On estimates for the energy levels of a quantum billiard

Valerii A. Pchelintsev
Tomsk State University

III International Conference "Mathematical Physics, Dynamical Systems, Infinite-Dimensional Analysis"
Dolgoprudny, Russia
July 05-13, 2023

[0] This talk is based on the joint work with Vladimir Gol'dshtein and Alexander Ukhlov

1. Dirichlet spectral problem.

Main problem:

Lower and upper estimates of the ground state energy of a quantum billiard (a bounded planar domain $\Omega \subset \mathbb{R}^2$).

The problem for a quantum billiard:

$$-\frac{\hbar}{2m}\Delta u_n = E_n u_n, \quad u_n|_{\partial\Omega} = 0,$$

where $u_n(x,y)$ is the wave function and E_n is the energy of a particle in the billiard with the boundary $\partial\Omega$;
$\Delta = \partial_{xx} + \partial_{yy}$ is the two-dimensional Laplace operator.

The classical Dirichlet spectral problem for the Laplacian:

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega \\ u = 0 & \text{on} \ \ \partial\Omega. \end{cases}$$

The weak statement of this spectral problem: $u \in W_0^{1,2}(\Omega)$ solves the previous problem iff

$$\int\limits_{\Omega} \nabla u(x) \cdot \nabla v(x)\ dx = \lambda \int\limits_{\Omega} u(x)v(x)\ dx$$

$\forall v \in W_0^{1,2}(\Omega)$.

By the min-max principle the first eigenvalue of the Dirichlet-Laplacian can be represented as

$$\lambda_1(\Omega) = \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int\limits_\Omega |\nabla u(x)|^2 \, dx}{\int\limits_\Omega |u(x)|^2 \, dx}.$$

Moreover, $\lambda_1(\Omega)^{-\frac{1}{2}}$ is equal to the best constant $A_{2,2}(\Omega)$ in the following Poincaré-Sobolev inequality

$$\left(\int\limits_\Omega |u(x)|^2 \, dx\right)^{\frac{1}{2}} \leq A_{2,2}(\Omega) \left(\int\limits_\Omega |\nabla u(x)|^2 \, dx\right)^{\frac{1}{2}}, \quad u \in W_0^{1,2}(\Omega).$$

The Sobolev space $W^{1,p}(\Omega)$, $1 \leq p < \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^{1,p}(\Omega)\| = \left(\int\limits_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left(\int\limits_{\Omega} |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

The Sobolev space $W_0^{1,p}(\Omega)$, $1 \leq p < \infty$, is the closure in the $W^{1,p}(\Omega)$-norm of the space $C_0^{\infty}(\Omega)$ of all infinitely continuously differentiable functions with compact support in Ω.

Short historical review.

Explicit values of $\lambda(\Omega)$ are known only for several particular domains. For example:

- Rectangle $a \times b$: $\lambda_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \quad m, n = 1, 2, \ldots;$
- Disc of the radius R: $\lambda_{m,n} = \left(\frac{j_{m,n}}{R}\right)^2, \quad m = 0, 1, 2, \ldots, \; n = 1, 2, \ldots;$

$j_{m,n}$ is the nth zero of the mth Bessel function J_m.

- Annulus $a \le r \le b$: $\lambda_{m,n} = \left(\frac{k_{m,n}}{a}\right)^2, \quad m = 0, 1, 2, \ldots, \; n = 1, 2, \ldots;$

$k_{m,n}$ is the nth root of

$$Y_m(k_{m,n}) J_m\left(\frac{k_{m,n} b}{a}\right) - J_m(k_{m,n}) Y_m\left(\frac{k_{m,n} b}{a}\right) = 0.$$

- Rayleigh-Faber-Krahn inequality:

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) = \frac{(j_{0,1})^2}{R_*^2},$$

where $j_{0,1} \approx 2.4048$ is the first positive zero of the Bessel function J_0 and Ω^* is a disc of the same area as Ω with R_* as its radius.

