THE INTERACTION OF SHOCKS IN 2D PRESSURELESS MEDIUM

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PRESENTATION PLAN

- **1D case, solutions as measures, existence and uniqueness, variational principle**
- **2D case, the formation of singularities along the curves in R^2, collision of the curves**
- **Variational description and 2D Riemann problem**
- **Numerical calculations and the existence of shocks hierarchy,**
- **The usage in astrophysics – the Universe at large scales**

1. Зельдович Я.Б. Распад однородного вещества на части под действием тяготения // Астрофизика 6:2, 319–335 (1970). Zel'dovich Ya.B. Gravitational instability: An approximate theory for large density perturbations // Astron. Astrophys. 5, 84 – 89 (1970).
2. Крайко А. Н., О поверхностях разрыва в среде, лишенной собственного давления // Прикл. Матем. Мех. 43:3, 500–510 (1979). Kraiko A.N. On discontinuity surfaces in a medium devoid of “proper” pressure // J. Appl. Math. Mech. 43:3, 539–549.
3. Bouchut F. On zero-pressure gas dynamics // in: B. Perthame (Ed.), Advances in Kinetic Theory and Computing, series on Advances in Mathematics and Applied Sciences. Singapore.: World Scientific 22, 171190 (1994).

1D CASE, SOLUTIONS AS MEASURES, EXISTENCE AND UNIQUENESS, VARIATIONAL PRINCIPLE

1D PRESSURELESS GAS DYNAMICS SYSTEM

Consider the so-called pressureless gas dynamics (PGD) system

$\boxed{\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2)_x = 0}$, ρ-density, u-velocity.

Consider the Cauchy problem for PGD: $\rho(0,x) = \rho_0(x), u(0,x) = u_0(x)$.

DEFINITION. Let's call the generalized solution to Cauchy problem for PGD the pair of Radon measures $P_t(x)$ and $I_t(x)$, weakly continuous w.r.t. t, $I_t(x)$ is absolutely continuous w.r.t. $P_t(x)$, $u(t,x) \equiv dI_t(x)/dP_t(x)$ s.t. for any $f, g \in C_0^1(\mathbb{R}), 0 < t_1 < t_2$

$$\int_{\mathbb{R}} f(x)\,dP_{t_2}(x) - \int_{\mathbb{R}} f(x)\,dP_{t_1}(x) = \int_{t_1}^{t_2} d\tau \int_{\mathbb{R}} f'(x)\,dI_\tau(x)$$

$$\int_{\mathbb{R}} g(x)\,dI_{t_2}(x) - \int_{\mathbb{R}} g(x)\,dI_{t_1}(x) = \int_{t_1}^{t_2} d\tau \int_{\mathbb{R}} g'(x)\,u(\tau,x)\,dI_\tau(x)$$

and $P_t \to P_0$, $I_t \to I_0$ weakly as $t \to +0$.

EQUIVALENCE IN SMOOTH CASE

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \dfrac{\partial}{\partial x}(\rho u) = 0 \\ \dfrac{\partial}{\partial t}(\rho u) + \dfrac{\partial}{\partial x}(\rho u^2) = 0 \\ u(0,x) = u_0(x), \rho(0,x) = \rho_0(x) \end{cases} \Longleftrightarrow \begin{cases} \dfrac{\partial}{\partial t}\rho + \dfrac{\partial}{\partial x}(\rho u) = 0 \\ \dfrac{\partial}{\partial t}u + u\dfrac{\partial}{\partial x}u = 0 \end{cases}$$

$$\begin{aligned} \dot{x} &= u \\ \dot{u} &= 0 \\ \dot{\rho} + \rho\frac{\partial u}{\partial x} &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \mathrm{x} &= \mathrm{a} + \mathrm{t}\cdot\mathrm{u}_0(\mathrm{a}) \\ \mathrm{u} &= \mathrm{u}_0(\mathrm{a}) \\ \rho &= \frac{\rho_0(\mathrm{a})}{1 + \mathrm{t}\cdot\mathrm{u}_0'(\mathrm{a})} \end{aligned}$$

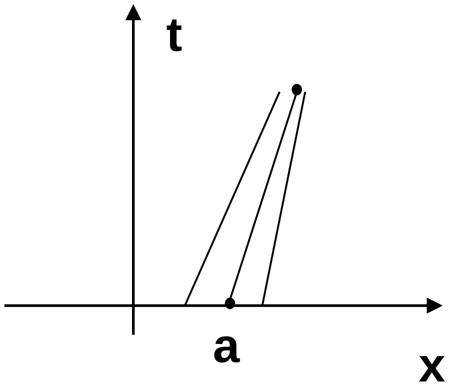
The Riemann problem:

$$\big(\rho(0,x), u(0,x)\big) = (\rho^-, u^-) \text{ as } x<0; \boxed{(P_0, u^*) \text{ as } x=0}; (\rho^+, u^+) \text{ as } x<0.$$

THE SOLUTIONS WITH SINGULARITIES

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = 0$$



The solution of Riemann problem:

$$\rho = \rho^- + (\rho^+ - \rho^-) H(x - s(t)) + P(t)\delta(x - s(t))$$

$$u = u^- + (u^+ - u^-) H_{\dot{s}}(x - s(t))$$

Rankine-Hugoniot conditions:

$$[\rho]\dot{s} - [\rho u] - \dot{P} = 0$$

$$[\rho u]\dot{s} - [\rho u^2] - (\dot{s}P)^{\cdot} = 0$$

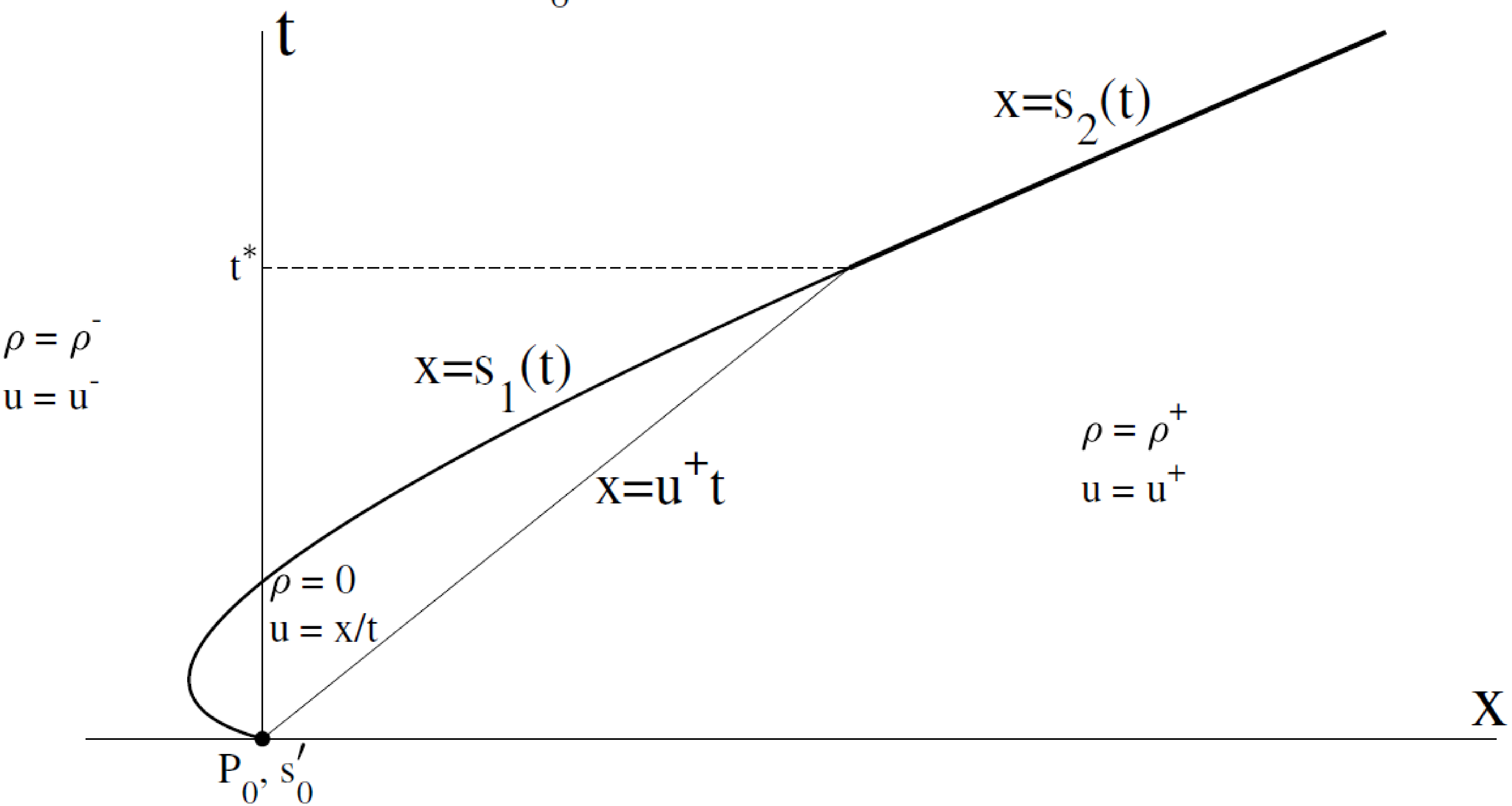
if $P = 0$ then only contact discontinuities are possible:

$$u^+ = u^- = \dot{s}$$

1. Weinan E, Yu. G. Rykov, and Ya. G. Sinai, The Lax–Oleinik variational principle for some one-dimensional system of quasilinear equations // Russ. Math. Surv. 50, 220–222 (1995).
2. E. Grenier, Existence globale pour la systeme des gaz sans pression // C. R. Acad. Sci., Ser. 1: Math. 321, 171–174 (1995).

THE TYPICAL FORM OF RIEMANN SOLUTION

$$s_0' < u^+ < u^-$$



The interaction of shocks is simple: when meeting they glue together and move according to conservation of mass and momentum.

EXISTENCE AND UNIQUENESS THEOREMS

Let ρ and u are generalized solutions of PGD:

$\rho_t + (\rho u)_x = 0, (\rho u)_t + (\rho u^2)_x = 0,$ ρ-density, u-velocity.

THEOREM. Assume $P_0(x) \geq 0 \in \mathrm{M}_{loc}(\mathbb{R})$ and $u_0(x)$ is bounded and measurable w.r.t. P_0. Then the generalized solution exists. If the additional conditions are true: 1) Oleinik's condition $\dfrac{u(t,x_2)-u(t,x_1)}{x_2-x_1} \leq \dfrac{1}{t}$ for all $x_1 < x_2$ and a.e. $t > 0$ and 2) the measure ρu^2 weakly converges to $\rho_0 u_0{}^2$ as $t \to 0,$ then the generalized solution is unique.

