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Moments method in optimal control investigation and state estimation for fractional-order systems

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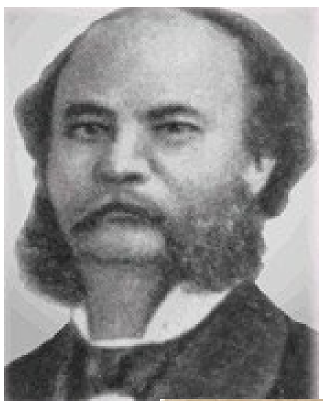
Fractional calculus: history and fundamentals

1695:

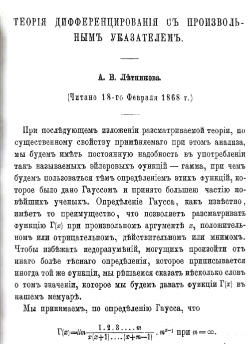
$$\left.\frac{d^n f(x)}{dx^n}\right|_{n=1/2} = ?$$

1822:

$$\frac{d^p f(x)}{dx^p} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda^p d\lambda \int_{-\infty}^{\infty} f(t)\cos\left(tx - t\lambda + \frac{p\pi}{2}\right)dt$$



1868:



ТЕОРІЯ ДИФФЕРЕНЦИРОВАНІЯ СЪ ПРОИЗВОЛЬНЫМЪ УКАЗАТЕЛЕМЪ.

А. В. Лѣтникова.

(Читано 18-го Февраля 1868 г.)

При послѣдующемъ изложеніи разсматриваемой теоріи, по существенному свойству примѣняемаго при этомъ анализа, мы будемъ имѣть постоянную надобность въ употребленіи такъ называемыхъ эйлеровыхъ функцій — гамма, при чемъ будемъ пользоваться тѣмъ опредѣленіемъ этихъ функцій, которое было дано Гауссомъ и принято большею частію новѣйшихъ ученыхъ. Опредѣленіе Гаусса, какъ извѣстно, имѣетъ то преимущество, что позволяетъ разсматривать функцію $\Gamma(x)$ при произвольномъ аргументѣ x, положительномъ или отрицательномъ, дѣйствительномъ или мнимомъ. Чтобы избѣжать недоразумѣній, могущихъ произойти отъ инаго болѣе тѣснаго опредѣленія, которое приписывается иногда той же функціи, мы рѣшаемся сказать нѣсколько словъ о томъ значеніи, которое мы будемъ давать функціи $\Gamma(x)$ въ нашемъ мемуарѣ.

Мы принимаемъ, по опредѣленію Гаусса, что

$$\Gamma(x) = lim \frac{1.2.3\ldots.m}{x(x+1)\ldots.(x+m-1)}.m^{x-1} \text{ при } m=\infty.$$

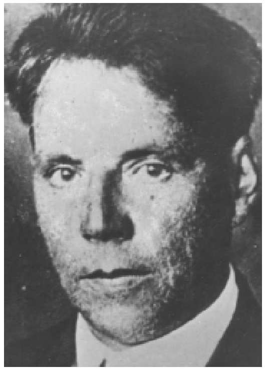
1

$$\Delta^\alpha f(x) = \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(x \mp k\xi)$$

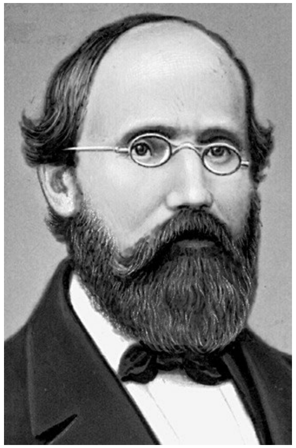
$$D_x^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \lim_{\xi\to 0}\frac{\Delta^\alpha f(x)}{\xi^\alpha}$$

Fractional calculus: history and fundamentals

Riemann-Liouville derivative:

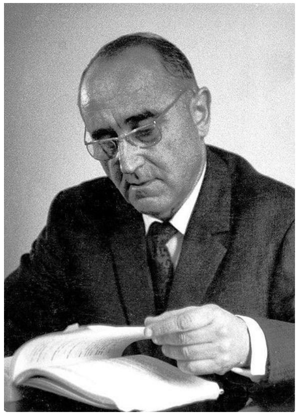


$$ {}_{t_0}^{RL}D_t^{\alpha}q(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}\frac{q(\tau)d\tau}{(t-\tau)^{\alpha}}, \qquad \alpha \in (0,1]. $$



Gerasimov-Caputo-Dzherbashyan (more often – Caputo) derivative:

$$ {}_{t_0}^{C}D_t^{\alpha}q(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{dq(\tau)}{d\tau}\frac{d\tau}{(t-\tau)^{\alpha}}, \qquad \alpha \in (0,1], $$



$$ {}_{t_0}^{C}D_t^{\alpha}q(t) = {}_{t_0}^{RL}D_t^{\alpha}[q(t) - q(t_0)]. $$

Fractional derivative multi-parametric definitions

Hilfer derivative:

$$ {}^{Hil}_{t_0}D_t^{\alpha,\mu} q(t) = I^{\mu(1-\alpha)} \frac{d}{dt} I^{(1-\mu)(1-\alpha)} q(t),\ I^{\sigma} q(t) = \frac{1}{\Gamma(\sigma)} \int_{t_0}^{t} \frac{q(\tau)}{(t-\tau)^{1-\sigma}} d\tau, $$

$$ \mu_i \in [0,1], \qquad {}^{Hil}_{t_0}D_t^{\alpha,0} q(t) = {}^{RL}_{t_0}D_t^{\alpha} q(t), \qquad {}^{Hil}_{t_0}D_t^{\alpha,1} q(t) = {}^{C}_{t_0}D_t^{\alpha} q(t). $$

Katugampola derivative:

$$ {}^{K}_{t_0}D_t^{\alpha,\rho} q(t) = \frac{\rho^{\alpha} t^{1-\rho}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^{t} \frac{\tau^{\rho-1} q(\tau)}{(t^{\rho}-\tau^{\rho})^{\alpha}} d\tau, \qquad \rho > 0, $$

$$ {}^{K}_{t_0}D_t^{\alpha,0} q(t) = {}^{H}_{t_0}D_t^{\alpha} q(t), \qquad {}^{K}_{t_0}D_t^{\alpha,1} q(t) = {}^{RL}_{t_0}D_t^{\alpha} q(t). $$

Erdelyi-Kober derivative:

$$ {}^{EK}_{t_0}D_{\beta}^{\alpha,\delta} q(t) = \left(\alpha + 1 + \frac{t}{\beta}\frac{d}{dt}\right) I_{\beta}^{\alpha+\delta,1-\delta} q(t), $$

$$ I_{\beta}^{\alpha,\delta} q(t) = \frac{\beta t^{-\beta(\alpha+\delta)}}{\Gamma(\delta)} \int_{t_0}^{t} \frac{\tau^{\beta(\alpha+1)-1} q(\tau)}{(t^{\beta}-\tau^{\beta})^{1-\delta}} d\tau, \qquad \alpha \in [0,1], \delta \in (0,1], \beta > 0. $$

Fractional derivative «non-singular» definitions

Caputo-Fabrizio derivative:

$$ {}^{CF}_{t_0}D^{\alpha}_{t} q(t) = \frac{M(\alpha)}{1-\alpha} \int_{t_0}^{t} q'(\tau) \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) d\tau, \qquad \alpha \in (0,1). $$

Atangana-Baleanu derivative:

$$ {}^{AB}_{t_0}D^{\alpha}_{t} q(t) = \frac{M(\alpha)}{1-\alpha} \int_{t_0}^{t} q'(\tau) E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right) d\tau, $$

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} \frac{z^k}{k!}, \qquad \alpha \in (0,1). $$

Fractional calculus: applications

- Tautochrone problem (N. Abel, 1823)
- ODE solution and multi-fold integral calculations (A.V. Letnikov, N.Ya. Sonine, P.A. Nekrasov etc., 1868-1890)
- Electrical scheme calculations (O. Heaviside, 1892; T. Bromvich, 1919; S. Manabe, 1961-1963)
- Hereiditary mechanics and physics (A. Gemnt, A.N. Gerasimov, G. Scott-Blair, Yu.N. Rabotnov, 1930-1940's; F. Mainardi, M. Caputo, R. Bagley, P. Torvik, 1960-1980's)
- Electrochemical cells and systems calculation (R.Sh. Nigmatullin, 1950-1980's)
- Disordered and complex media models (J. Liouville, I. Podlubny, B. Ross, P. Grigolini, R.R. Nigmatullin, R. Hilfer, G.M. Zaslavskii, V.E. Tarasov, V.V. Uchaikin etc.)
- **Fractional-order dynamics and control** (Y.Q. Chen, I. Podlubny, I. Petras, D. Matignon, S. Manabe, D.F.M. Torres, O.P. Agrawal etc.)