- Payne-Weinberger inequality:

$$\lambda_1(\Omega) \leq \frac{\pi j_{0,1}^2}{|\Omega|} \left[1 + \left(\frac{1}{J_1^2(j_{0,1})} - 1\right)\left(\frac{|\partial\Omega|^2}{4\pi|\Omega|} - 1\right)\right],$$

where Ω is a simply connected planar domain, $|\Omega|$ is the Lebesgue measure of Ω, $|\partial\Omega|$ is the Hausdorff measure of the boundary of Ω and J_1 denotes the Bessel function.

2. Main results.

We suggest the method is based on the following diagram proposed in (V. Gol'dstein and L. Gurov, 1994):

$$\begin{array}{ccc} L^{1,2}(\Omega) & \xrightarrow{\varphi^*} & L^{1,2}(\Omega') \\ \downarrow & & \downarrow \\ L^s(\Omega) & \xleftarrow{(\varphi^{-1})^*} & L^r(\Omega'). \end{array}$$

φ^* is a bounded composition operator on Sobolev spaces, $\varphi^*(u) = u \circ \varphi$;

$(\varphi^{-1})^*$ is a bounded composition operator on Lebesgue spaces, $(\varphi^{-1})^*(v) = v \circ \varphi^{-1}$.

$$\|u \mid L^s(\Omega)\| \leq A_{s,2}(\Omega) \|\nabla u \mid L^2(\Omega)\|, \; u \in W_0^{1,2}(\Omega).$$

Recall that:

- For any conformal homeomorphism $|\varphi'(z)|^2 = J(z,\varphi) > 0$.
- Any conformal homeomorphism $\varphi : \Omega' \to \Omega$ generates an isometry

$$\varphi^* : L^{1,2}(\Omega) \to L^{1,2}(\Omega').$$

Indeed, for $u \in L^{1,2}(\Omega)$ we have:

$$\|\varphi^*(u) \mid L^{1,2}(\Omega')\| = \left(\int\limits_{\Omega'} |\nabla (u \circ \varphi(z))|^2 \, d\nu \right)^{\frac{1}{2}}$$

$$= \left(\int\limits_{\Omega'} |\nabla u|^2(\varphi(z)) \cdot |\varphi'(z)|^2 \, d\nu \right)^{\frac{1}{2}} = \left(\int\limits_{\Omega'} |\nabla u|^2(\varphi(z)) \cdot J(z,\varphi) \, d\nu \right)^{\frac{1}{2}}$$

$$= \left(\int\limits_{\Omega} |\nabla u|^2(w) \, d\nu \right)^{\frac{1}{2}} = \|u \mid L^{1,2}(\Omega)\|.$$

Estimates of a norm of a composition operator

$$(\varphi^{-1})^* : L^r(\Omega') \to L^s(\Omega)$$

generated by the composition rule $(\varphi^{-1})^*(v) = v \circ \varphi^{-1}$ are based on the notion of conformal regular domains.

A domain $\Omega \subset \mathbb{R}^2$ is called a conformal α-regular about a domain $\Omega' \subset \mathbb{R}^2$ if there exists a conformal mapping $\varphi : \Omega' \to \Omega$ such that $\|\varphi' \mid L^\alpha(\Omega')\| < \infty$, $\alpha \in (2, \infty]$.

The exponent of integrability α does not depend on the choice of a conformal mapping (by the Riemann Mapping Theorem) and depends solely on the hyperbolic metric on Ω.