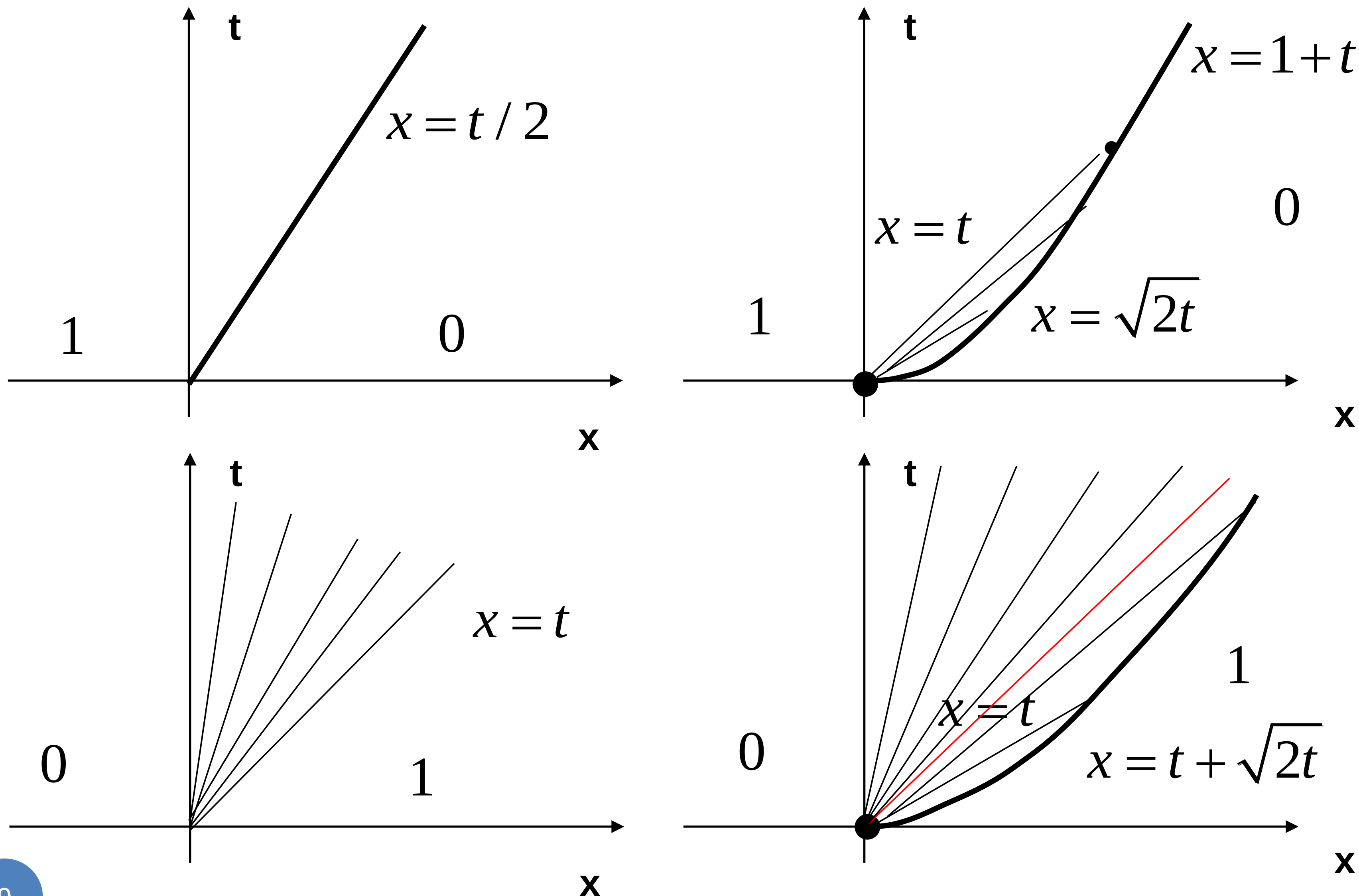
Another form: 2) $\to$ 3) $\lim\limits_{t\to+0} P_t\left\{x\in\mathbb{R}; \left|\dfrac{x-x_0}{t} - u_0(x_0)\right| \leq \varepsilon\right\} = P_0(x_0) > 0.$

1. F. Huang and Z. Wang, Well posedness for pressureless flow // Comm. Math. Phys. 222, 117–146 (2001).

2. J. Li and G. Warnecke, Generalized characteristics and the uniqueness of entropy solutions to zero-pressure gas dynamics // Adv. Differ. Equat. 8, 961–1004 (2003).

3. [illegible] Hynd, Sticky particle dynamics on the real line // Not. Am. Math. Soc. 66, 162–168 (2019).

THE EXAMPLE OF RIEMANN PROBLEM SOLUTION WITH INITIAL DELTA FUNCTION



VARIATIONAL REPRESENTATION

Consider the function:

$$F(t,x;a)=\int_{0-0}^{a-0}\left[u_0(s)-\frac{x-s}{t}\right]P_0(ds).$$

Let us seek the global minima $a(t,x)$. In case such minimum is unique

$$u(t,x)=\frac{x-a(t,x)}{t},\ \rho(t,x)\,dx=P_0(da),\ x=a+t\cdot u_0(a);$$ if not then

there is the velocity discontinuity point and $P_t(x;dx)=\int_{a_{\min}(t,x)}^{a_{\max}(t,x)}P_0(da).$

1. О.А. Олейник, Задача Коши для нелинейных дифференциальных уравнений первого порядка с разрывными начальными условиями // Труды Моск. Мат. Об-ва. 5, 433–454 (1956).
2. Weinan E, Yu. G. Rykov, and Ya. G. Sinai, Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in sion particle dynamics // Comm.Math.Phys. 177, 349–380 (1996).

THE MOVEMENT OF CONCENTRATION POINT

If for each t the point $(t, x(t))$ is a concentration point, then:

$$F(t, x; a_{\max}) - F(t, x; a_{\min}) = \int_{a_{\min}(t,x(t))}^{a_{\max}(t,x(t))} \left[u_0(s) - \frac{x(t) - s}{t} \right] P_0(ds) = 0.$$

Differentiating w.r.t. t, we obtain

$$0 = \int_{a_{\min}(t,x(t))}^{a_{\max}(t,x(t))} \left[\frac{s - \dot{x}(t)}{t} - \frac{s - x(t)}{t^2} \right] P_0(ds) =$$

$$\frac{1}{t} \left\{ -\dot{x}(t) \int_{a_{\min}(t,x(t))}^{a_{\max}(t,x(t))} P_0(ds) + \int_{a_{\min}(t,x(t))}^{a_{\max}(t,x(t))} u_0(s) P_0(ds) \right\}.$$

THE REFORMULATION OF VARIATIONAL DESCRIPTION

Fix $t > 0$. Assume A is the set of segments of the type $\left(a_{\min}(t,x), a_{\max}(t,x)\right)$, and $B = A \cup \{\text{set of all segments in } \mathbb{R} \setminus A\}$.

Let $F\left(t,x;[a_1,a_2]\right) = \int_{a_1}^{a_2}\left[u_0(s) - \frac{x-s}{t}\right]P_0(ds)$, define $\delta F / \delta a$:

$$\frac{\delta F}{\delta a} = \lim_{[a_1,a_2]\downarrow \min} \frac{1}{a_2 - a_1}\int_{a_1}^{a_2}\left[u_0(s) - \frac{x-s}{t}\right]P_0(ds) = 0,\ a \in [a_1,a_2] \in B.$$

Then for any $t > 0$ the generalized solution is characterized by the relation

$$\forall a \in \mathbb{R}\ \exists x : \frac{\delta F(t,x;a)}{\delta a} = 0.$$

2D CASE, THE FORMATION OF SINGULARITIES ALONG THE CURVES IN R^2, COLLISION OF THE CURVES

2D PRESSURELESS GAS DYNAMICS

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2)+\frac{\partial}{\partial y}(\rho uv)=0$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho v^2)=0$$

$$\Longleftrightarrow$$

$$\dot{x}=u$$

$$\dot{y}=v$$

$$\dot{u}=0\,;\,\dot{v}=0$$

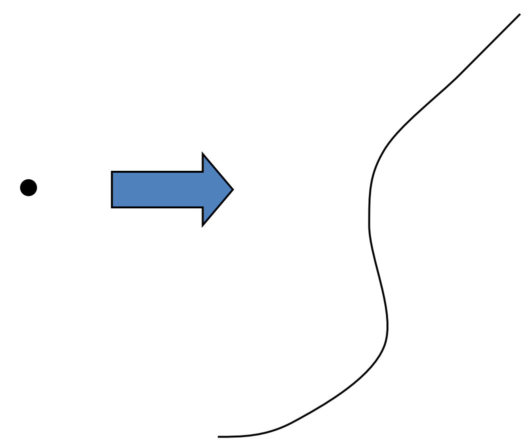
$$\dot{\rho}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

$$\mathrm{x}=\mathrm{a}+\mathrm{t}\cdot\mathrm{u}_0(\mathrm{a},\mathrm{b})$$

$$\mathrm{y}=\mathrm{b}+\mathrm{t}\cdot\mathrm{v}_0(\mathrm{a},\mathrm{b})$$

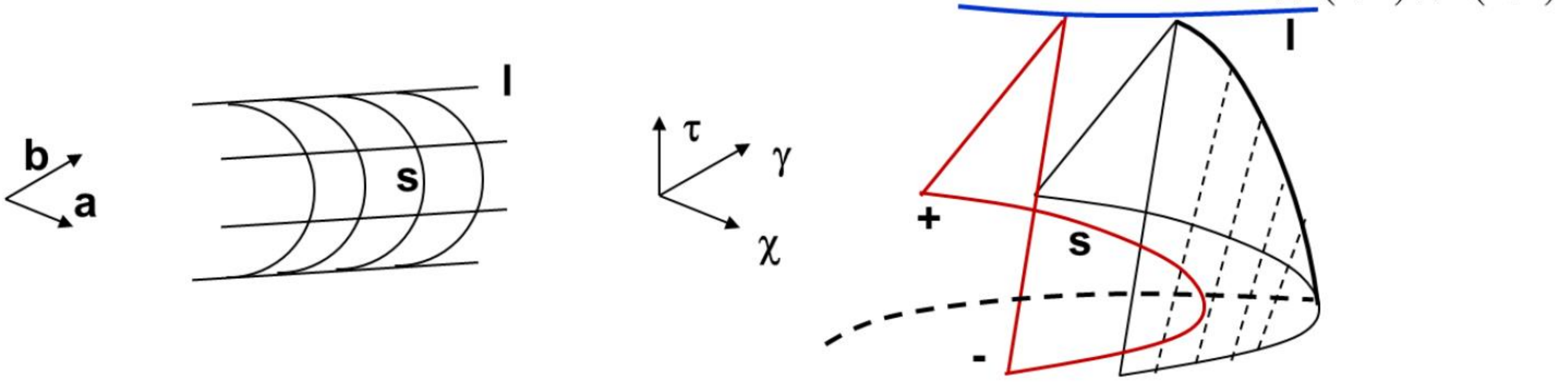
$$\mathrm{u}=\mathrm{u}_0(\mathrm{a},\mathrm{b})\,;\,\mathrm{v}=\mathrm{v}_0(\mathrm{a},\mathrm{b})$$

$$\rho=\frac{\rho_0(\mathrm{a},\mathrm{b})}{1+\mathrm{t}\cdot\mathrm{divU}_0+\mathrm{t}^2\cdot\mathrm{rotU}_0}$$

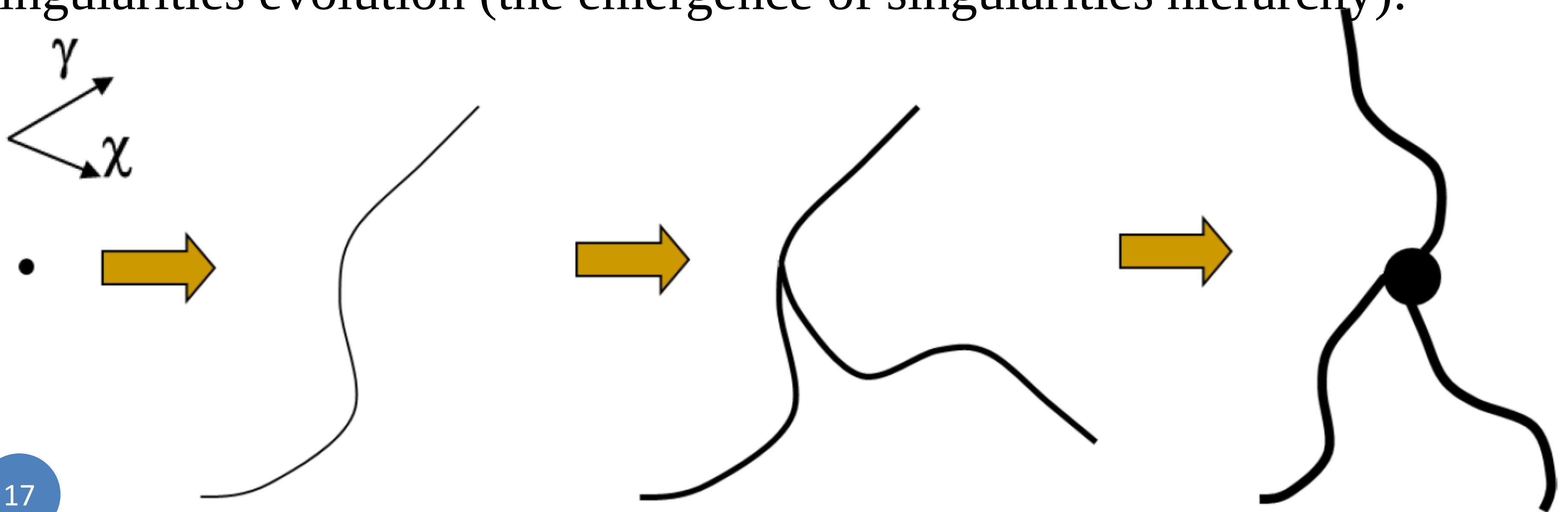


THE SINGULARITIES FORMATION

The singularities formation along the curves in R^2:



Singularities evolution (the emergence of singularities hierarchy):



RANKINE-HUGONIOT CONDITIONS

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2)_x + (\rho u v)_y = 0 \\ (\rho v)_t + (\rho u v)_x + (\rho v^2)_y = 0 \end{cases}$$

Rankine-Hugoniot conditions:

$$\dot{P}_t = \chi_l \left\{ V\left(\rho^+ - \rho^-\right) - \left(\rho^+ v^+ - \rho^- v^-\right) \right\} - \gamma_l \left\{ U\left(\rho^+ - \rho^-\right) - \left(\rho^+ u^+ - \rho^- u^-\right) \right\}$$

$$\dot{I}_t = \chi_l \left\{ V\left(\rho^+ u^+ - \rho^- u^-\right) - \left(\rho^+ u^+ v^+ - \rho^- u^- v^-\right) \right\} - \gamma_l \left\{ U\left(\rho^+ u^+ - \rho^- u^-\right) - \left(\rho^+ \left(u^+\right)^2 - \rho^- \left(u^-\right)^2\right) \right\}$$

$$\dot{J}_t = \chi_l \left\{ V\left(\rho^+ v^+ - \rho^- v^-\right) - \left(\rho^+ \left(v^+\right)^2 - \rho^- \left(v^-\right)^2\right) \right\} - \gamma_l \left\{ U\left(\rho^+ v^+ - \rho^- v^-\right) - \left(\rho^+ u^+ v^+ - \rho^- u^- v^-\right) \right\}$$

$$\dot{\chi} = U = \frac{I_t}{P_t} \quad , \quad \dot{\gamma} = V = \frac{J_t}{P_t}$$

1. Li J., Zhang T., Yang S. L. The Two-Dimensional Riemann Problem in Gas Dynamics. London.: Longman, 1998.
2. Рыков Ю.Г. Особенности типа ударных волн в среде без давления, решения в смысле теории меры и в смысле Коломбо // Препринты ИПМ им. М.В. Келдыша. 1998. №30.
3. Yu. G. Rykov. On the non-hamiltonian character of shocks in 2-D pressureless gas // Bollettino dell'Unione Matematica Italiana, 5-B,1, 55–78 (2002).

INTEGRAL FORM OF RANKINE-HUGONIOT CONDITIONS

Let fix $t > 0$. Along the singularities curves the following relations are true:

$$\int_0^t \left[\chi(t,l) - a_+(s,l) - tu_0^+(s,l)\right]\rho_0(a_+,b_+)\left[(a_+)_s(b_+)_l - (b_+)_s(a_+)_l\right]ds =$$

$$\int_0^t \left[\chi(t,l) - a_-(s,l) - tu_0^-(s,l)\right]\rho_0(a_-,b_-)\left[(a_-)_s(b_-)_l - (b_-)_s(a_-)_l\right]ds;$$

$$\int_0^t \left[\gamma(t,l) - b_+(s,l) - tv_0^+(s,l)\right]\rho_0(a_+,b_+)\left[(a_+)_s(b_+)_l - (b_+)_s(a_+)_l\right]ds =$$

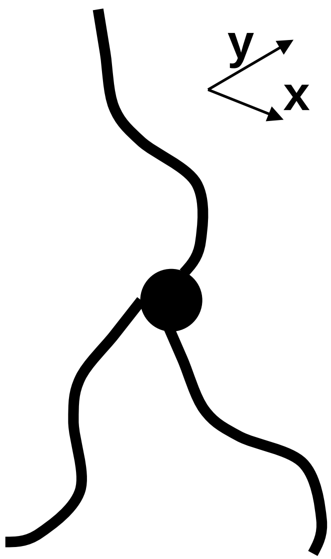
$$\int_0^t \left[\gamma(t,l) - b_-(s,l) - tv_0^-(s,l)\right]\rho_0(a_-,b_-)\left[(a_-)_s(b_-)_l - (b_-)_s(a_-)_l\right]ds.$$

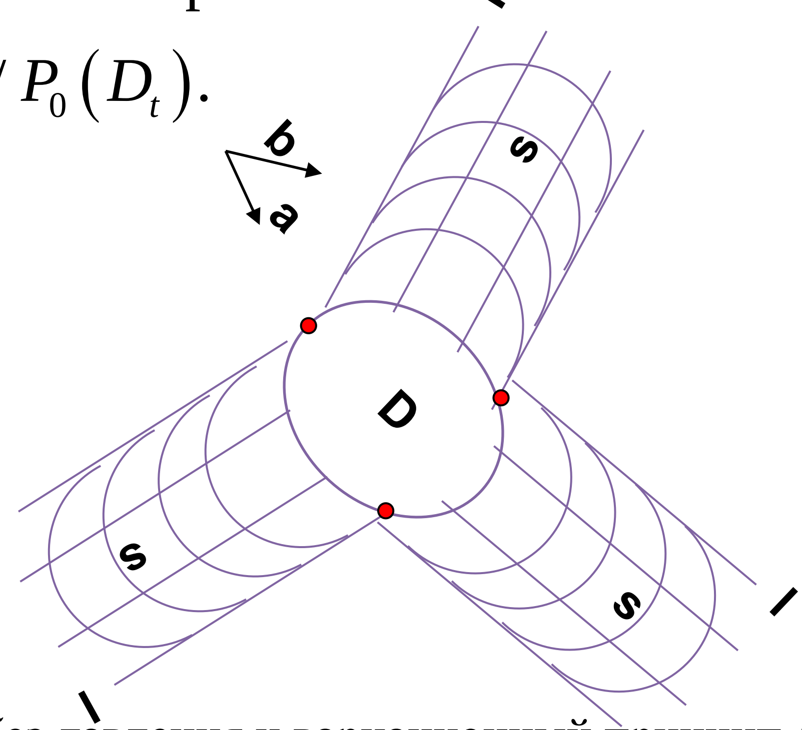
1. Рыков Ю.Г. Особенности типа ударных волн в среде без давления, решения в смысле теории меры и в смысле Коломбо // Препринты ИПМ им. М.В. Келдыша. 1998. №30.
2. Yu. G. Rykov. On the non-hamiltonian character of shocks in 2-D pressureless gas // Bollettino dell'Unione Matematica Italiana, 5-B,1, 55–78 (2002).

THE FORMATION OF POINT SINGULARITY

THEOREM. Suppose there exists the geometrical configuration shown below. The Rankine-Hugoniot conditions for curves are fulfilled. Then the following relations at a point are true

$$\left(U_{Point}, V_{Point}\right) = \left(I_0\left(D_t\right), J_0\left(D_t\right)\right) / P_0\left(D_t\right).$$





1. Рыков Ю.Г. Двумерная газовая динамика без давления и вариационный принцип // Препринты ИПМ им. М. В. Келдыша. 2016. № 94.

2. Аптекарев А. И., Рыков Ю. Г. Детализация механизма образования особенностей в системе уравнений газовой динамики без давления // ДАН, 484:6, 655–658 (2019). Aptekarev A.I., Rykov Yu.G. Detailed description of the evolution mechanism for singularities in the system of pressureless gas dynamics // Doklady Mathematics, 99:1, 79–82 (2019).

VARIATIONAL DESCRIPTION AND 2D RIEMANN PROBLEM

THE FORMULATION OF VARIATIONAL DESCRIPTION

Let $\boldsymbol{a}=(a,b)$, $\boldsymbol{x}=(x,y)$, $\boldsymbol{u}_0=(u_0,v_0)$.

Fix $t>0$. Assume A_1 is the set of disjoint domains $\{G_\alpha \subset \mathbb{R}^2\}$,

A_2 is the set of domains $\{G_\beta \subset \mathbb{R}^2\}$ of the form: $s\in\left[s_\beta^1,s_\beta^2\right]$,

$l\in\left[l_\beta^1,l_\beta^2\right]$, $G_\beta=\left\{\boldsymbol{a}_\beta(s,l), s\in\left[s_\beta^1,s_\beta^2\right], l\in\left[l^1,l^2\right]\subset\left[l_\beta^1,l_\beta^2\right]\right\}$,

and $B=A_1\cup A_2\cup\left\{\text{the set of all domains in } \mathbb{R}^2\setminus(A_1\cup A_2)\right\}$.

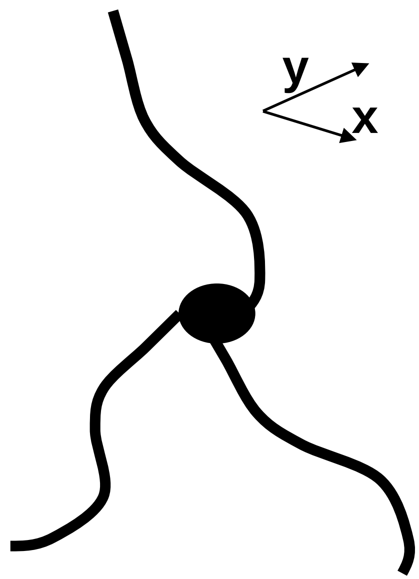
Let $\boldsymbol{F}(t,x;G)=\iint\limits_G\left[\boldsymbol{u}_0(\boldsymbol{a})-\dfrac{\boldsymbol{x}-\boldsymbol{a}}{t}\right]P_0(d\boldsymbol{a})$, $G\in B$. Define $\delta\boldsymbol{F}/\delta\boldsymbol{a}$:

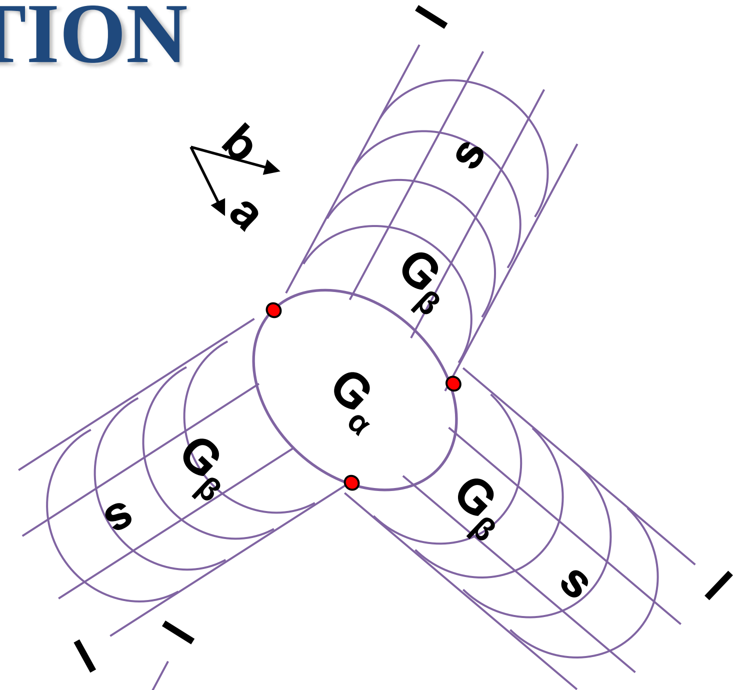
$$\frac{\delta\boldsymbol{F}}{\delta\boldsymbol{a}}=\lim_{|G|\downarrow\min}\frac{1}{|G|}\iint\limits_G\left[\boldsymbol{u}_0(\boldsymbol{a})-\frac{\boldsymbol{x}-\boldsymbol{a}}{t}\right]P_0(d\boldsymbol{a}),\ \boldsymbol{a}\in G\in B.$$

