THE ASYMPTOTIC STABILITY OF THE HYBRID SYSTEMS

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The linear autonomious hybrid system with bounded delay

Continuous-discrete systems of functional-differential equations (also called hybrid) are systems whose state is described by two groups of interrelated variables: some variables $x = x(t)$ are functions of continuous time and satisfy differential equations; others $y = y(n)$ are functions of discrete time and satisfy difference equations.

The linear autonomious hybrid system with bounded delay

$$\begin{cases} \dot{x}(t) + \displaystyle\int_0^h x(t-s)dR(s) = Bx(n) + Cy(n), \ t \in [n, n+1], \\ x(t) = \psi(t), \quad t \in (-h; 0), \\ y(n+1) + Dy(n) = Ex(n), \quad n \in \mathbb{N}_0, \end{cases} \tag{1}$$

where $\psi \in L_1^n(-h, 0)$,

$$R(s) \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times n}, \quad C \in \mathbb{R}^{n\times m},$$

$$D \in \mathbb{R}^{m\times m}, \quad E \in \mathbb{R}^{m\times n}.$$

The simpliest linear autonomious hybrid system

$$\begin{cases} \dot{x}(t) + Ax(t) = Bx(n) + Cy(n), \quad t \in [n, n+1), \\ y(n+1) + Dy(n) = Ex(n), \quad n \in \mathbb{N}_0, \end{cases} \tag{2}$$

$$A, B \in \mathbb{R}^{n\times n}, \quad C \in \mathbb{R}^{n\times m},$$

$$D \in \mathbb{R}^{m\times m}, \quad E \in \mathbb{R}^{m\times n}.$$

The equiavalent form:

$$\begin{cases} \dot{x}(t) + Ax(t) = Bx\big([t]\big) + Cy\big([t]\big), \\ y\big([t]+1\big) + Dy\big([t]\big) = Ex\big([t]\big), \end{cases}, \quad t \geqslant 0, \tag{3}$$

where $[t]$ — the integer part of t.

Stability of the simpliest hybrid system

The system (2) is called *exponentially stable* if for any initial data $x(0) \in \mathbb{R}^n$, $y(0) \in \mathbb{R}^m$ there exist $M, \sigma > 0$ such that

- $\|x(t)\| < Me^{-\sigma t}$,
- $\|y(n)\| < Me^{-\sigma n}$.

The system (2) is called *asymptotically stable* if

- $\lim_{t\to\infty} \|x(t)\| = 0$
- $\lim_{n\to\infty} \|y(n)\| = 0$

for any initial data $x(0) \in \mathbb{R}^n$, $y(0) \in \mathbb{R}^m$.

Applying the inverse Caushy operator of differential subsystem

$$\begin{cases} x(n+1) = e^{-A}x(n) + S\big(Bx(n) + Cy(n)\big), \\ y(n+1) + Dy(n) = Ex(n). \end{cases} \tag{4}$$

$$S = \int_0^1 e^{-As} ds.$$

Theorem 1

The hybrdid system (2) is asymptotically stable if and only if the auxiliary system of difference equations (4) is asymptotically stable.

Auxiliary system of difference equations

$$u(n) = \begin{Bmatrix} x(n) \\ y(n) \end{Bmatrix}, \quad M = \left[\begin{array}{c|c} -e^{-A} - SB & -SC \\ \hline -E & D \end{array} \right],$$

$$u(n+1) + Mu(n) = 0. \tag{5}$$

Characterisitic equation

$$det(zI + M) = 0$$

Example No.1.

$$\ddot{x}(t) + \omega^2 x(t) = bx\big([t]\big), \quad t \geqslant 0. \tag{6}$$

$$x_1 = x, \quad x_2 = \dot{x}, \quad u = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$M = \begin{bmatrix} -\cos\omega - \frac{1-\cos\omega}{\omega^2}b & -\frac{\sin\omega}{\omega} \\ \frac{\sin\omega}{\omega}(\omega^2 - b) & -\cos\omega \end{bmatrix}.$$

Example No.1.

$$M = \begin{bmatrix} -\cos\omega - \frac{1-\cos\omega}{\omega^2}b & -\frac{\sin\omega}{\omega} \\ \frac{\sin\omega}{\omega}(\omega^2 - b) & -\cos\omega \end{bmatrix}.$$

Characterisitic equation

$$z^2 + \operatorname{Sp} M z + \det M = 0.$$

$$\det M = 1 - \frac{1-\cos\omega}{\omega^2}b,$$

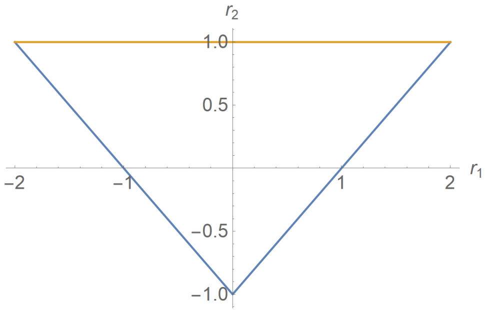
$$\operatorname{Sp} M = -2\cos\omega - \frac{1-\cos\omega}{\omega^2}b.$$

The stability criterion of the polynome

$$z^2 + r_1 z + r_2 = 0 \tag{7}$$

Theorem 2

For all roots of the polynome (7) the inequality $|z| < 1$ holds if and only if $|r_1| - 1 < r_2 < 1$.