In the case of conformal α-regular domains $\Omega \subset \mathbb{R}^2$ we have the embedding

$$L^r(\Omega, h) \hookrightarrow L^s(\Omega), \;\; s = \frac{\alpha - 2}{\alpha} r.$$

Theorem A. *Let a domain* Ω *be conformal* α*-regular about* Ω'*. Then*

$$\frac{1}{\lambda_1(\Omega)} \leq A^2_{\frac{2\alpha}{\alpha-2},2}(\Omega')\left\|\varphi' \mid L^{\alpha}(\Omega')\right\|^2, \ \alpha < \infty,$$

where

$$A^2_{\frac{2\alpha}{\alpha-2},2}(\Omega') \leq \inf_{p\in\left(\frac{\alpha}{\alpha-1},2\right)} \left(\frac{p-1}{2-p}\right) \frac{\left(\sqrt{\pi}\cdot\sqrt[p]{2}\right)^{-1}|\Omega'|^{\frac{\alpha-2}{2\alpha}}}{\sqrt{\Gamma(2/p)\Gamma(3-2/p)}}.$$

In case $\alpha = \infty$

$$\frac{1}{\lambda_1(\Omega)} \leq \frac{\left\|\varphi' \mid L^{\infty}(\Omega')\right\|^2}{\lambda_1(\Omega')}.$$

Example. The diffeomorphism

$$\varphi(z) = \sin z, \quad z = x + iy,$$

is conformal and maps the rectangle

$$Q := [-\pi/2, \pi/2] \times [-d, d]$$

in the z-plane onto the interior of the ellipse

$$\Omega_e := \left(\frac{u}{\cosh d}\right)^2 + \left(\frac{v}{\sinh d}\right)^2 = 1$$

in the w-plane with slits from the foci $(\pm 1, 0)$ to the tips of the major semi-axes $(\pm \cosh d, 0)$.

Then by Theorem A in the case $\alpha = \infty$ we have

$$\lambda_1(\Omega_e) \geq \frac{2\left(1 + \frac{\pi^2}{4d^2}\right)}{1 + \cosh 2d}.$$

Rayleigh-Faber-Krahn inequality:

$$\lambda_1(\Omega_e) \geq \frac{2(j_{0,1})^2}{\sinh 2d}.$$

If $0 < d < 1/2$ then

$$\frac{1 + \frac{\pi^2}{4d^2}}{1 + \cosh 2d} > \frac{(j_{0,1})^2}{\sinh 2d}.$$

Table: Values of $\lambda_1(\Omega_e)$

d:	1/2	1/3	1/4	1/8
RFK:	9,841	16,127	22,195	45,786
Our:	8,548	20,807	38,050	156,456

V. I. Burenkov, V. Gol'dshtein, A. Ukhlov (2015):

Stability Theorem. Let a domain Ω be conformal α-regular about Ω'. Then for any $k \in \mathbb{N}$

$$|\lambda_k(\Omega) - \lambda_k(\Omega')| \leq c_k \gamma_\alpha V_\alpha(\Omega, \Omega').$$

Here $c_k = \max\left\{\lambda_k^2(\Omega), \lambda_k^2(\Omega')\right\}$,

$$\gamma_\alpha \leq \inf_{p \in \left(\frac{4\alpha}{3\alpha-2}, 2\right)} \left(\frac{p-1}{2-p}\right)^{\frac{p-1}{p}} \frac{\left(\sqrt{\pi} \cdot \sqrt[p]{2}\right)^{-1} |\Omega'|^{\frac{\alpha-2}{4\alpha}}}{\sqrt{\Gamma(2/p)\Gamma(3-2/p)}}$$

and

$$V_\alpha(\Omega, \Omega') = \inf_{\varphi} \left[\left(|\Omega'|^{\frac{1}{\alpha}} + \|\varphi' \mid L^\alpha(\Omega')\|\right) \|1 - \varphi' \mid L^2(\Omega')\|\right],$$

where the infimum is taken over all conformal mappings $\varphi : \Omega' \to \Omega$.

The case $\Omega' = \mathbb{D}$.