Some basic results in optimal control theory for fractional-order systems

- Fractional-order generalizations of Euler-Lagrange equations (Agrawal 2002 etc.)
- Fractional-order analogues of Lagrange and Hamilton approaches (Agrawal 2004, 2006, 2007; Almeida 2009; Torres, Malinowska 2008, 2010, 2014 etc.)
- Pontryagin maximum principle for some types of fractional-order systems (Kamocki 2014, 2015)
- Moment method application for linear fractional-order systems (Kubyshkin and Postnov, 2012-2016)

A majority of the basic results obtained for Riemann-Liouville and Caputo operators (there are some papers about Riesz, Hadamard and some multiparametric operators). The optimal control problems often considered with continuous control and quality functional based on quadratic superposition of state and control.

Research motivation: - investigation of the problems with an obvious restriction on control norm and with discontinuous (step-wise) control;

- analysis of influence of fractional operator type on the correctness and solvability of optimal control problems and on the system dynamics.

General problem statement

Dynamics equation in general case:

$$ {}_{t_0}D_t^{\alpha} q_i(t) = a_{ij} q_j(t) + b_{ij} u_j(t) + C_i, \quad t \in (t_0, T], \qquad i, j = 1, \dots, N, $$

One-dimensional dynamics equations in case of Erdelyi-Kober operator:

$$ D_{\beta}^{\alpha,\delta} q(t) = u(t), \qquad t \in (0, T], $$

$$ t^{-\beta\delta} D_{\beta}^{\alpha,\delta} q(t) - \lambda q(t) = u(t), \qquad t \in (0, T], $$

$$ t^{-\delta} D_{1}^{\alpha,\delta} q(t) - \lambda t^{\nu} I_{1}^{\alpha-\nu,\nu} q(t) = u(t), \qquad t \in (0, T]. $$

Initial conditions:

$$ q_i(t_0) = q_i^0 \qquad \text{local type} $$

$$ \lim_{t \to t_0 +} \left[\; I^{\sigma_i} q_i(t) \right] = s_i^0 \qquad \text{non} - \text{local type} $$

Terminal conditions:

$$ q_i(T) = q_i^T $$

The moment problem

<u>Optimal control problem.</u> It's needed to find a control *u(t)* such that the system considered pass from the given initial state to the given terminal state and

- (A) the control norm $\|u\|$ be minimal at T given, or
- (B) control time T be minimal at $\|u\| \leq l$, where $l > 0$ is given.

***l*-problem of moments.** Let the system of functions $g_i(t) \in L_{p'}(0,T]$ and number set c_i (which called moments) be given and $\sum_{i=1}^{N} c_i^2 \neq 0$. Let the number $l > 0$ also be given. It's needed to find a function $u(t) \in L_p(0,T]$ such that the following conditions hold:

$$\int_0^T g_i(\tau)u(\tau)d\tau = c_i, \qquad i = 1, \dots, N.$$

$$\|u(t)\| \leq l.$$

The key condition which define the correctness of moment problem is an existence of norm of functions $g_i(t) \in L_{p'}(0,T]$.

Moment problem correctness and solvability

Solvability of moment problem defined by linear independency of functions $g_i(t)$ or (equivalently) by the positivity of the constant λ_N, where

$$\min_{\xi_1,\dots,\xi_N}\left(\int_0^T\left|\sum_{i=1}^N \xi_i g_i(t)\right|^{p'} dt\right)^{1/p'} = \left(\int_0^T\left|\sum_{i=1}^N \xi_i^* g_i(t)\right|^{p'} dt\right)^{1/p'} = \frac{1}{\lambda_N}.$$

Remark. For integer-order systems the correctness condition usually is trivial and for fractional-order systems it usually non only have the sense but also cause the solvability of problem.

Moment problem correctness and solvability

As in case of integer-order systems one can use solutions of system (*), which formally represent at $t = T$ the finite-dimensional moment problem.

Theorem. *l*-problem of moments for Caputo-Fabrizio system (*) is correct and solvable.

Theorem. *l*-problem of moments for Caputo, Riemann-Liouville, Hilfer, Hadamard, Atangana-Baleanu system (*) is correct and solvable iff the following conditions hold:

$$\alpha_k > \frac{1}{p}.$$

Moment problem for 1D Erdelyi-Kober systems

In case of Erdelyi-Kober systems we can use the known solutions for the dynamics equations [Kiryakova1994,2011; Luchko2007] and reduce the optimal control problem to the moment problem.

Theorem. *l*-problem of moments for Erdelyi-Kober systems is correct and solvable iff the following conditions hold:

- $\delta > \frac{1}{p}$, $\beta(\alpha + 1) > \frac{1}{p}$ for the system $D_{\beta}^{\alpha,\delta} q(t) = u(t)$;

- $\delta > \frac{1}{p}$, $\beta(\alpha + \delta + 1) > \frac{1}{p}$ for the system $t^{-\beta\delta} D_{\beta}^{\alpha,\delta} q(t) - \lambda q(t) = u(t)$;

- $\alpha - \nu + \delta + 1 > \frac{1}{p}$, $2\delta + 2\nu > \frac{1}{p}$, $\alpha + \delta + 1 > \frac{1}{p}$
 for the system $t^{-\delta} D_{1}^{\alpha,\delta} q(t) - \lambda t^{\nu} I_{1}^{\alpha-\nu,\nu} q(t) = u(t)$.

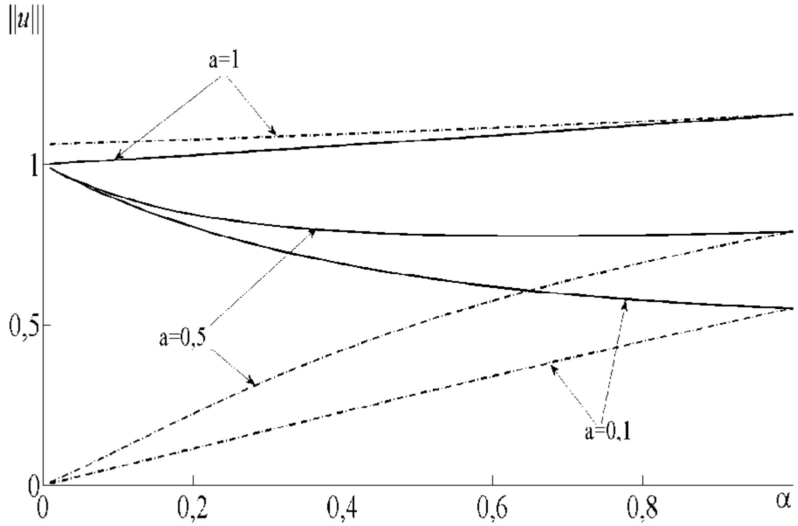
.

Remark. In case of $\beta = 1$, $\alpha = 0$ for the 1st and 2nd of systems mentioned above only condition $\delta > \frac{1}{p}$ holds non-trivial, which correspond to the case with Riemann-Liouville analogue unvestigated later.

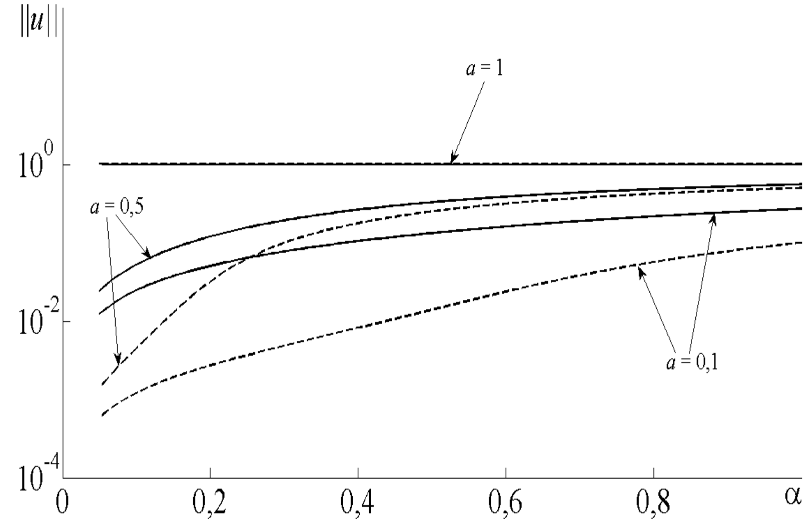
Operator type influence on optimal control properties: illustration

$$u(t) \in L_\infty(t_0, T]$$

Caputo double integrator



Hadamard double integrator



Dashed lines correspond to Riemann-Liouville double integrator.