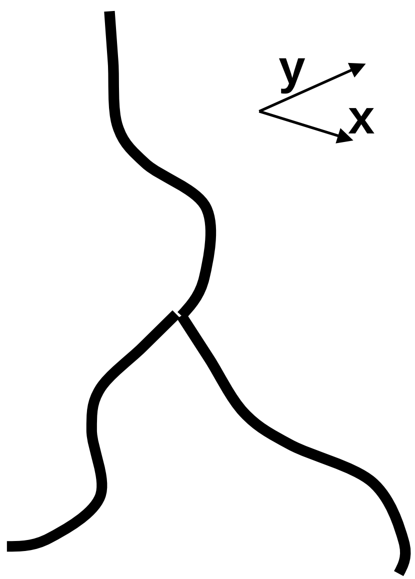
Then for any $t>0$ the generalized solution is characterized by

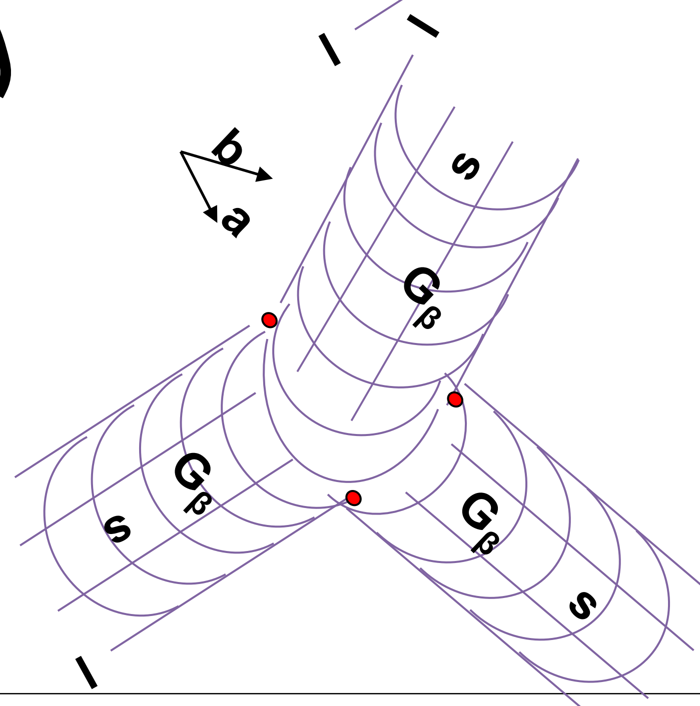
the relation: $\forall\boldsymbol{a}\in\mathbb{R}^2\ \exists\boldsymbol{x}:\dfrac{\delta\boldsymbol{F}(t,x;G)}{\delta\boldsymbol{a}}=0.$

EXPLANATORY PICTIRE TO VARIATIONAL DESCRIPTION









SELF-SIMILAR SOLUTIONS (1)

Introduce the notations: $\boldsymbol{\chi} = (\chi, \gamma)$; $\boldsymbol{u} = (u, v)$; $\boldsymbol{U} = (U, V)$;

$\boldsymbol{I}_t = (I_t, J_t) = \boldsymbol{U} \cdot P_t$. Then Rankine-Hugoniot conditions read:

$$\begin{cases} \dot{P}_t = \chi_l \{V[\rho] - [\rho v]\} - \gamma_l \{U[\rho] - [\rho u]\} \\ \dot{\boldsymbol{I}}_t = \chi_l \{V[\rho \boldsymbol{u}] - [\rho v \boldsymbol{u}]\} - \gamma_l \{U[\rho \boldsymbol{u}] - [\rho u \boldsymbol{u}]\} \\ \dot{\boldsymbol{\chi}} = \boldsymbol{U} \end{cases} \qquad [f] \equiv f^+ - f^-$$

This system has the first integral:

$$P_t \{u^+ v^- - u^- v^+ + [v]U - [u]V\} = C(l).$$

Entropy conditions:

$\boldsymbol{N} \equiv (\chi_l, -\gamma_l)$; $\boldsymbol{u}^+ \cdot N < \boldsymbol{U} \cdot N < \boldsymbol{u}^- \cdot N$.

Introduce the self-similar variables:

$$\boldsymbol{X} = (X, Y); \boldsymbol{\chi} = t\boldsymbol{X}(l/t), P_t = tm(l/t).$$

SELF-SIMILAR SOLUTIONS (2)

Self-similar form ($\bar{l} \equiv l/t$, 'prime' is the differentiation w.r.t. $\bar{l}$):

$$\begin{cases} \boldsymbol{X}' = \dfrac{\boldsymbol{X} - \boldsymbol{U}}{\bar{l}} \\ m' = \dfrac{m}{\bar{l}} - \dfrac{1}{\bar{l}^2}\left\{\rho^+ d^+ - \rho^- d^-\right\} \\ \boldsymbol{U}' = \dfrac{1}{m\bar{l}^2}\left\{\rho^+ d^+\left(\boldsymbol{U} - \boldsymbol{u}^+\right) - \rho^- d^-\left(\boldsymbol{U} - \boldsymbol{u}^-\right)\right\} \end{cases},$$

$d^\pm \equiv \left|\left(\boldsymbol{X} - \boldsymbol{U}\right) \quad \left(\boldsymbol{U} - \boldsymbol{u}^\pm\right)\right|$ and the first integral is

$$m \cdot \left|\left(\boldsymbol{u}^+ - \boldsymbol{u}^-\right) \quad \left(\boldsymbol{U} - \boldsymbol{u}^-\right)\right| = C\bar{l}.$$

The entropy conditions are $d^+ > 0, d^- < 0.$

Initial data: $\boldsymbol{X}\left(\bar{l}_0\right) = \boldsymbol{X}_0; m\left(\bar{l}_0\right) = m_0 > 0; \boldsymbol{U}\left(\bar{l}_0\right) = \boldsymbol{U}_0$.

GENERAL FORM OF RIEMANN PROBLEM SOLUTIONS

THEOREM. For general piecewise constant initial data

$\boldsymbol{u}_i, \rho_i \geq 0, i = 1, 2, 3, 4$ the solution of Riemann problem has the form:

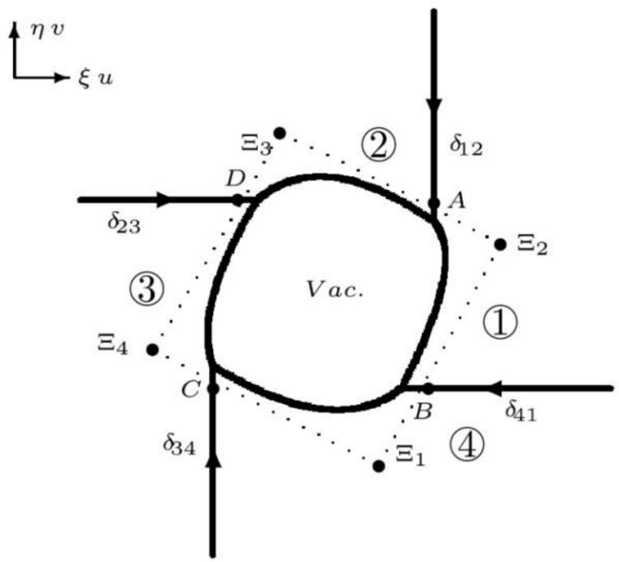


Fig. 4.3. The solution for Case 4.2(i).

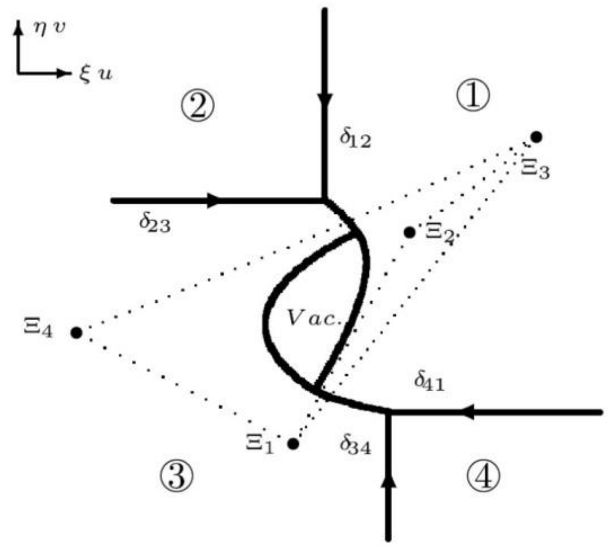


Fig. 4.6. The solution for Case 4.2(iii)b.

1. Li J., Zhang T., Yang S. L. The Two-Dimensional Riemann Problem in Gas Dynamics. London.: Longman, 1998.

2. Y. Pang, The Riemann problem for two-dimensional zero-pressure Euler equations // J. Math. Anal. Appl. 472, 2 (2019).

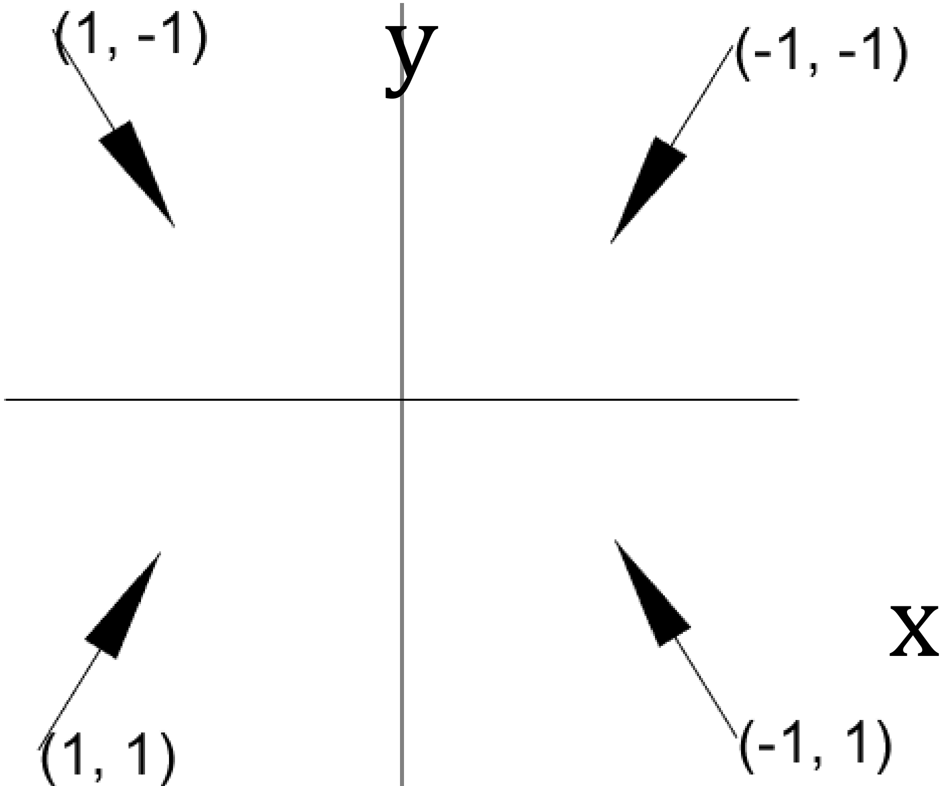
"Heavy" point is possible.

3. А.И. Аптекарев, Ю.Г. Рыков. Возникновение иерархии особенностей в средах без собственного перепада давления. Двумерный случай // Математические заметки, т. **112**, вып. 4 (2022), 486 – 499. A.I. Aptekarev, Yu.G. Rykov. Emergence of a hierarchy of singularities in zero-pressure media. 2D case // Math. Notes, 112:4 (2022), 495–504.