Theorem B. *Let $\Omega \subset \mathbb{R}^2$ be a conformal α-regular domain of area π. Then*

$$\lambda_1(\Omega) \leq \lambda_1(\mathbb{D}) + \lambda_1^2(\mathbb{D}_\rho)\gamma_\alpha V_\alpha(\mathbb{D}, \Omega),$$

where $\mathbb{D}_\rho$ is the largest disc inscribed in Ω and

$$\gamma_\alpha = \inf_{p \in \left(\frac{4\alpha}{3\alpha-2}, 2\right)} \left(\frac{p-1}{2-p}\right)^{\frac{2(p-1)}{p}} \frac{\pi^{-\frac{\alpha+2}{2\alpha}} 4^{-\frac{1}{p}}}{\Gamma(2/p)\Gamma(3-2/p)}.$$

Estimates of Dirichlet eigenvalues in quasidiscs.

Now we obtain integral estimates of conformal derivatives in special classes of domains $\Omega \subset \mathbb{R}^2$, namely in quasidiscs.

Recall that K-quasidiscs are images of the unit disc $\mathbb{D} \subset \mathbb{R}^2$ under K-quasiconformal homeomorphisms of the plane $\mathbb{R}^2$. This class includes all Lipschitz simply connected domains but also includes a class of fractal type domains like snowflakes. The Hausdorff dimension of the quasidisc's boundary can be any number in $[1, 2)$.

The suggested approach is based on the exact inverse Hölder inequality for Jacobians of (quasi)conformal mappings.

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a K-quasiconformal mapping. Then for every disc $\mathbb{D} \subset \mathbb{R}^2$ and for any $1 < \kappa < \frac{K}{K-1}$ the inverse Hölder inequality

$$\left(\iint\limits_{\mathbb{D}} |J_\varphi(x,y)|^\kappa \ dxdy \right)^{\frac{1}{\kappa}} \leq \frac{C_\kappa^2 K \pi^{\frac{1}{\kappa}-1}}{4} \exp\left\{ \frac{K\pi^2(2+\pi^2)^2}{2\log 3} \right\} \iint\limits_{\mathbb{D}} |J_\varphi(x,y)| \ dxdy.$$

holds, where

$$C_\kappa = \frac{10^6}{[(2\kappa-1)(1-\nu)]^{1/2\kappa}}, \quad \nu = 10^{8\kappa}\frac{2\kappa-2}{2\kappa-1}(24\pi^2 K)^{2\kappa} < 1.$$

So, we obtain estimates of conformal derivatives in domains allow quasiconformal extensions of conformal mappings (quasidiscs):

Let $\Omega \subset \mathbb{R}^2$ be a K-quasidisc of area π. Then

$$\lambda_1(\Omega) \leq \lambda_1(\mathbb{D}) + \lambda_1^2(\mathbb{D}_\rho) M(K) \|\varphi' - 1 \mid L^2(\mathbb{D})\|$$

where $\mathbb{D}_\rho$ is the largest disc inscribed in Ω and a constant $M(K)$ depends only on a quasiconformality coefficient K.

The quantity $M(K)$ depends only on a quasiconformality coefficient K of Ω:

$$M(K) = \inf_{2<\alpha<\alpha^*} \left\{ \inf_{p \in \left(\frac{4\alpha}{3\alpha-2}, 2\right)} \left(\frac{p-1}{2-p}\right)^{\frac{2(p-1)}{p}} \frac{\pi^{-\frac{\alpha+2}{2\alpha}} 4^{-\frac{1}{p}}}{\Gamma(2/p)\Gamma(3-2/p)} \right.$$
$$\left. \times \left(\frac{C_\alpha K \pi^{\frac{2-\alpha}{2\alpha}}}{2} \exp\left\{ \frac{K^2\pi^2(2+\pi^2)^2}{4\log 3} \right\} \cdot |\Omega|^{\frac{1}{2}} + \pi^{\frac{1}{\alpha}} \right) \right\},$$
$$C_\alpha = \frac{10^6}{[(\alpha-1)(1-\nu(\alpha))]^{1/\alpha}},$$