$a = 0,\ q^T = 0:$

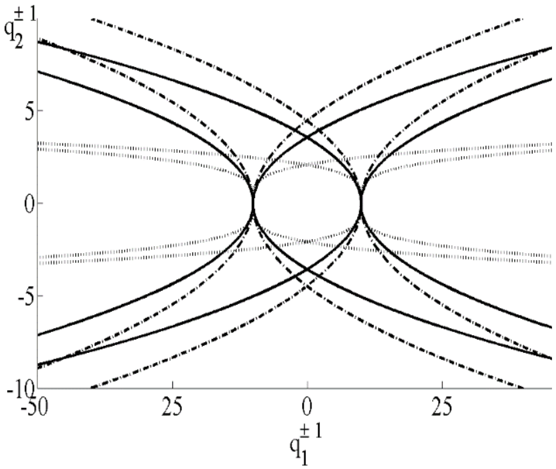
$$\lambda^C = \frac{|q^0|\Gamma(\alpha+1)}{(T-t_0)^\alpha} \qquad \lambda^{RL} = \frac{|s^0|\alpha}{T-t_0} \qquad \lambda^H = \frac{|s^0|\alpha}{\ln \frac{T}{t_0}}$$

Kubyshkin2014,2016; Postnov2017,2018

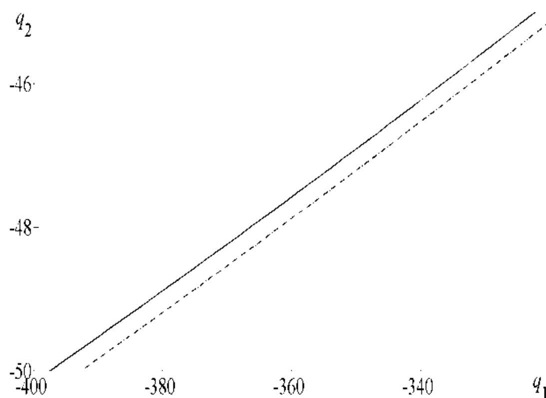
Operator type influence on dynamics of double integrator: illustration

Boundary trajectories $u(t) = \pm l$

Caputo system

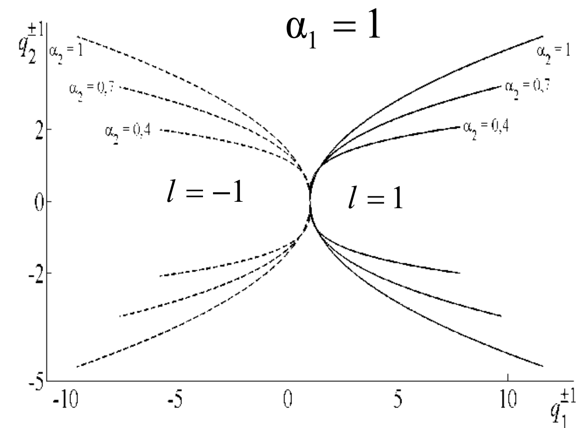


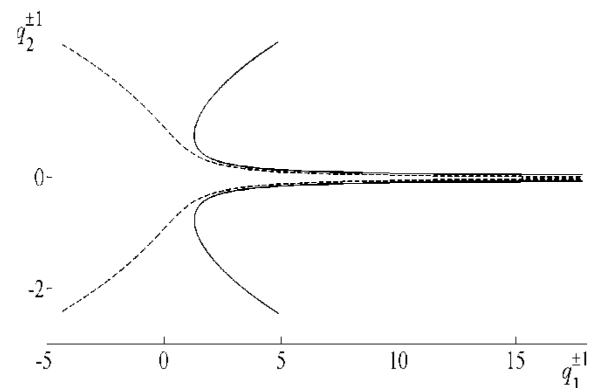
Riemann-Liouville system



$\alpha_1 = \alpha_2 = 0{,}5$

Hadamard system





$\alpha_1 = \alpha_2 = 0.5$

Solid lines:

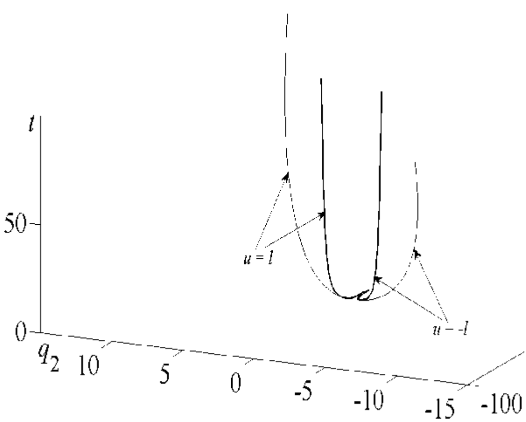
$\alpha_1 = \alpha_2 = 0{,}5$

Dash-doted lines:

$\alpha_1 = 1, \alpha_2 = 1/3$

Dashed lines:

$\alpha_1 = \alpha_2 = 1$



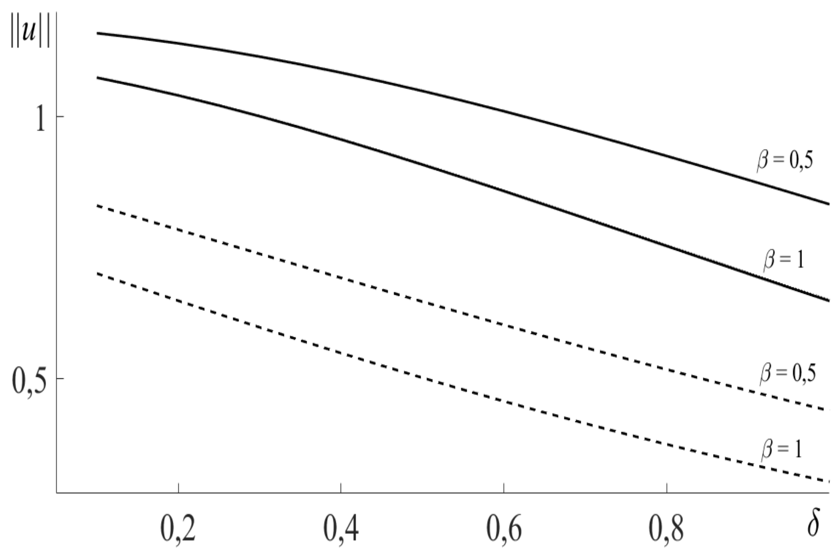
Hadamard system – bold lines,
Riemann-Liouville system – thin lines)

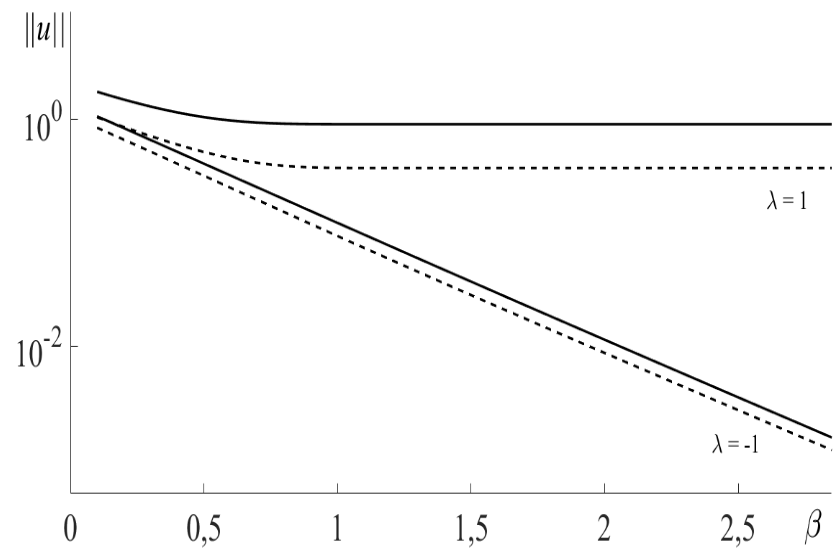
OCP for the Erdelyi-Kober 1D system

$$t^{-\beta\delta}\left(D_{\beta}^{\alpha,\delta}q\right)(t)-\lambda q(t)=u(t), \qquad t\in(0,T],\lambda\neq 0,u(t)\in L_{\infty}(0,T]$$

$$u(t)=\frac{c(T)}{T^{\beta\delta}\Gamma(\alpha+\delta+1)E_{\delta,\alpha+2\delta+1}(\lambda T^{\beta\delta})},$$

$$c(T)=q(T)-BT^{\beta(\delta-1)}E_{\delta,\alpha+2\delta}\left(\lambda T^{\beta\delta}\right)$$





$q_1^T=0, B=1, T=10, \lambda=1;$
Solid lines $-\alpha=0.7,$ dashed lines $-\ \alpha=1.5.$ The lines for $\beta=1$ and $\beta=2$ merge.

$q_1^T=0, B=1, T=10, \lambda=\pm 1$
Solid line – $\alpha=0.7, \delta=0.5,$
dashed line – $\alpha=1.5, \delta=0.8$

«Optimal observation» problem

The problem of «observation in stochastic circumstances» for integer-order systems was reduced to the problem of moments by N.N. Krasovskii in 1968.

Dynamics equation with initial conditions for the basic system:

$$ {}_{t_0}D_t^{\alpha_i} q_i(t) = \sum_{j=1}^{N} a_{ij}(t) q_j(t), \qquad t \in (t_0, T], \qquad i, j = 1, \dots, N, $$

$$ \lim_{t \to t_0+} [I^{1-\alpha_i} q_i(t)] = q_i^0 \ - \text{in case of } {}_{t_0}D_t^{\alpha_i} = {}_{t_0}^{RL}D_t^{\alpha_i}, $$

$$ q_i(t_0) = q_i^0 - \text{in case of } {}_{t_0}D_t^{\alpha_i} = {}_{t_0}^{C}D_t^{\alpha_i}. $$

Dynamics equation with initial conditions for the observed system:

$$ {}_{t_0}D_t^{\beta} z(t) = F(t) z(t) + \sum_{i=1}^{N} G_i(t) q_i(t) + \Delta(t), \qquad \Delta(t) = \sum_{i=1}^{\infty} \xi_i \delta(t - t_i), $$

$$ \lim_{t \to t_0+} \left[I^{1-\beta} z(t)\right] = z^0 \ - \text{in case of } {}_{t_0}D_t^{\beta} = {}_{t_0}^{RL}D_t^{\beta}, $$

$$ z(t_0) = z^0 \ - \text{in case of } {}_{t_0}D_t^{\beta} = {}_{t_0}^{C}D_t^{\beta}. $$

To find: $q_i(t) = \varphi[t, z(t)] + w$, $M\{w^2\} \to \min$.