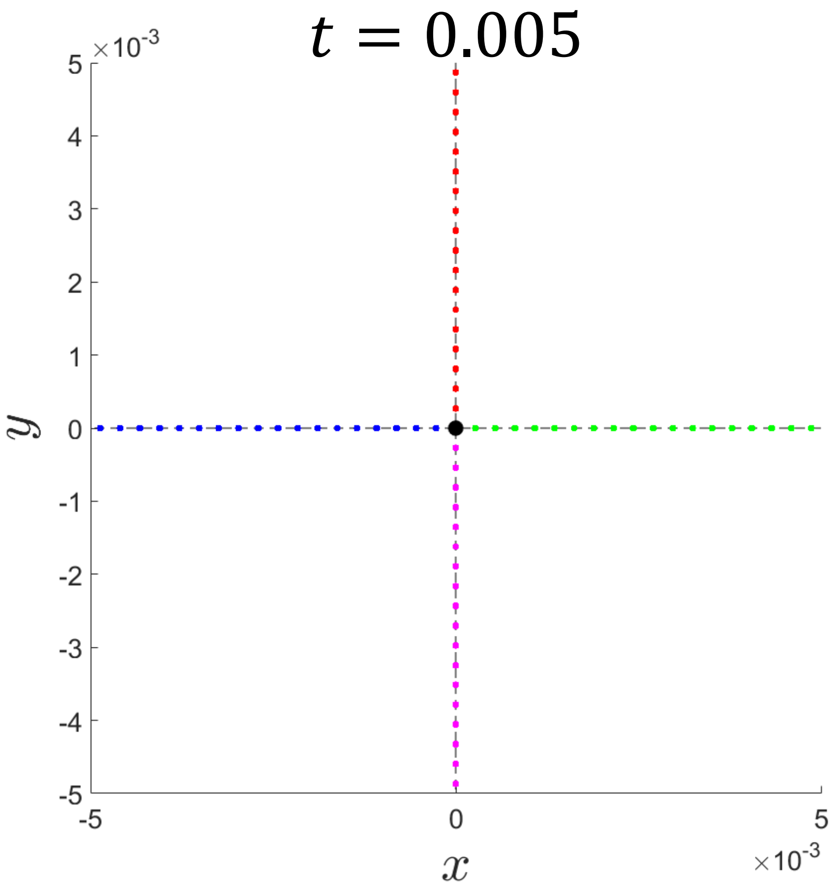
NUMERICAL CALCULATIONS,
THE EXISTENCE OF SHOCKS HIERARCHY

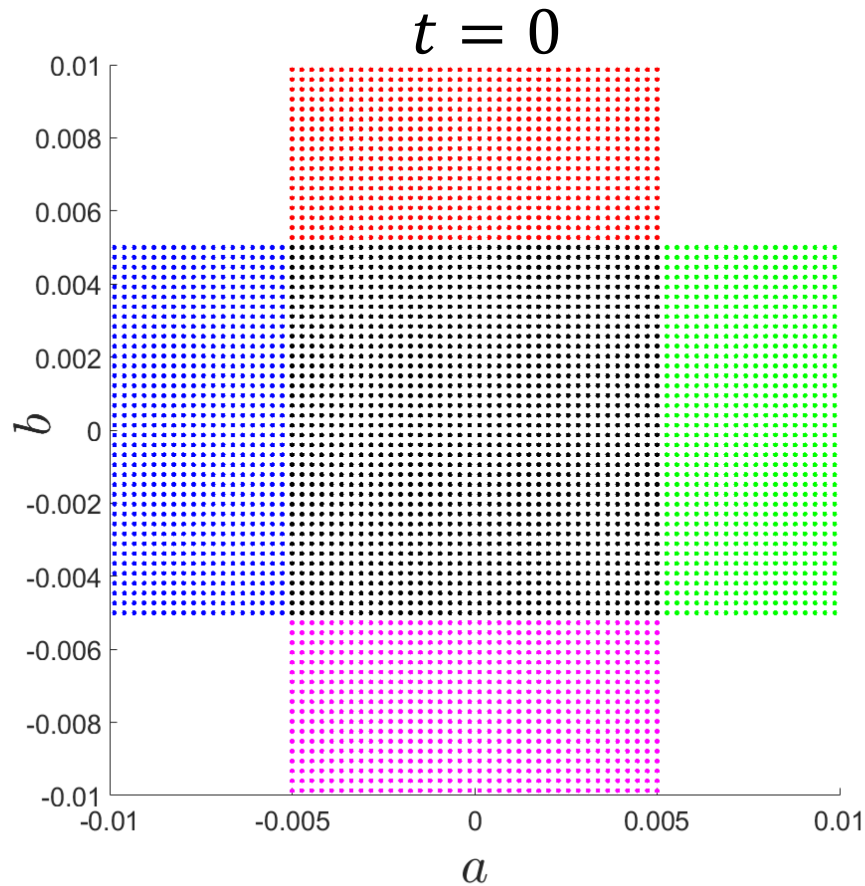
EXAMPLE 1

The calculations are due to N.V. Klyushnev



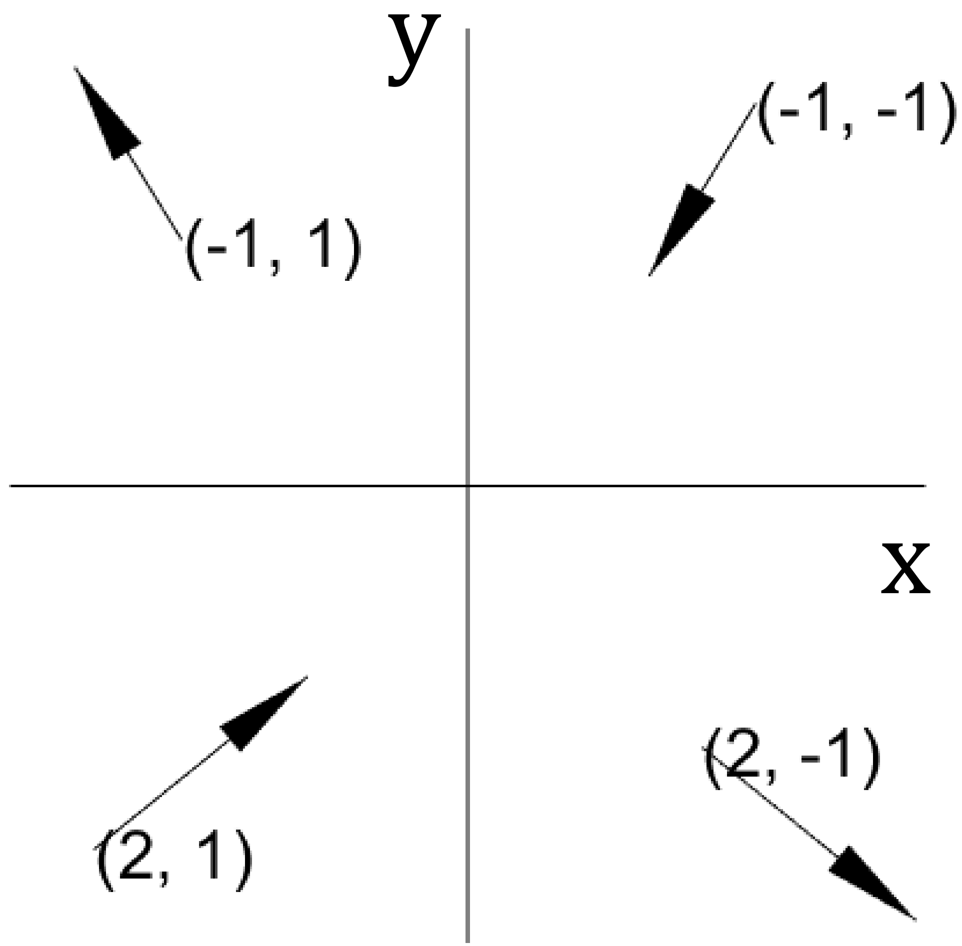
1. A. Chertock, A. Kurganov, Yu. Rykov. A new sticky particle method for pressureless gas dynamics // SIAM Journal on Numerical Analysis. 45, 2408-2441 (2007).
2. Клюшнев Н.В., Рыков Ю.Г. О модельных двумерных течениях газа без давления: вариационное описание и численный алгоритм в рамках динамики прилипания // Журн. вычисл. матем. и матем. физ., т. **63**, № 4, 2023, с. 639 – 656. N.V. Klyushnev, Yu.G. Rykov. On model 2D pressureless gas flows: variational description and numerical algorithm based on adhesion dynamics // Comp. Math. And Math. Phys., 63:4, 606–622 (2023).

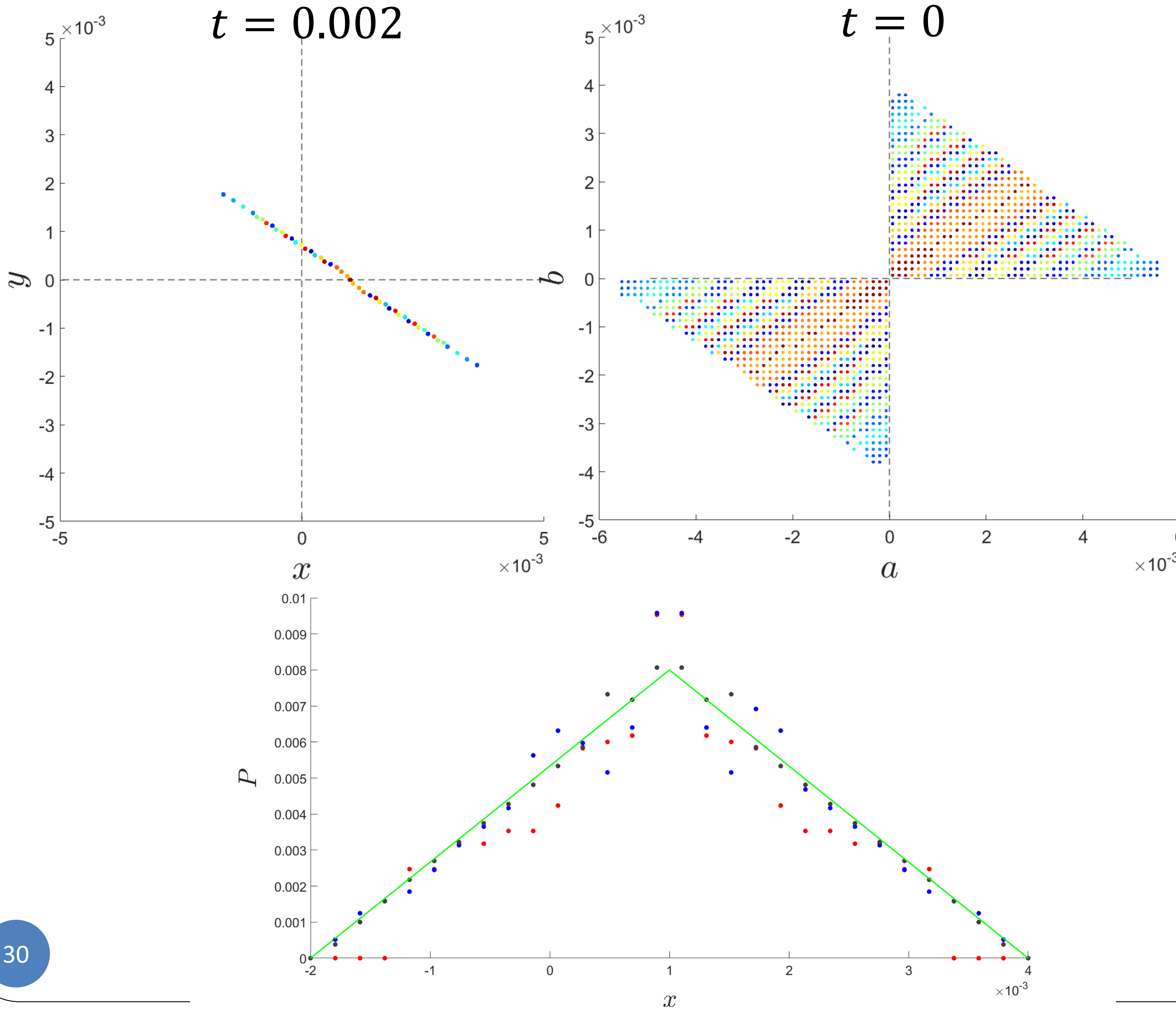




N	M_h	ρ_{12}	ρ_{23}
75	9.47×10^{-5}	9.73×10^{-3}	9.73×10^{-3}
150	9.74×10^{-5}	9.87×10^{-3}	9.87×10^{-3}
299	9.87×10^{-5}	9.93×10^{-3}	9.93×10^{-3}
Теория	1×10^{-4}	1×10^{-2}	1×10^{-2}

