu,\varepsilon})$, has one-dimensional global chaotic attractor $A(F_k)$, which is a ramified continuum with everywhere dense ramification points set of the cardinality c on the invariant circle $S^1 \times \{0\}$. Moreover, the following properties are fulfilled:*

(9.1) $A(F_k) = \overline{Per(F_k)} = \bigcup_{(x,y)\in M} \omega_{F_k}(x, y)$, *and all F_k-periodic points are not hyperbolic;*

Theorem about the global ramified attractor, II

(9.2) $A(F_k)$ *consists of two types of* C^1*-smooth arcs: on the circle* $S^1 \times \{0\}$ *the map* F_k *is mixing, and on each nondegenerate arc of the second type (the set of these arcs has continuum cardinality) the map* F_k^n *is not mixing for every* $n \geqslant 1$*;*

(9.3) F_k *possesses the property of dense intermittency (in the complement to the attractor) of attraction sets of different* ω*-limit sets, the union of which coincides with the global attractor* $A(F_k)$*.*

References

L.S. Efremova, Ramified continua as global attractors of C^1-smooth self-maps of a cylinder close to skew products, J. Difference Eq. and Appliq., **28** (2023) DOI: https://doi.org/10.1080/10236198.2023.2204144