«Optimal observation» problem

The solution for the observation equation:

$$z(t) = \tilde{Q}(t,t_0)z^0 + \int_{t_0}^{t} Q(t,\tau)\left[\sum_{i=1}^{N} G_i(\tau)q_i(\tau) + \Delta(\tau)\right]d\tau\,.$$

$F(t) - const \Rightarrow$

$$Q(t,\tau) = \frac{E_{\beta,\beta}\left[F(t-\tau)^{\beta}\right]}{(t-\tau)^{1-\beta}}, \qquad \tilde{Q}(t,t_0) = \begin{cases} Q(t,t_0) \text{ in case of } {}_{t_0}D_t^{\beta} = {}_{t_0}^{RL}D_t^{\beta}, \\ E_{\beta}\left[F(t-t_0)^{\beta}\right] \text{ in case of } {}_{t_0}D_t^{\beta} = {}_{t_0}^{C}D_t^{\beta} \end{cases}$$

Let 's introduce the notation :

$$\Xi(t) = z(t) - \tilde{Q}(t,t_0)z^0.$$

Then the problem can be reformulated in the form :

$$\varphi\left[\int_{t_0}^{t} Q(t,\tau)\sum_{i=1}^{N} G_i(\tau)q_i(\tau)\,d\tau\right] = q_i(t),$$

$$M\left\{\left[\varphi\left[\int_{t_0}^{t} Q(t,\tau)\,\Delta(\tau)d\tau\right]\right]^2\right\} \to \min.$$

«Optimal observation» problem

Let us look for an operation that restores the state of observation in the form :

$$\varphi[\Xi(t)] = \int_{t_0}^{t} \Xi(\tau) dV(\tau) .$$

Then the solution reduces to an equation that, for a fixed t, can be written in the form of a moment problem:

$$\int_{t_0}^{t} \left(\sum_{i=1}^{N} G_i(\varsigma) \sum_{j=1}^{N} \tilde{Z}_{ij}(\varsigma, t_0) q_j^0 \right) U(t, \varsigma) d\varsigma = \sum_{j=1}^{N} \tilde{Z}_{ij}(t, t_0) q_j^0 , \qquad (3)$$

where $U(t, \varsigma) = \int_{\varsigma}^{t} Q(\tau, \varsigma)\, dV(\tau)$.

The following estimation holds :

$$M\left\{ \left[\varphi \left[\int_{t_0}^{t} Q(t, \tau)\, \Delta(\tau) d\tau \right] \right]^2 \right\} \sim \int_{0}^{t} U^2(t, \varsigma) d\varsigma .$$

Consequently:

$$M\left\{ \left[\varphi \left[\int_{t_0}^{t} Q(t, \tau)\, \Delta(\tau) d\tau \right] \right]^2 \right\} \to \min \iff \|U\|_{L_2(0,T]} \to \min.$$

«Optimal observation» problem

Theorem. Let $G_k \in \mathrm{C}^1_\gamma = \left\{f(t) = (t - t_0)^{\gamma_k}\tilde{f}(t),\ \tilde{f} \in C^1[t_0, \infty)\right\}$. The "optimal observation" problem can be reduced to the moment problem of type (3) when the following conditions holds:

$$\gamma_k + \alpha_k > \frac{1}{p}, \qquad k = 1, \dots, N. \tag{4}$$

Theorem. When the conditions (4) holds the moment problem (3) is solvable and the solution can be written in the following form:

$$U(t,\varsigma) = \frac{c(t)}{\|g(\varsigma)\|_{L_2}} g(\varsigma),$$

где $g(\varsigma) = \sum_{i=1}^{N} G_i(\varsigma) \sum_{j=1}^{N} \tilde{Z}_{ij}(\varsigma, t_0) q_j^0$, $\qquad c(t) = \sum_{j=1}^{N} \tilde{Z}_{ij}(t, t_0) q_j^0$.

Problem Statement: Diffusion-Wave Equation

General equation:

$$r(x)\ {}_0D_t^{\alpha} Q(x,t) = \frac{\partial}{\partial x}\left[w(x)\frac{\partial Q(x,t)}{\partial x}\right] - q(x)Q(x,t) +$$

$$+u(x,t) + f(x,t), \qquad t > 0, \quad 0 \le x \le L, 0 < \alpha < 2$$

${}_0D_t^{\alpha} Q(x,t)$ - Caputo fractional derivative of order α.

Initial conditions:

$$Q(x,0) = \varphi^0(x), \qquad \frac{\partial Q(x,0)}{\partial t} = \varphi^1(x).$$

Boundary conditions:

$$\left[a_{1,2} Q(x,t) + b_{1,2}\frac{\partial Q(x,t)}{\partial x}\right]\Bigg|_{x=0,L} = h_{1,2}(t) + u_{1,2}(t)$$

Final condition:

$$Q(x,T) = Q^*(x) \qquad \text{and/or} \qquad \frac{\partial Q(x,T)}{\partial t} = A(x).$$

Admissible controls

$u_{1,2}(t)$ - boundary controls; $U(t) = \big(u_1(t), u_2(t)\big)$

$u_{1,2}(t) \in L_\infty[0,T]$: $\quad \|U(t)\|_\infty = \operatorname*{vraimax}_{t\in[0,T]} \max_{i=1,2} |u_i(t)|$

$u_{1,2}(t) \in L_p[0,T]$, $\qquad 1 < p < \infty$:

$$\|U(t)\|_p = \left(\int_0^T [|u_1(t)|^p + |u_2(t)|^p] dt \right)^{1/p}$$

$u(x,t)$ - distributed control, p-integrable or sufficiently bounded on $\Omega = [0,L] \times [0,T]$

Two kinds of boundary control can be considered:

1) local, when the controls $u_{1,2}(t)$ are included directly in boundary conditions;

2)non-local, when the controls expressed by the fractional derivative from the function included in boundary conditions, $\tilde{u}_{1,2}(t) = {}_0D_t^\alpha u_{1,2}(t)$.

Problem statement

Optimal control problem (OCP):

(A) the time T is given and it's needed to find controls $u(x,t)$ and/or $U(t)$, which pass the system from initial to final state and the norm $\|u(x,t)\|$ and/or $\|U(t)\|$ will be minimal;

(B) to find controls $u(x,t)$ and/or $U(t)$, such that the system will pass from initial to final state in minimal time at the given restriction on control norm $\|u(x,t)\| \leq l$ and/or $\|U(t)\| \leq l$, $l > 0$.

Remark. Usually OCP for distributed and boundary controls are investigated separately.

Solution of Diffusion-Wave Equation

An explicit solution of diffusion wave equation exist (Sandev, 2010):

$$Q(x,t) = \sum_{n=1}^{\infty} a_n(t) X_n(x) + \sum_{n=1}^{\infty} \left(\mathcal{E}_{0+;\alpha,\alpha}^{-\lambda_n;1,1} \tilde{f}_n\right)(t) X_n(x) + v(x,t),$$

where $X_n(x)$ and λ_n - eigenfunctions and eigennumbers for the corresponding Sturm-Liouville problem;

$$v(x,t) = v_1(x)[h_1(t) + u_1(t)] + v_2(x)[h_2(t) + u_2(t)]$$

$$\left(\mathcal{E}_{0+;\alpha,\alpha}^{-\lambda_n;1,1} \tilde{f}_n\right)(t) = \int_0^t \frac{E_{\alpha,\alpha}[-\lambda_n (t-\tau)^\alpha]}{(t-\tau)^{1-\alpha}} \tilde{f}_n(\tau) d\tau \,,$$

$$\tilde{f}_n(t) = \frac{1}{\|X_n(x)\|^2} \int_0^L \tilde{f}(x,t) X_n(x) dx \,,$$

$$\tilde{f}(x,t) = u(x,t) + f(x,t) + \frac{\partial}{\partial x}\left[w(x)\frac{\partial v(x,t)}{\partial x}\right] -$$

$$-q(x)v(x,t) - r(x)\ {}_0D_t^\alpha v(x,t).$$

The *l*-problem of moments

Using the explicit solution one can reduce the OCP for diffusion-wave equation to the form of generalized infinite-dimensional *l*-problem of moments:

$$\int_0^T g_n(t,T)W_n(t)dt = c_n(T), \qquad n = 1,2,\dots,$$

$$\|W_n\| \le l,$$

$$g_n(t,T) = \frac{E_{\alpha,\alpha}[-\lambda_n(T-t)^\alpha]}{(T-t)^{1-\alpha}} \quad \text{and/or} \quad g_n(t,T) = \frac{E_{\alpha,\alpha-1}[-\lambda_n(T-t)^\alpha]}{(T-t)^{2-\alpha}},$$

$$W_n(t) = u^n(t) + \left(v_1' w_n' - \left(q(x)v_1(x)\right)_n + \lambda_n\left(q(x)v_1(x)\right)_n\right)u_1(t) +$$

$$+\left(v_2' w_n' - \left(q(x)v_2(x)\right)_n + \lambda_n\left(q(x)v_2(x)\right)_n\right)u_2(t),$$

In general case there are no conditions on solvability and explicit solutions of infinite-dimensional problem of moments.