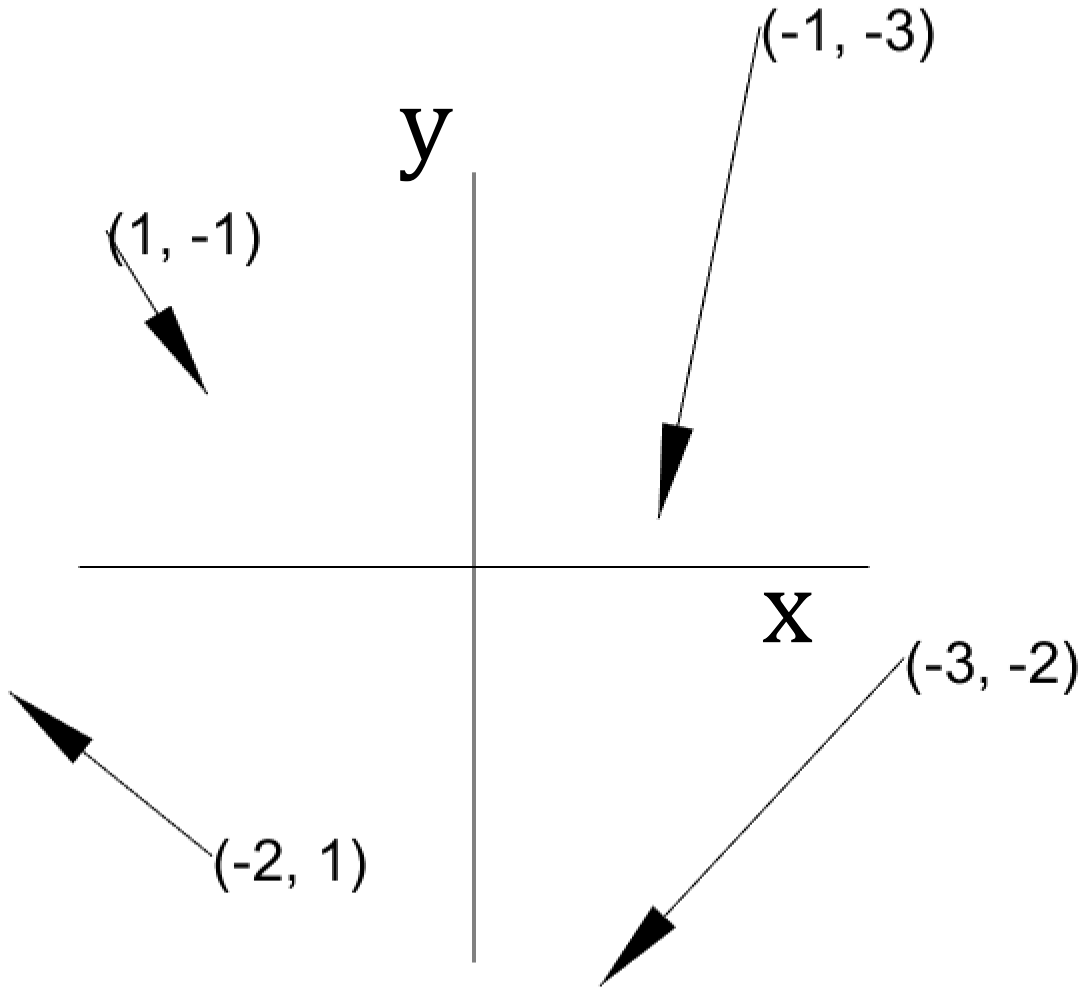
EXAMPLE 2

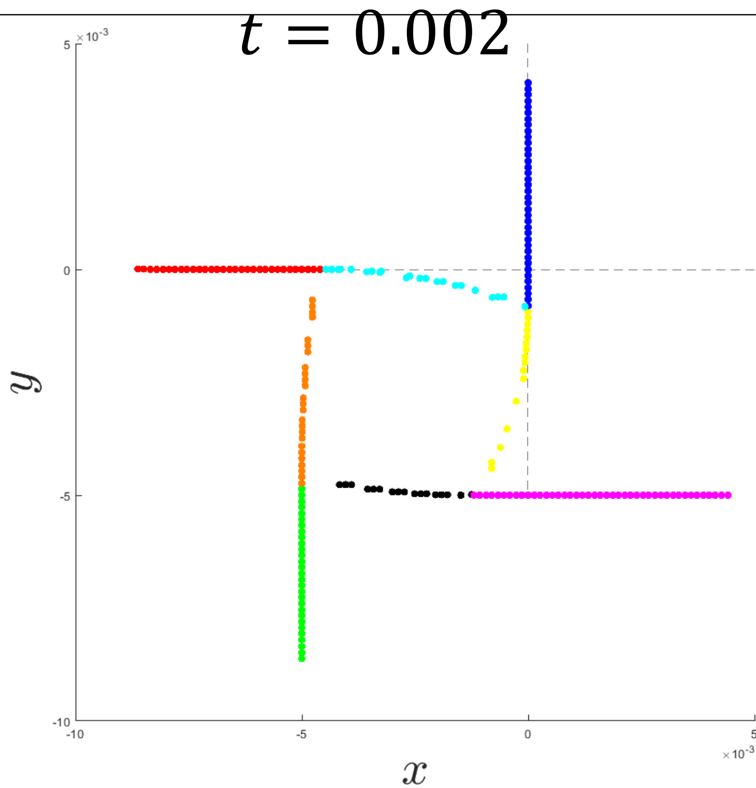


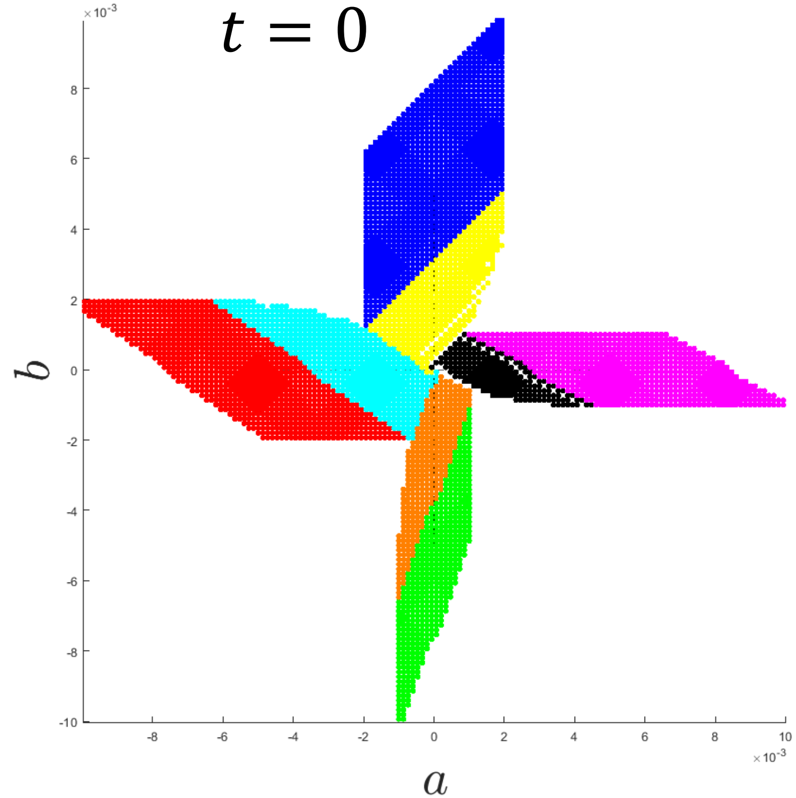


$t = 0.002$
$t = 0$
y
x
b
a
$\times 10^{-3}$
P

EXAMPLE 3







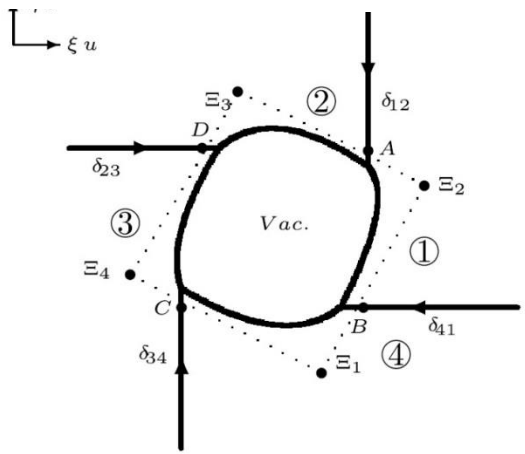
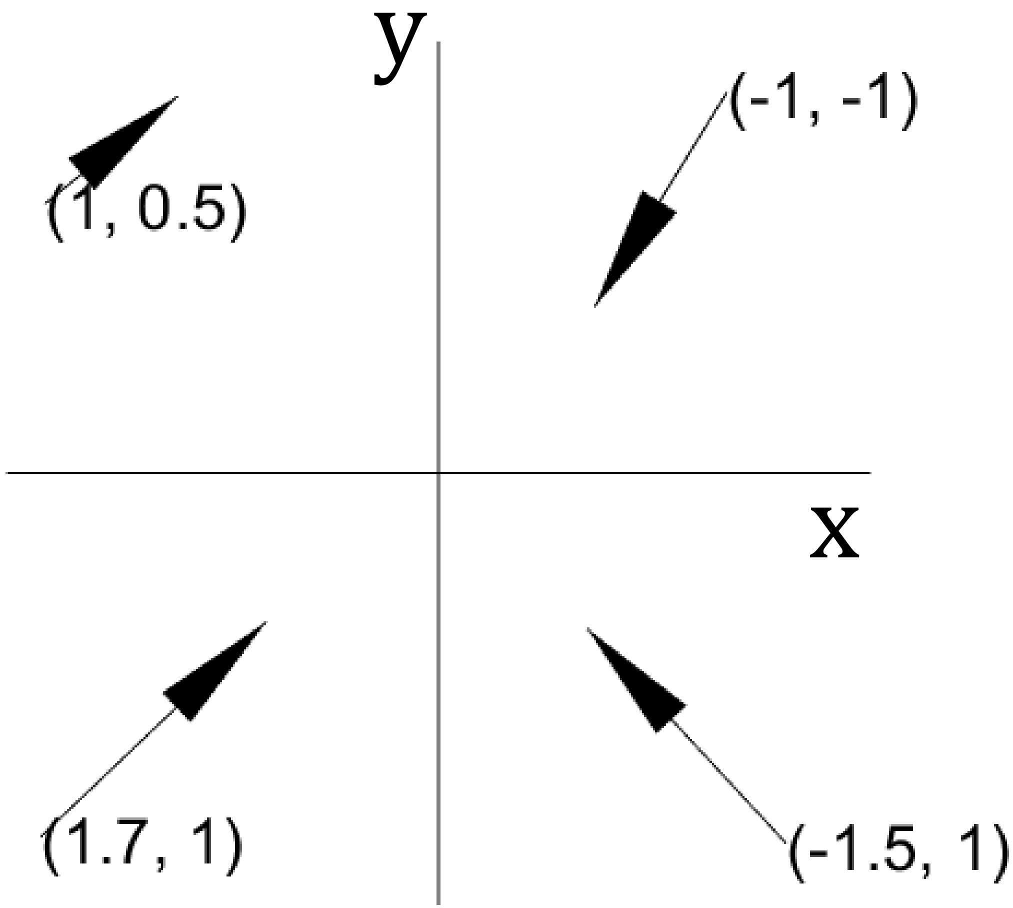


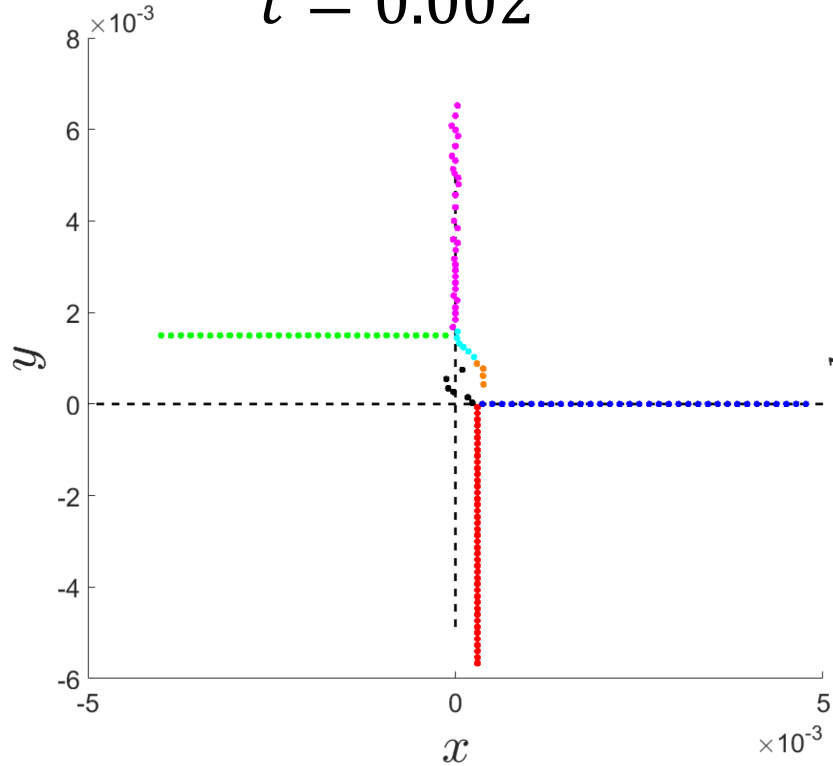
Fig. 4.3. The solution for Case 4.2(i).