Example No.1. The stability criterion

$$\ddot{x}(t) + \omega^2 x(t) = bx\big([t]\big), \quad t \geqslant 0. \tag{7}$$

$$\det M = 1 - \frac{1 - \cos\omega}{\omega^2} b,$$

$$\operatorname{Sp} M = -2\cos\omega - \frac{1 - \cos\omega}{\omega^2} b.$$

$$|\operatorname{Sp} M| - 1 < \det M < 1.$$

Theorem 3

The equation (6) is asymptotically stable if and only if $0 < b < \omega^2$.

Example No.2.

$$\ddot{x}(t) + \omega^2 x(t) = bx\big([t]-1\big), \quad t \geqslant 0. \tag{8}$$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) + ax_1(t) = by(n), \quad t \in [n, n+1), \\ y_1(n+1) = x_1(n). \end{cases} \tag{9}$$

$$M = \begin{bmatrix} -\cos\omega & -\frac{\sin\omega}{\omega} & -\frac{1-\cos\omega}{\omega^2}b \\ \omega\sin\omega & -\cos\omega & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Example No.2. The stability criterion

$$\ddot{x}(t) + \omega^2 x(t) = bx\big([t]-1\big), \quad t \geqslant 0. \tag{8}$$

Theorem 4

The equation (8) *is asymptotically stable iff* $b \neq 0$ *and*

$$\cos\omega \notin \left\{-1, \frac{1}{2}, 1\right\}$$

and

$$\begin{cases} 0 < b < \omega^2 \dfrac{1+\cos\omega}{1-\cos\omega}, & \text{if} \quad \cos\omega < 0, \\ 0 < b < \omega^2 \dfrac{1-2\cos\omega}{1-\cos\omega}, & \text{if} \quad \cos\omega \in \left[0, \dfrac{1}{2}\right), \\ \omega^2 \dfrac{1-2\cos\omega}{1-\cos\omega} < b < 0, & \text{if} \quad \cos\omega \in \left(\dfrac{1}{2}, 1\right). \end{cases}$$

The example of linear autonomious hybrid system with delay

$$\begin{cases} \dot{x}(t) + ax(t-1) = y(n), \quad t \in [n, n+1), \\ x(t) = \psi(t), \quad t \in [-1, 0), \\ y(n) = -bx(n), \quad n \in \mathbb{N}_0, \end{cases} \tag{10}$$

where $a, b \in \mathbb{R}$ and the initial function ψ is considered to be summable.

Differential equation with discrete delay argument and concentrated delay

$$\begin{cases} \dot{x}(t) + ax(t-1) + bx\big([t]\big) = 0, \quad t \in \mathbb{R}_+, \\ x(t) = \psi(t), \quad t \in [-1, 0), \end{cases} \tag{11}$$

where $a, b \in \mathbb{R}$ and the initial function ψ is considered to be summable.

The hybrid system (10) is equivalent to the equation (11).

Stability of the equation (11)

The equation (11) is called *exponentially stable* if for any initial data $x(0) \in \mathbb{R}$, $\psi \in L_1[-1, 0]$ there exist $M, \sigma > 0$ such that

$$\left| x(t) \right| < M e^{-\sigma t}.$$

The equation (11) is called *asymptotically stable* if

$$\lim_{t \to \infty} \left| x(t) \right| = 0$$

for any initial data $x(0) \in \mathbb{R}$, $\psi \in L_1[-1, 0]$.

Path shift operator

$$(Sx)(\tau) = x(1)(1 - b\tau) - a\int_0^{\tau} x(s)\,ds.$$

The operator S acts in $C[0,1]$.

Denote the solution of the equation (11) on the interval $[n, n+1]$ by $x_n = x_n(\tau)$, where $\tau = t - n$.

For any $n \geqslant 1$ the equality holds:

$$x_n = Sx_{n-1} = \cdots = S^n x_0.$$

Gelphand's formula:

$$\rho(S) = \lim_{n\to\infty} \sqrt[n]{\|S^n\|}.$$

Main result

$$\begin{cases} \dot{x}(t) + ax(t-1) + bx\big([t]\big) = 0, & t \in \mathbb{R}_+, \\ x(t) = \psi(t), & t \in [-1, 0), \end{cases} \tag{11}$$

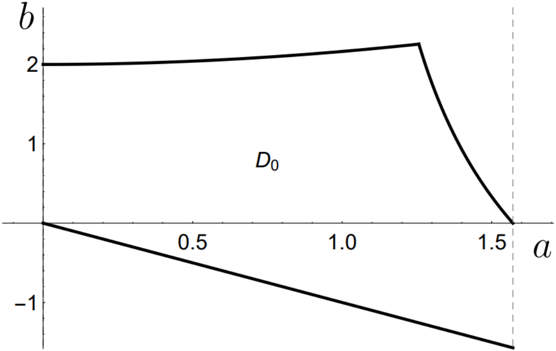
Theorem 5

Let $a \neq 0$. The following statements are equivalent:

- *the equation (11) is asymptotically stable,*
- *the equation (11) is exponentially stable,*
- *the spectral radius of the operator S is less than one,*
- *all eigenvalues λ of the operator S lie inside the unit circle.*

Stability domain in the string $a \in [0, \pi/2]$

$$\dot{x}(t) + ax(t-1) + bx\big([t]\big) = 0, \quad t \in \mathbb{R}_+, \tag{11}$$



Theorem 6

Suppose that $a \in [0, \pi/2]$. Then the equation (11) is asymptotically stable iff the point (a, b) belongs to the domain D_0.

Thank you!