We can obtain the classical finite-dimensional *l*-problem of moments in case when we use an approximate (truncated on *N*-th member) solution of diffusion-wave equation ($\{l, N\}$-problem of moments).

Finite-dimensional *l*-problem of moments

The key condition for the correct problem statement: $g_n(t,T) \in L_{p'}[0,T]$.

$$\|g_n(t,T)\|_{p'} = \left\| \frac{E_{\alpha,\alpha}[-\lambda_n (T-t)^{\alpha}]}{(T-t)^{1-\alpha}} \right\|_{p'} \Rightarrow \alpha > \frac{1}{p}$$

$$\|g_n(t,T)\|_{p'} = \left\| \frac{E_{\alpha,\alpha-1}[-\lambda_n (T-t)^{\alpha}]}{(T-t)^{2-\alpha}} \right\|_{p'} \Rightarrow \alpha > \frac{1}{p} + 1 \Leftrightarrow \{\alpha\} > \frac{1}{p}$$

Solvability of the moment problem is provided by linear independence of functions $g_n(t,T)$ and caused by correctness condition evaluation.

Theorem. Let the $\{l, N\}$-problem of moments given and $W_n(t) \in L_p[0,T]$, $p > 1$. This problem is correct and solvable iff:

- $\alpha > \frac{1}{p}$ in case of final condition only for the system state;
- $\{\alpha\} > \frac{1}{p}$ in case of final conditions contain the condition for temporal derivative of the system state.

Solutions of finite-dimensional l-problem of moments: OCP A vs OCP B

If the theorem conditions are met then one can obtain the solution with minimal norm Λ_N (OCP A solution).

Time-optimal (OCP B) solution can be constructed from the condition:

$$\Lambda_N(T^*) \le l \qquad \Leftrightarrow \qquad \frac{1}{\Lambda_N} \ge \frac{1}{l}.$$

The functional

$$\frac{1}{\Lambda_N} = \min_{\xi_i} \rho_{\xi}(T) = \rho_{\xi^*}(T) =$$

$$= \left(\int_0^T \left| \sum_{i=1}^{N-1} \xi_i^* \left[g_i(t,T) - \frac{c_i(T)}{c_N(T)} g_N(t,T) \right] + \frac{1}{c_N(T)} g_N(t,T) \right|^{p'} dt \right)^{1/p'}$$

contains the integrand depending parametrically on the upper limit of integral.

Time-optimal solutions of finite-dimensional l-problem of moments

In case of $W_n(t) \in L_\infty[0,T]$ the estimation holds:

$$\rho_{\xi^*}(T) \leq r(T)$$

$$r(T) = \sum_{i=1}^{N-1} \xi_i^* \left(\int_0^T |g_i(t,T)|dt + \left|\frac{c_i(T)}{c_N(T)}\right| \int_0^T |g_i(t,T)|dt \right) + \left|\frac{1}{c_N(T)}\right| \int_0^T |g_N(t,T)|dt$$

Theorem. Let $W_n(t) \in L_\infty[0,T]$, $c_N(T) \neq 0$. Then $r(T)$ is bounded at every finite $T > 0$ and for $T \to \infty$ the following estimations hold:

$$\lim_{T\to\infty} r(T) = \sum_{i=1}^{N-1} |\xi_i^*| \left(\frac{1}{\lambda_i} + \left|\frac{Q_i^*}{\lambda_N Q_N^*}\right| \right) + \frac{1}{|\lambda_N Q_N^*|}$$

when the final condition is the condition only on the system state,

$$\lim_{T\to\infty} r(T) = 0$$

when the final condition concerns only on the first derivative of the system state,

$$\lim_{T\to\infty} r(T) = \sum_{i=1}^{N/2} \frac{|\xi_{2i-1}^*|}{\lambda_{2i-1}}$$

when the final conditions concern both of the system state and its first derivative.

Corollary. The time-optimal solution can be found not for every l given.

Time-optimal solutions of finite-dimensional l-problem of moments: integer-order case

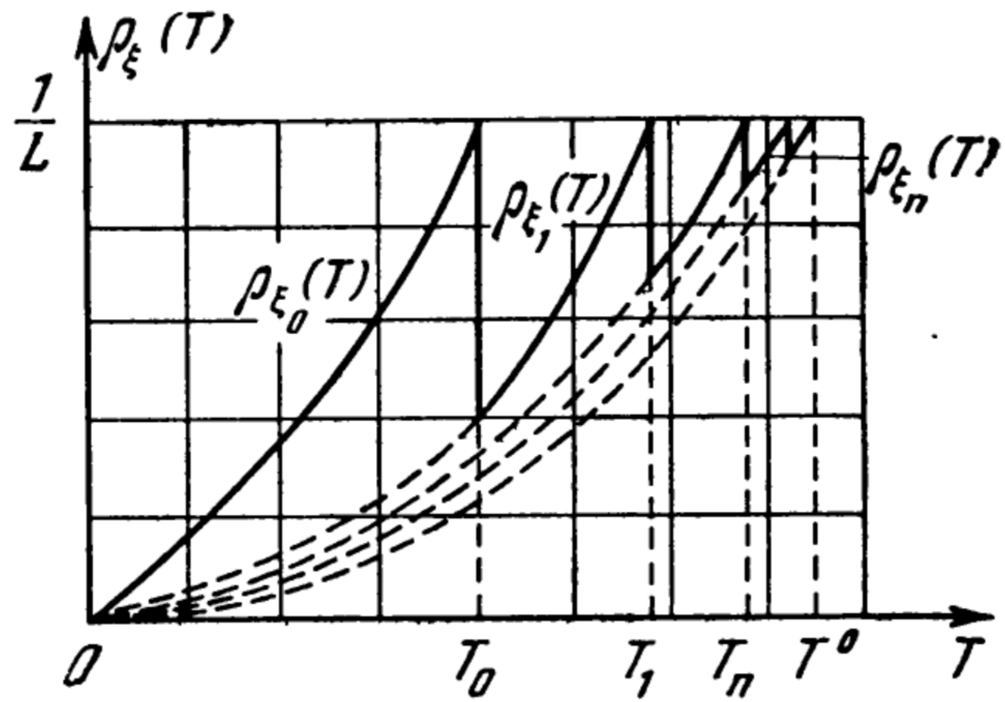
A.G. Butkovskii, 1965: l-problem of moments for classical heat equation

$$ve^{-\mu_n^2\varphi_0} = \mu_n^2 \int_0^{\varphi_0} u(\xi)\, e^{-\mu_n^2(\varphi_0-\xi)}\, d\xi, \quad n = 1, 2, \ldots$$

$$\frac{\nu}{\mu_n^2} = \int_0^{\varphi} u(\xi)\, e^{\mu_n^2 \xi}\, d\xi, \quad n = 1, 2, \ldots,$$

Here the integrand didn't depend on the upper limit and the functional is unbounded:

$$\rho_\xi(T) \xrightarrow[T\to\infty]{} \infty$$



Example 1: OCP A for sub-diffusion equation one-side boundary control $u(t) \in L_2[0, T]$

Sub-diffusion equation:

$$ {}_0D_t^{\alpha} Q(x,t) = \frac{\partial^2 Q(x,t)}{\partial x^2}, \qquad \frac{1}{2} < \alpha \leq 1 $$

$$ Q(0,t) = u(t), \qquad Q(L,t) = Q^T $$

$$ Q(x,0) = Q_0 $$

$$ Q(x,T) = Q^T, \qquad Q^T > Q_0 $$

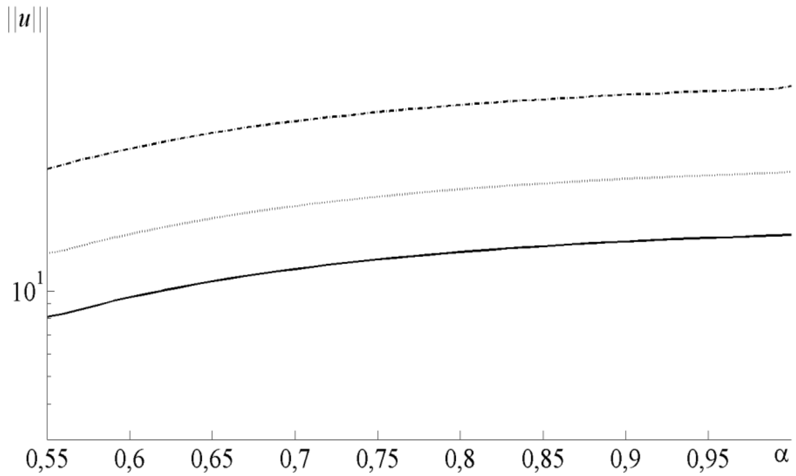
Local control: $W(t) = u(t) \in L_2[0, T]$