…ang, The Riemann problem for two-dimensional zero-pressure Euler equations // J. Math. …nal. Appl. 472, 2 (2019).

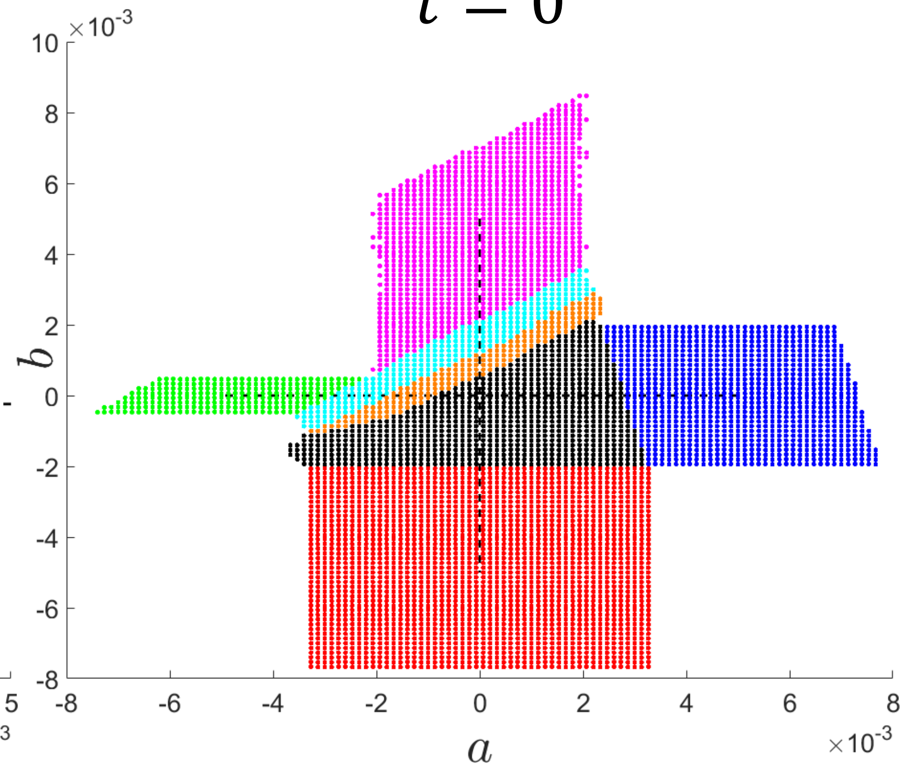
EXAMPLE 4



$t = 0.002$



$t = 0$



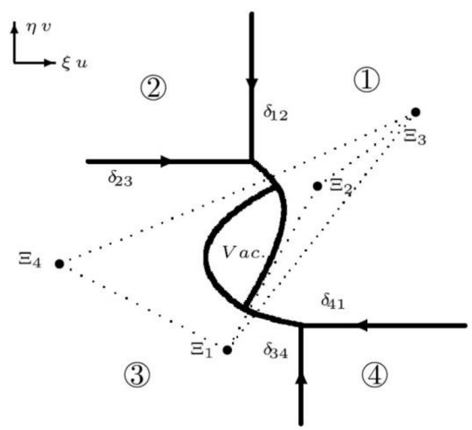
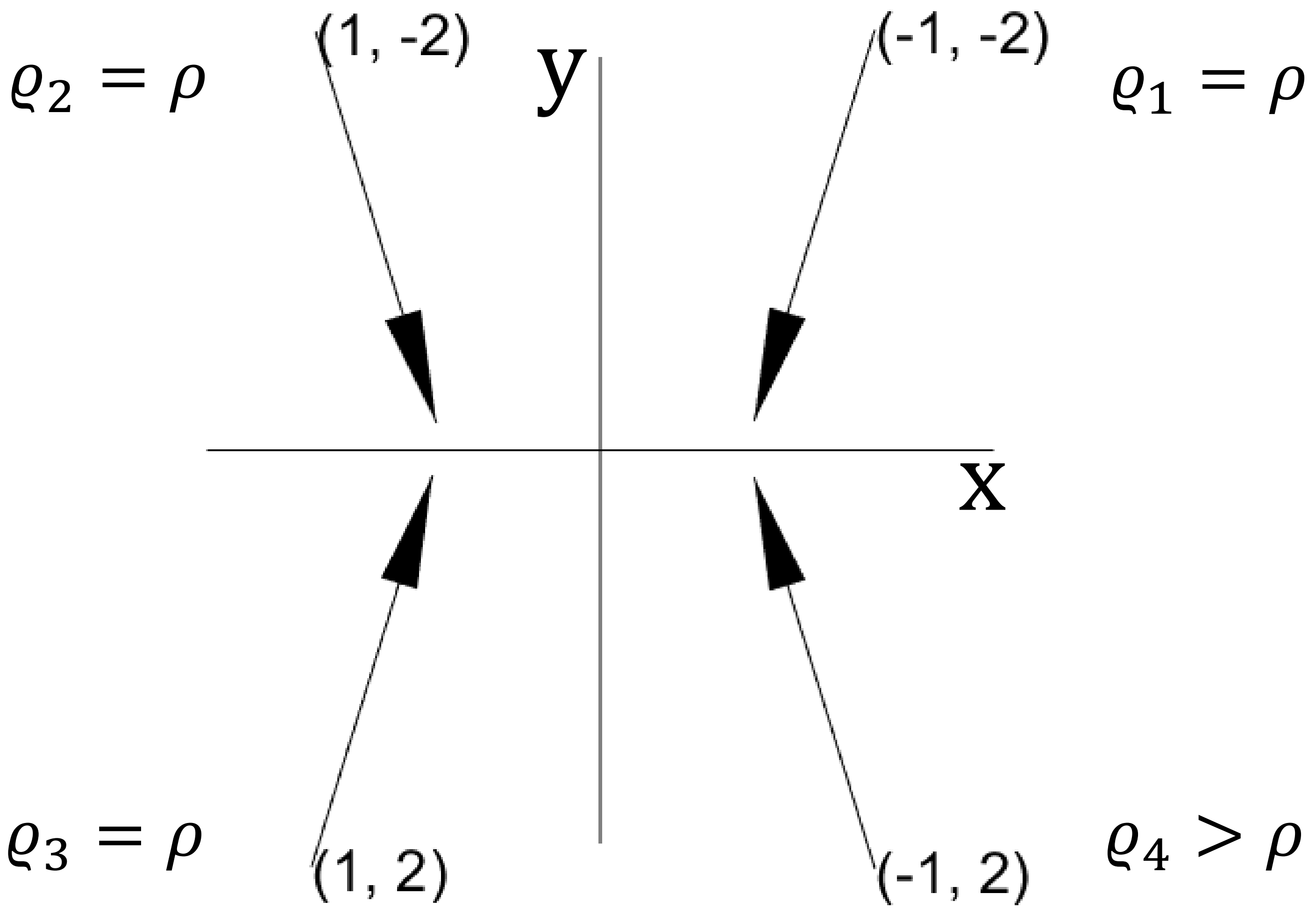


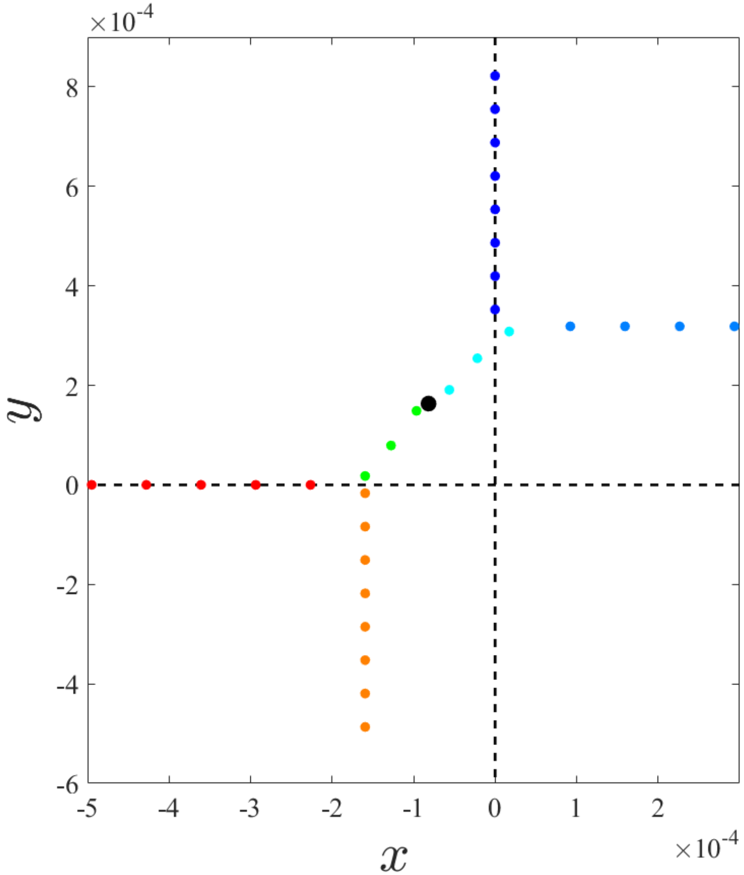
Fig. 4.6. The solution for Case 4.2(iii)b.

ang, The Riemann problem for two-dimensional zero-pressure Euler equations // J. Math. nal. Appl. 472, 2 (2019).