where $\alpha* = \min\left(\frac{K^2}{K^2-1}, \gamma*\right)$ and $\gamma*$ is the unique solution of the equation

$$\nu(\alpha) := 10^{4\alpha}((\alpha-2)/(\alpha-1))(24\pi^2K^2)^\alpha = 1.$$

In the book Henrot A. *Extremum Problems for Eigenvalues of Elliptic Operators, Frontiers in Mathematics, 2006* were formulated open problems on high eigenvalues of the Dirichlet Laplacian.

We give a partial answer:

Theorem C. *Let Ω be a conformal α-regular domain such that $\mathbb{D} \subseteq t\Omega$, $t > 0$. Then for any $k \in \mathbb{N}$ the following inequalities*

$$\lambda_k(\mathbb{D}) - t^4 \lambda_k^2(\mathbb{D}) \gamma_\alpha V_\alpha(\mathbb{D}, \Omega) \leq \lambda_k(\Omega) \leq t^2 \lambda_k(\mathbb{D})$$

hold.

Corollary 1. *Let* Ω *be a conformal* α*-regular domain and* $\mathbb{D} \subseteq t\Omega$ *for* $t > 0$*. Then for any* $m, n \in \mathbb{N}$*,* $m < n$*, the following inequality*

$$\frac{\lambda_n(\Omega)}{\lambda_m(\Omega)} \geq \frac{\lambda_n(\mathbb{D}) - t^4 \lambda_n^2(\mathbb{D}) \gamma_\alpha V_\alpha(\mathbb{D}, \Omega)}{t^2 \lambda_m(\mathbb{D})}$$

holds.

Example. For $k \geq 2$, the diffeomorphism

$$\varphi(z) = z + \frac{1}{k} z^k, \quad z = x + iy,$$

is conformal and maps the unit disc $\mathbb{D}$ onto the domain Ω_k bounded by an epicycloid of $(k-1)$ cusps, inscribed in the circle $|w| = (k+1)/k$.

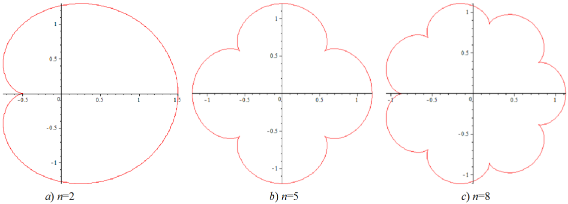

Figure: Image of $\mathbb{D}$ under $\varphi(z) = z + \frac{1}{k} z^k$.

Note that $\mathbb{D} \not\subset \Omega_k$. However, if put $t = k^2/(k-1)^2$ then $\mathbb{D} \subseteq t\Omega_k$. Then by Corollary 1 we have

$$\frac{\lambda_n(\Omega_k)}{\lambda_m(\Omega_k)} \geq \frac{\lambda_n(\mathbb{D}) - \frac{k^8}{(k-1)^8}\lambda_n^2(\mathbb{D})\gamma_\alpha V_\alpha(\mathbb{D}, \Omega_k)}{\frac{k^4}{(k-1)^4}\lambda_m(\mathbb{D})}.$$

Burenkov V.I., Gol'dshtein V., Ukhlov A. *Conformal spectral stability for the Dirichlet-Laplace operator.* Math. Nachr., **288** (2015), 1822–1833.

Gol'dshtein V., Pchelintsev V., Ukhlov A. *Integral estimates of conformal derivatives and spectral properties of the Neumann-Laplacian.* J. Math. Anal. Appl., **463** (2018), 19–39.

Gol'dshtein V., Pchelintsev V., Ukhlov A. *On conformal spectral gap estimates of the Dirichlet-Laplacian.* St. Petersburg Math. J., **31**:2 (2020), 325–335.

THANKS