$\mathrm{Non-local\ control}$: $W(t) = \tilde{u}(t) = {}_0D_t^{\alpha} u(t) \in L_2[0, T]$

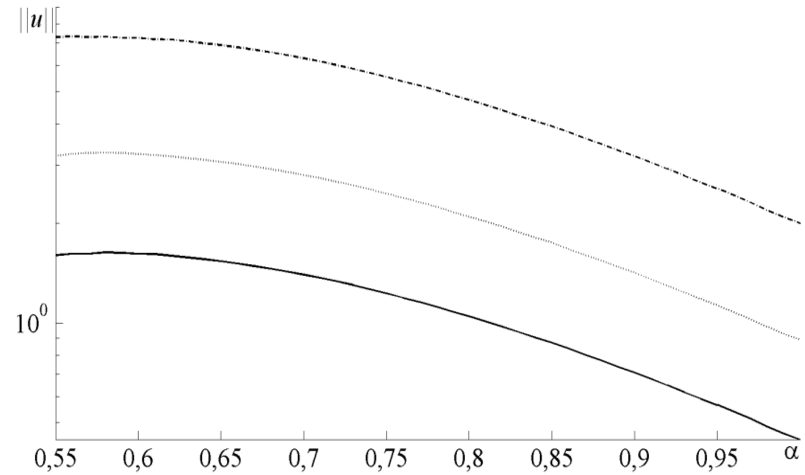
$\{l, N\}$-problem of moments investigated for $N = 3$.

Example 1

Local control



Non-local control



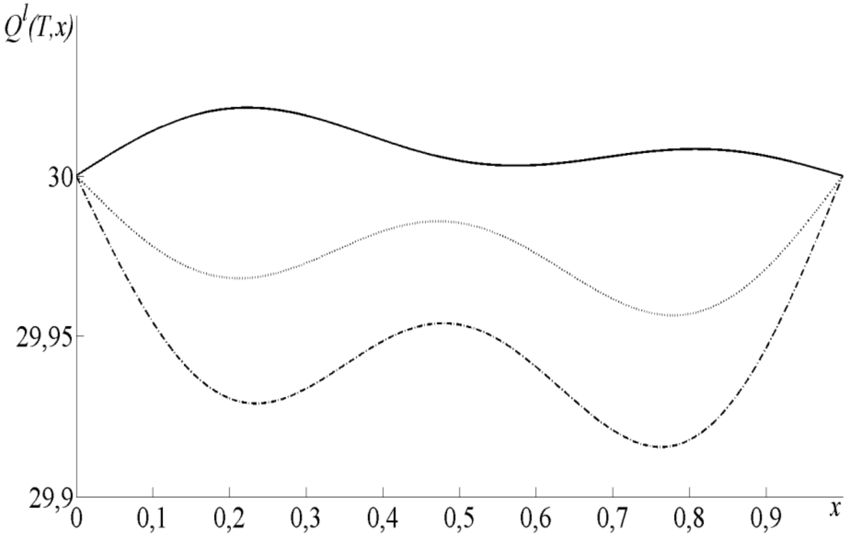
$Q_0 = 30$ – solid line

$Q_0 = 50$ – doted line

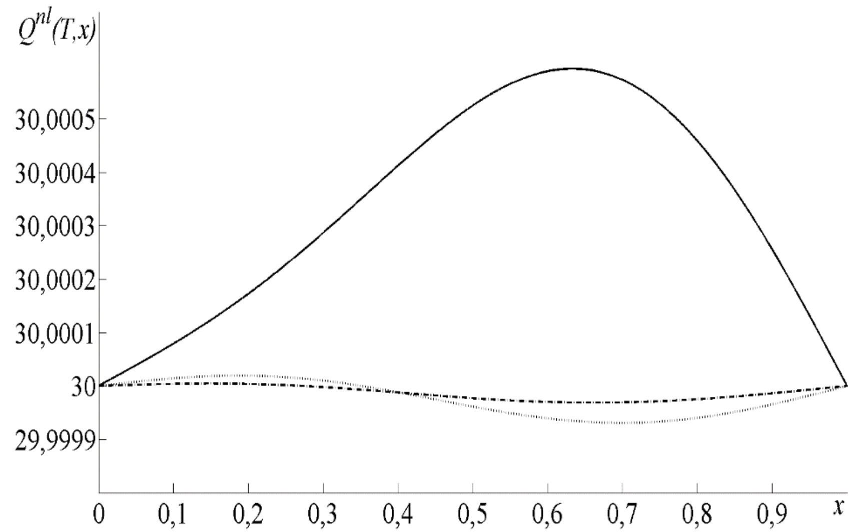
$Q_0 = 100$ – dash-doted line

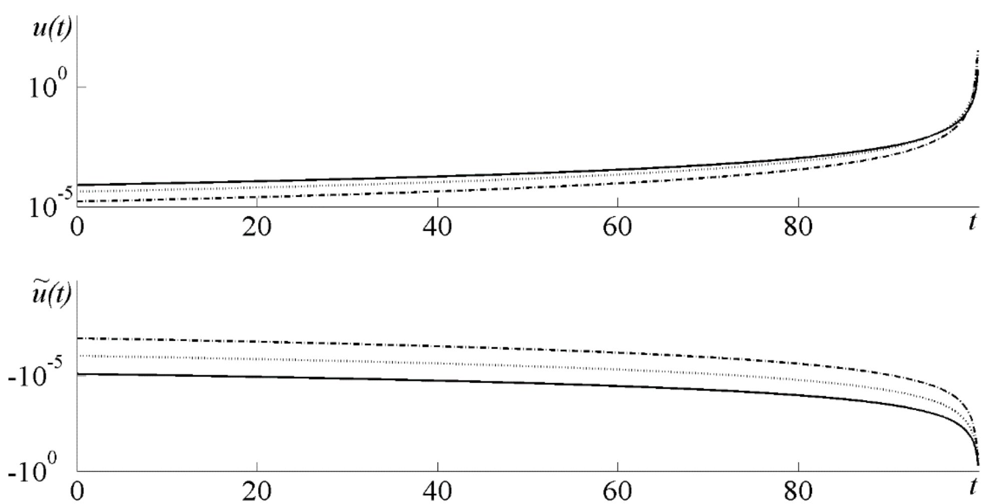
Example 1

Local control



Non-local control





$\alpha = 0{,}6$ – solid line

$\alpha = 0{,}8$ – doted line

$\alpha = 0{,}9$ – dash-doted line

$Q_0 = 10, Q^T = 30$

Example 2:
OCP B for sub-diffusion equation with two-side boundary control $U(t) \in L_\infty[0,T]$

Sub-diffusion equation:

$$ {}_0D_t^\alpha Q(x,t) = \frac{\partial^2 Q(x,t)}{\partial x^2}, \qquad 0 < \alpha \le 1 $$

$$ Q(0,t) = Q(L,t) = u(t) $$

$$ Q(x,0) = Q_0 $$

$$ Q(x,T) = Q^T, \qquad Q^T > Q_0 $$

$$ U(t) = u(t) \in L_\infty[0,T] $$

$\{l, N\}$-problem of moments investigated for $N = 3$.

Example 2: computational scheme

$$\int_0^T g_n(t,T)U(t)dt = c_n(T), \qquad n = 1, \dots, 3, \qquad 0 < \alpha \le 1$$

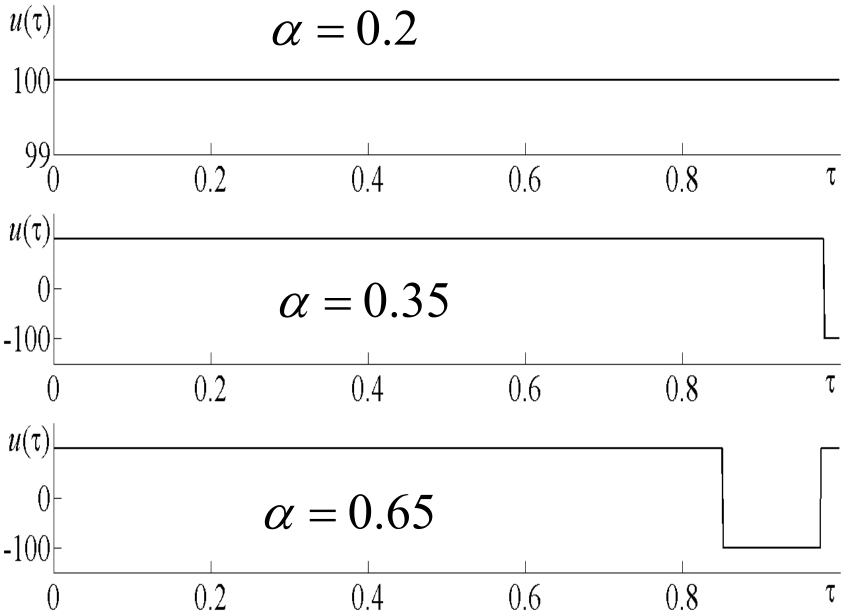
$$\frac{1}{\Lambda_3} = \min_{\xi_1,\xi_2} \int_0^T \left| \left(1 - \frac{c_1}{c_3}\right) \xi_1 g_1(t,T) + \left(1 - \frac{c_2}{c_3}\right) \xi_2 g_2(t,T) + \frac{1}{c_3} g_3(t,T) \right| dt \equiv \min_{\xi_1,\xi_2} \rho_\xi(T)$$