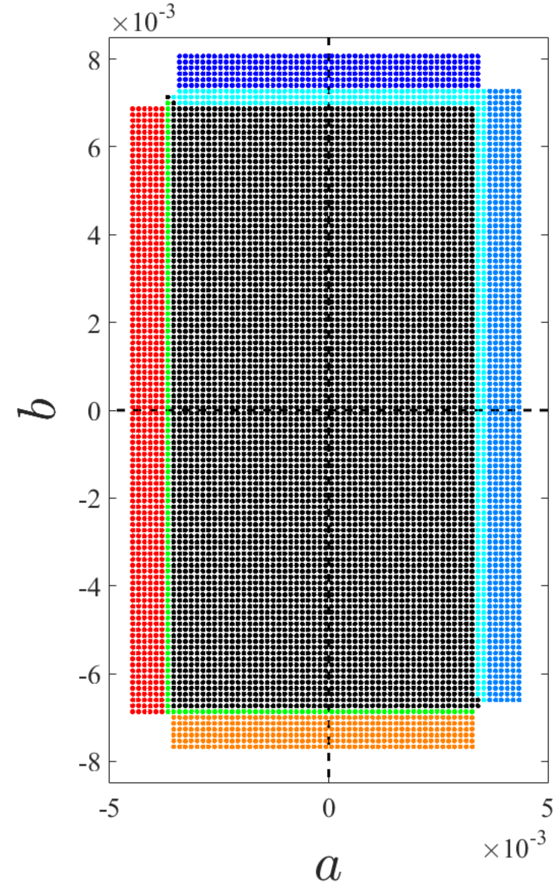
EXAMPLE 5
(DYNAMIC HIERARCHY OF SHOCKS)



$t = 0.0035$



$t = 0$



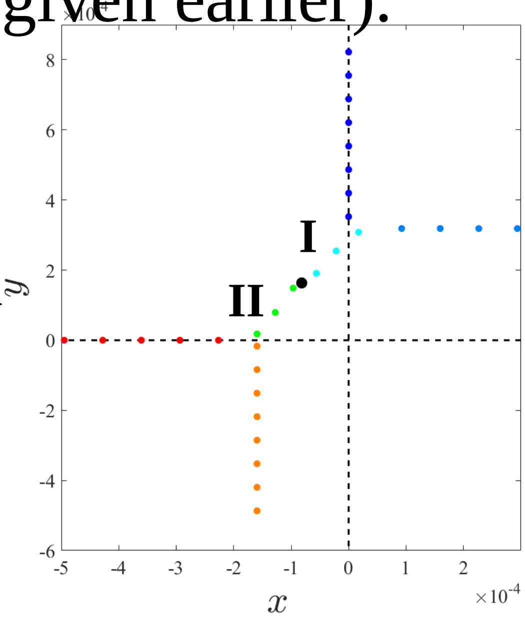
N	M_h	$\Im_h^x$	$\Im_h^y$	M_1	$\Im_1^x$	$\Im_1^y$	M_2	$\Im_2^x$	$\Im_2^y$
150	9.983	-0.08782	0.7232	0.2880	0.1813	0.1991	0.6805	-0.3996	-0.5163
299	9.781	-0.2633	0.4330	0.5324	0.2622	0.5063	0.5380	-0.3029	-0.4339
Теория	9.805	-0.2282	0.4565	0.4626	0.2070	0.4788	0.4626	-0.2394	-0.4139

THE EXISTENCE OF SHOCKS HIERARCHY (1)

Ю.Г. Рыков. Об эволюции иерархии ударных волн в двумерной изобарической среде // Известия РАН, принято к публикации (2023).

THEOREM. Let $u > 0, v > 0, \rho > 0, R > \rho$ are constants. Consider the Riemann data: $u_1 = u_4 = -u; u_2 = u_3 = u; v_1 = v_2 = -v; v_3 = v_4 = v;$ $\rho_i = \rho, i \neq 4; \rho_4 = R$. Then there exists such line $\boldsymbol{x} = t\boldsymbol{X}^*$ that for any $t > 0$ the measure P_t has delta function at point $\boldsymbol{x}$. At this the point $\boldsymbol{X}^*$ is the intersection point of curves that are under Rankine-Hugoniot conditions (satisfy the PDE system given earlier).

SKETCH OF THE PROOF. The aimed configuration of the solution is shown in the picture on the right. The main question is: **whether the shock waves I and II intersect providing the fulfillment of entropy conditions?**



THE EXISTENCE OF SHOCKS HIERARCHY (2)

Two systems describe the shock waves I and II respectively:

$$\begin{cases} \boldsymbol{X}_1' = \dfrac{\boldsymbol{X}_1 - \boldsymbol{U}_1}{\bar{l}} \quad , \quad \boldsymbol{X}_{10} = \left(0, \lambda(\rho, R) v\right) \\ m_1' = \dfrac{m_1}{\bar{l}} - \dfrac{1}{\bar{l}^2}\left\{\rho d_1^{\,+} - R d_1^{\,-}\right\} \quad , \quad m_{10}\,, \boldsymbol{U}_{10} \\ \boldsymbol{U}_1' = \dfrac{1}{m\bar{l}^2}\left\{\rho d_1^{\,+}\left(\boldsymbol{U} + (-u, v)\right) - R d_1^{\,-}\left(\boldsymbol{U} + (u, -v)\right)\right\} \end{cases} ;$$

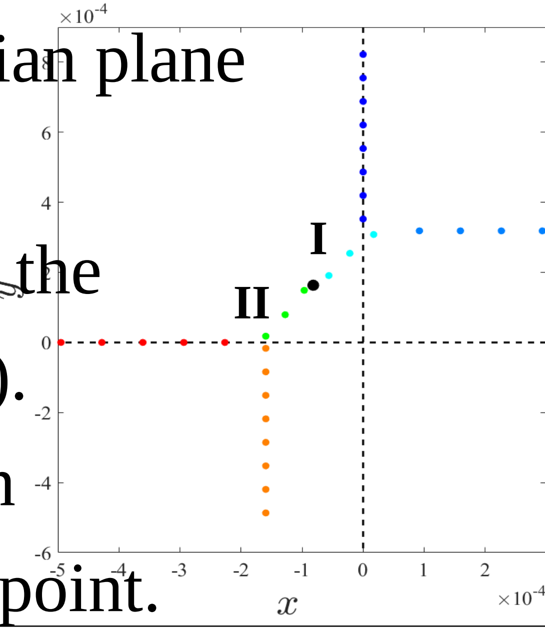
$$\begin{cases} \boldsymbol{X}_2' = \dfrac{\boldsymbol{X}_2 - \boldsymbol{U}_2}{\bar{l}} \quad , \quad \boldsymbol{X}_{20} = \left(-\lambda(\rho, R) u, 0\right) \\ m_2' = \dfrac{m_2}{\bar{l}} - \dfrac{1}{\bar{l}^2}\left\{R d_2^{\,+} - \rho d_2^{\,-}\right\} \quad , \quad m_{20}\,, \boldsymbol{U}_{20} \\ \boldsymbol{U}_2' = \dfrac{1}{m\bar{l}^2}\left\{R d_2^{\,+}\left(\boldsymbol{U} + (u, -v)\right) - \rho d_2^{\,-}\left(\boldsymbol{U} + (-u, v)\right)\right\} \end{cases} .$$

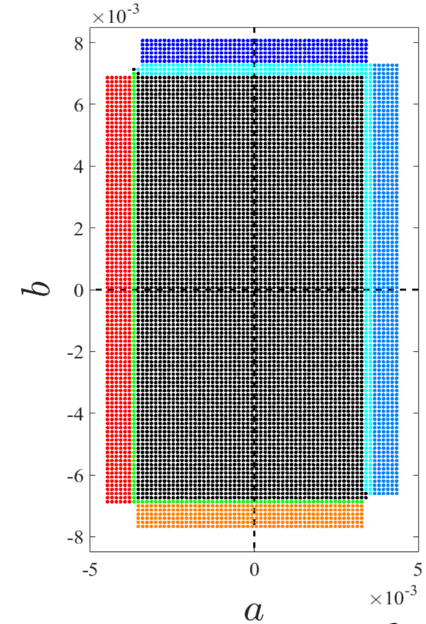
THE EXISTENCE OF SHOCKS HIERARCHY (3)

It can be shown that the shock waves I and II intersect the line $L: \boldsymbol{X}\cdot(v,u)=0$. Consider the following transformation

$$X_2=-\frac{u}{v}Y_1\,,Y_2=-\frac{v}{u}X_1\,,m_2=m_1\,,U_2=-\frac{u}{v}V_1\,,V_2=-\frac{v}{u}U_1.$$

Under this transformation the systems that correspond to shocks I and II turn to each other. And so do the initial conditions. At the same time L stays in place. Thus I and II intersects at a point belonging to L. In lagrangian plane the areas corresponded to shocks I and II encompass the whole domain (grey color). Hence all this grey domain should concentrate in one point.





THE USAGE IN ASTROPHYSICS – THE UNIVERSE AT LARGE SCALES

EXAMPLE 6
(RANDOM INITIAL DISTRIBUTION)

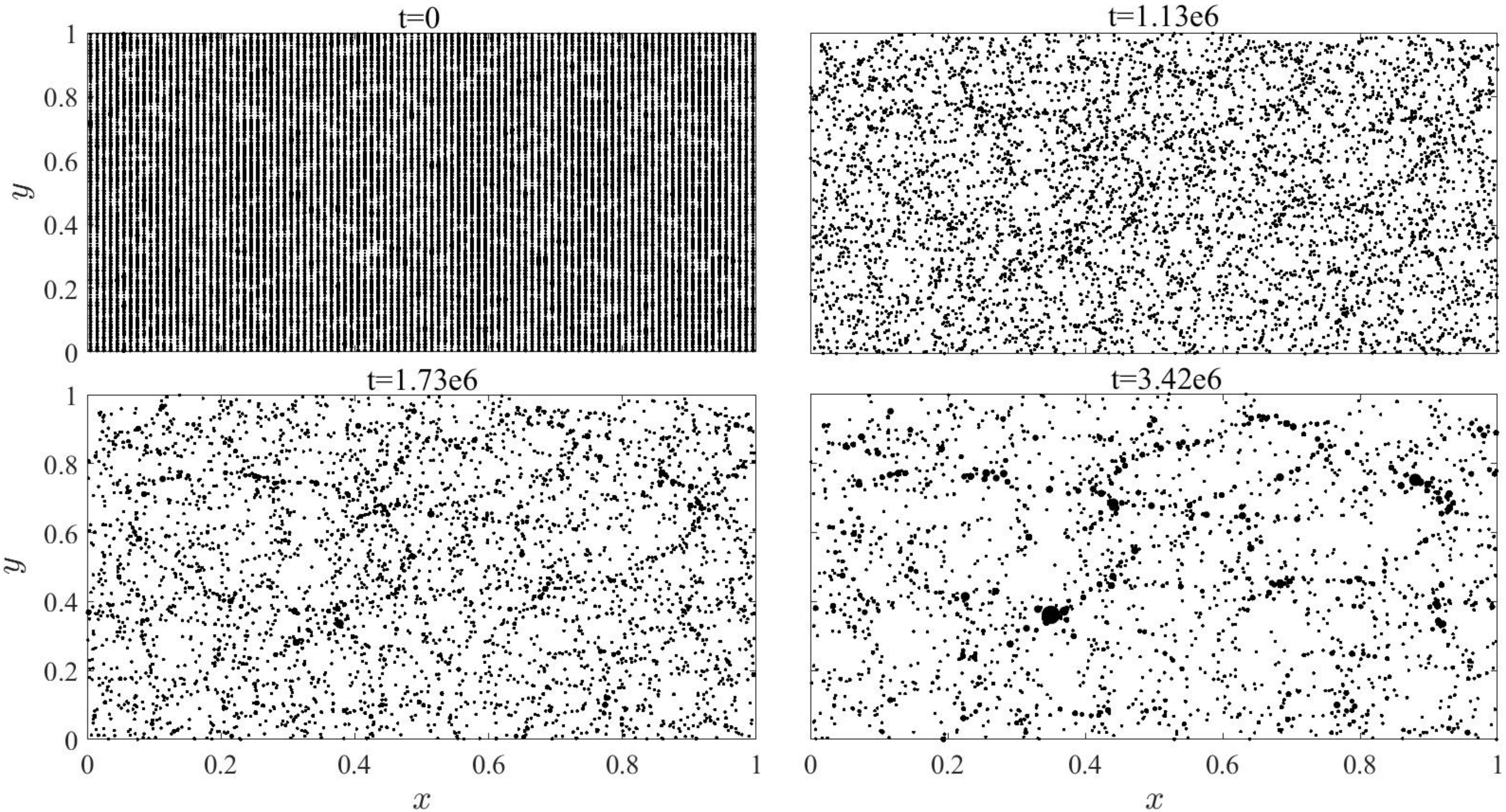
$$(u, v, \varrho):$$

$$\varrho = N(1,0.3),$$

$$\bar{\varrho} = \frac{1}{|\Omega|} \int_{\Omega} \varrho dxdy,$$

$$\begin{cases} u = -\phi_x, \\ v = -\phi_y, \\ \Delta\phi = 4\pi G(\varrho - \bar{\varrho}) \end{cases}$$

CALCULATION RESULTS



THANK YOU FOR ATTENTION