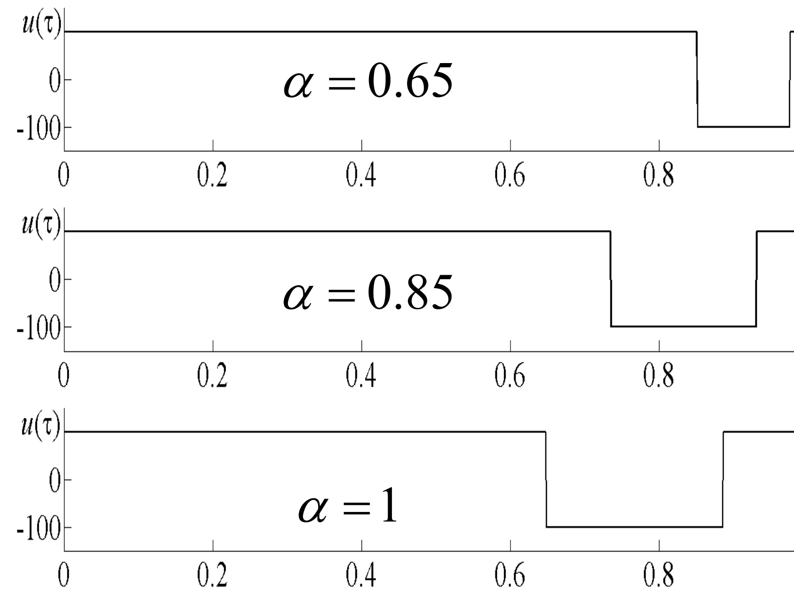
Scheme of calculations

1. Initialization: $\xi_{1,2} = \xi_{1,2}^0, \{\tilde{T}_k |\, k = 1, \dots, M\}$
2. For each $\tilde{T}$ the unconditional optimization problem solved: $\min_{\xi_1,\xi_2} \rho_\xi(\tilde{T}) \to \tilde{\xi}_{1,2}$.
3. For each set $\tilde{T}, \tilde{\xi}_{1,2}$ the following condition checked: $\rho_\xi(T) \le {}^1/_l$.

Stopping rule: $\frac{\left|\tilde{T}^k - \tilde{T}^{k-1}\right|}{\tilde{T}^{k-1}} < \varepsilon.$

Example 2



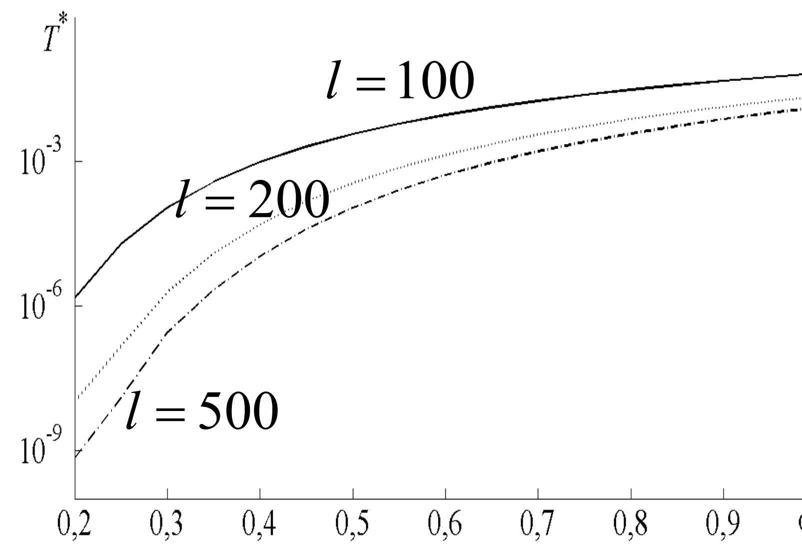


$$\tau = t / T^*$$

$$Q_0 = 10$$

$$Q^T = 30$$

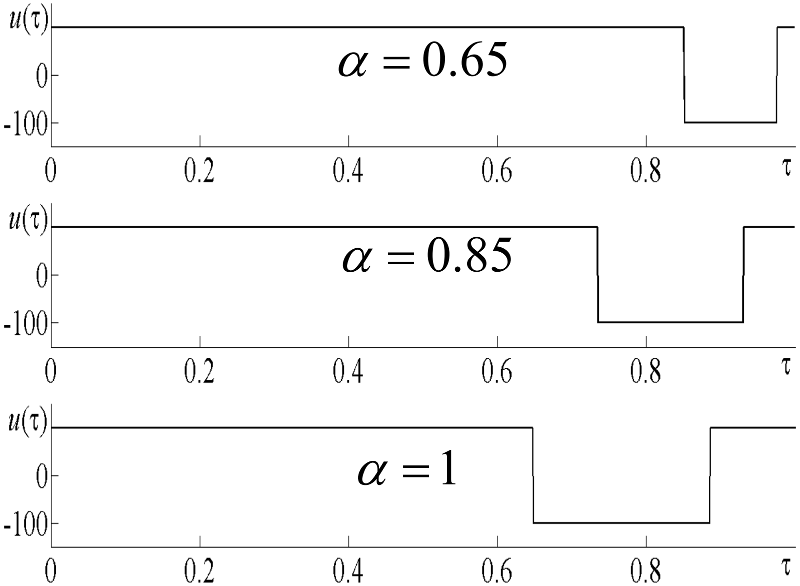
$$l = 100$$

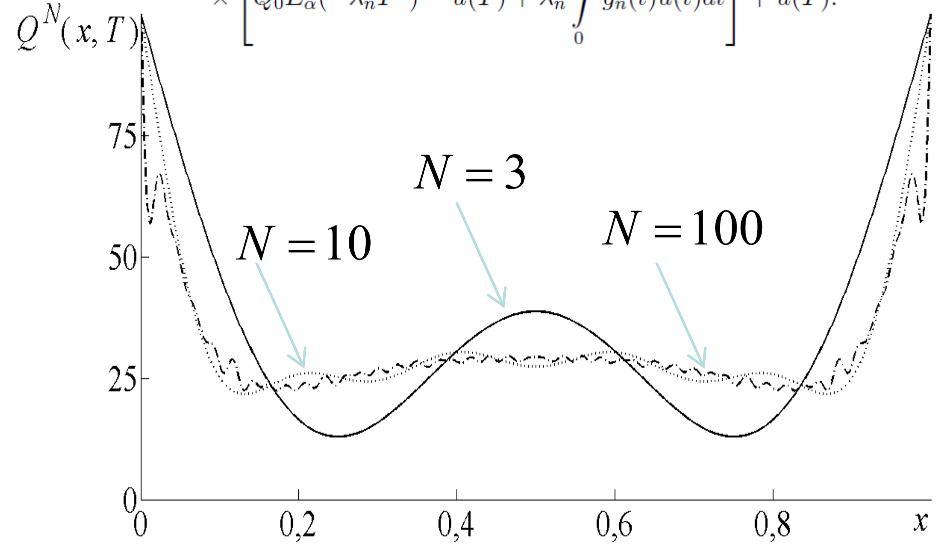


Example 2

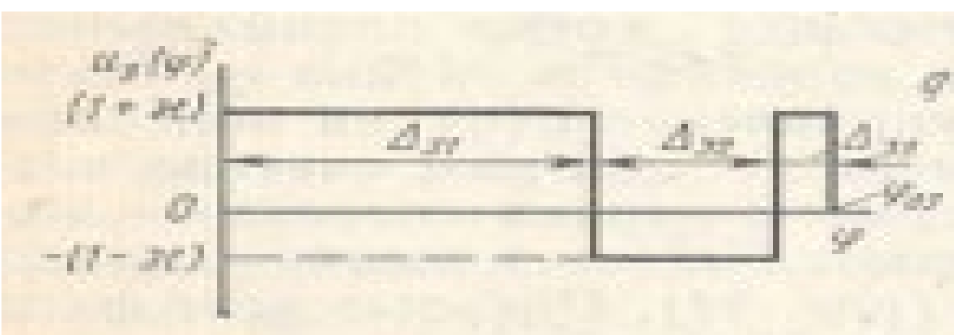
$$Q^N(x,T) = \frac{2}{\pi}\sum_{n=1}^{N}\frac{\sin\sqrt{\lambda_n}x}{n}(1-(-1)^n)\times$$

$$\times\left[Q_0E_\alpha(-\lambda_nT^\alpha) - u(T) + \lambda_n\int_0^T g_n(t)u(t)dt\right] + u(T).$$





A.G. Butkovskii results:



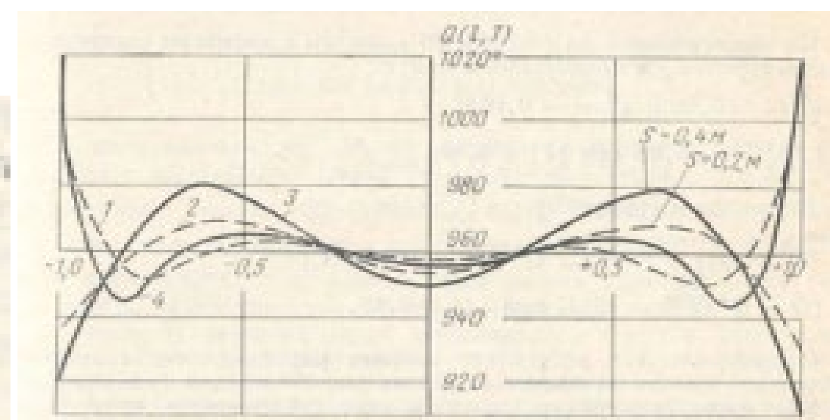


Рис. 50. Температурные распределения в пластине, полученные при нагреве, описываемом функциями $u_2(\varphi)$ 1 и 2 и $u_3(\varphi)$ 3 и 4

Example 3: OCP for super-diffusion equation with two-side boundary control $U(t) \in L_\infty[0,T]$

Super-diffusion equation :

$$ {}_0D_t^\alpha Q(x,t) = \frac{\partial^2 Q(x,t)}{\partial x^2}, \qquad 1 < \alpha < 2 $$

$$ Q(0,t) = Q(L,t) = u(t) $$

$$ Q(x,0) = Q_0 $$

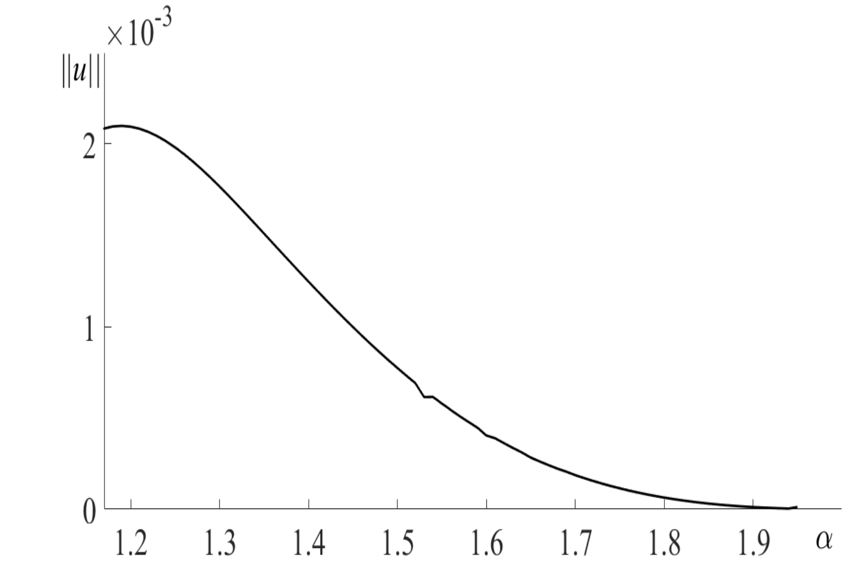
$$ Q(x,T) = Q^T, \qquad Q^T > Q_0 $$

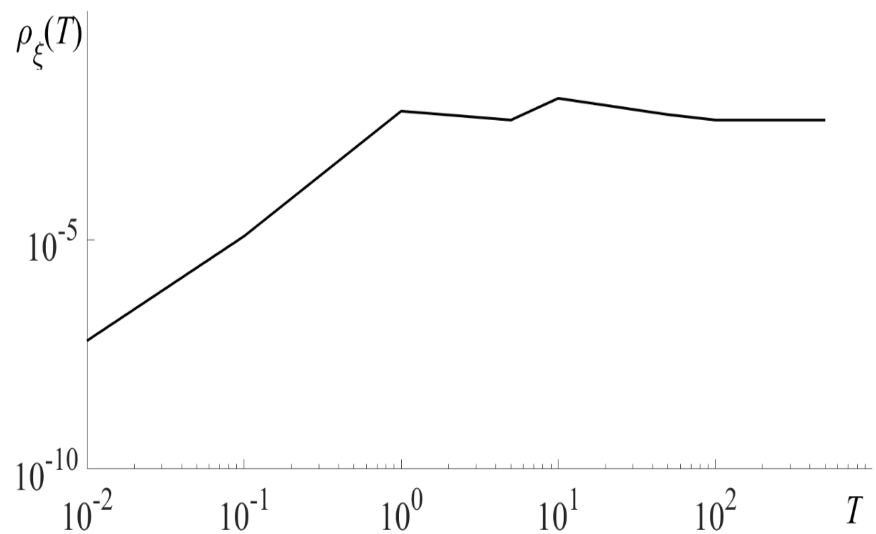
$$ U(t) = u(t) \in L_\infty[0,T] $$

$\{l, N\}$-problem of moments investigated for $N = 3$

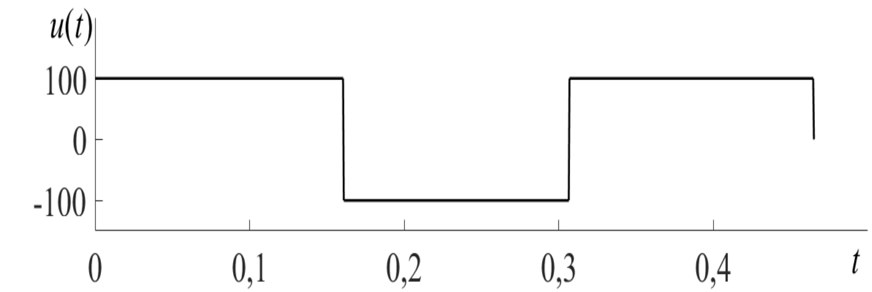
Example 3

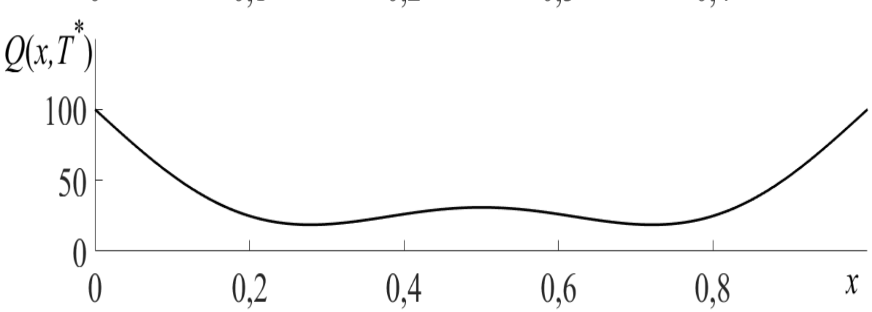
OCP A

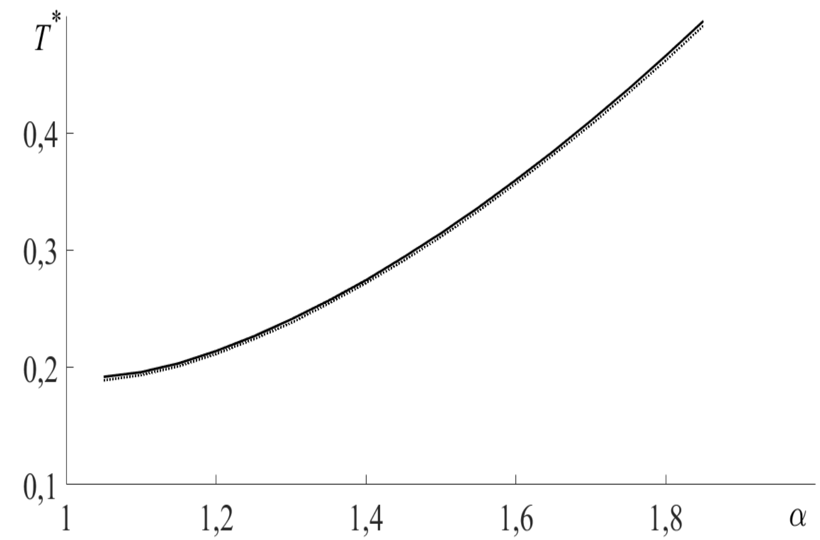




OCP B







Summary

- The optimal control problem for fractional-order systems investigated and reduced to the *l*-problem of moments
- The correctness and solvability conditions for the *l*-problem of moments obtained
- The generalization of Krasovskii “optimal observation” problem derived and analyzed
- The optimal control problem investigated using moments method for the system modelled by diffusion-wave equation with fractional-order time derivative
- An approximation considered leading to finite-dimensional problem of moments and for the last one the correctness and solvability proved
- It’s demonstrated that for fractional-order system the time-optimal problem can be unsolvable when the norm-optimal control exist