Affine super Yangian, Weyl groupoid and beyond

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Outline

1. Motivation
   - Introduction
   - Super Yangian and Quantum Loop Superalgebra
2. Our Results/Contribution
   - Main Results
   - Quantum affine Weyl groupoid and classification of Hopf superalgebra structures
   - Main Results for representations
   - Affine super Yangian

The object of our study is the Super Yangians, which related to the rational integrable models of Quantum Field Theory and have relations with other mathematical objects.

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We describe associative superalgebra and partially Hopf superalgebra structures on super Yangians and its affine counterparts. We give classifiation of Hopf superagebra structures of super Yangian of special linear superalgebra. We describe also an isomorphism (of associative superalgeras) between the completions of the Yangian of a special linear Lie superalgebra and the quantization of a loop superalgebra of a special linear Lie superalgebra. This last result is natural generalization of result by Sachin Gautam and Valerio Toledano Laredo on isomorphism between completions of Yangian of general linear algebra and quantum loop algebra of $L\mathfrak{gl}_n$ ([4], see also the paper [6] for futher development of this theory). S. Gautam, V. Toledano Laredo Meromorphic tensor equivalence for Yangians and quantum loop algebras. – Publ. Math. Inst. Hautes ´ Etudes Sci., 125(2017), 267–337.

We also consider separetly the problem on categories representation equivalence. The first main our results related to

The report presents a work that is a natural continuation of [3], [10]. At the end it is presented some results related to the super Yangian of affine Kac-Moody algebras [12].

Short description. Basic problems.

Frame subtitles are optional. Use upper- or lowercase letters.

- Lie Superalgebras
- 2. Weyl groupoid .
- 3. Super Yangians and Quantum Loop Superalgebras.
- 4. Isomorphism between completions od Super Yangian and Quantum Loop Superalgebra.
- 5. Drinfeld comultiplication.
- 6. Representation Theory of Super Yangian and Quantum Loop Superalgebra.
- 7. Main result. Description of Hopf superalgebra structures.
- 8. Affine super Yangian

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Lie Superalgebras and Quantum Superalgebras.

- Bisuperalgebra structures.
- Problem of quantization
- Problem of description representations

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Definition

The Lie superalgebra $\mathfrak{g} = A(m,n)$ is generated by the generators: $h_i, x_i^{\pm}$, $i \in I$. The generators $x_{m+1}^{\pm}$ are odd, while the remaining generators are even, that is, the parity function p takes on the following values:
$p(h_i) = 0, i \in I, p(x_j^{\pm}) = 0, j \neq m+1, p(x_{m+1}^{\pm}) = 1$. These generators satisfy the following defining relations:
$[h_i, h_j] = 0, \quad [h_i,\, x_j^{\pm}] = \pm a_{ij} x_j^{\pm}$,
$[x_i^+, x_j^-] = \delta_{ij} h_i$,
$[[x_m^{\pm}, x_{m+1}^{\pm}], [x_{m+1}^{\pm}, x_{m+2}^{\pm}]] = 0$ if α_{m+1} (This definition is suitable for arbitrary Dynkin diagram if α_{m+1} be an arbitrary odd simple root.)
$ad^{1-\tilde{a}_{ij}}(x_i^{\pm})x_j^{\pm} = 0.$
$[x_i^{\pm}, [x_i^{\pm}, x_j^{\pm}]] = 0$ if $|i-j| = 1$, $[x_i^{\pm}, x_j^{\pm}] = 0$ elsewhere $(i \neq j)$.

Lie bisuperalgebras and quantization.

In short, a Lie bialgebra is a Lie algebra $\mathfrak{g}$ such that the dual vector space $\mathfrak{g}^*$ is also a Lie algebra. Moreover, the bracket on $\mathfrak{g}$ induced by the operation of the bracket on $\mathfrak{g}^*$ it is one cocycle with coefficients in the module $\mathfrak{g} \otimes \mathfrak{g}$,where module structure is defined by $a \cdot (b \otimes c) = [a \otimes 1 + 1 \otimes a, b \otimes c] = [a, b] \otimes c + b \otimes [a, c]$. The main object of my report are Yangian and quantum loop algebra which are quantization of the following bialgebras, correspondingly: current Lie algebra $\mathfrak{g}[\mathfrak{t}]$ and loop Lie algebra $\mathfrak{g}[t, t^{-1}]$ with cobrackets which can be defined using some standard r-matrices or by equivalent way using language of Manin triples. Let $A_\hbar$ be a QUE superalgebra: $A_\hbar/\hbar A_\hbar \cong U(\mathfrak{g})$. Then the Lie superalgebra $\mathfrak{g}$ has a natural structure of a Lie superbialgebra defined by $\delta(x) = \hbar^{-1}(\Delta(\tilde{x}) - \Delta^{op}(\tilde{x})) \mod \hbar$,where $x \in \mathfrak{g}$, $\tilde{x} \in A_\hbar$ is a preimage of x, Δ is a comultiplication in A and $\Delta^{op} := \tau_{U(\mathfrak{g}),U(\mathfrak{g})} \circ \Delta$, where $\tau_{U(\mathfrak{g}),U(\mathfrak{g})}$ be a (super)permutation of tensor multipliers.

More precisely, we will deal with the superanalogues of the Lie bialgebras introduced above, that is, $\mathfrak{g}$ will be some Lie superalgebra. More precisely, we restrict ourselves to considering the particular case when $\mathfrak{g} = sl(m+1, n+1) = A(m, n)$ is a special linear superalgebra.

- Lie bisuperalgebra $\mathfrak{A} = \mathfrak{g}[\mathfrak{u}]$, where $\mathfrak{g} = \mathfrak{sl}(m+1, n+1)$
- Cobracket is defined by $\delta(a) = [a(u) \otimes 1 + 1 \otimes a(v), \frac{\mathfrak{t}}{u-v}$, where $\mathfrak{t}$ is Casimir operator. .

Two main examples

- Lie bisuperalgebra $\mathfrak{B} = \mathfrak{g}[u, u^{-1}]$.
- Cobracket

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Basic definitions. Super Yangian

Definition

Let, $\mathfrak{g} = A(m,n)$, $Y_\hbar(\mathfrak{g})$ be an associative (super)algebra generated by generators $\{x^{\pm}_{i,r}, h_{i,r}\}_{i\in I, r\in\mathbb{Z}_+}$, which satisfied the following defining relations:

Y1) For all $i,j \in I$ and $r,s \in \mathbb{Z}_+$ $[h_{i,r}, h_{j,l}] = 0$.

Y2) For all $i,j \in I$ and $s \in \mathbb{Z}_+$ $[h_{i,0}, x^{\pm}_{j,s}] = \pm d_i a_{ij} x^{\pm}_{j,s}$.

Y3) For all $i,j \in I$ and $r,s \in \mathbb{Z}_+$
$[h_{i,r+1}, x^{\pm}_{j,s}] - [h_{i,r}, x^{\pm}_{j,s+1}] = \pm\frac{d_i a_{ij}\hbar}{2}(h_{i,r}x^{\pm}_{j,s} + x^{\pm}_{j,s}h_{i,r})$.

Y4) For every $i,j \in I$ and $r,s \in \mathbb{Z}_+$
$[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] - [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}] = \pm\frac{d_i a_{ij}\hbar}{2}(x^{\pm}_{i,r}x^{\pm}_{j,s} + x^{\pm}_{j,s}x^{\pm}_{i,r})$.

Y5)For every $i,j \in I$ and $r,s \in \mathbb{Z}_+$ $[x^{+}_{i,r}, x^{-}_{j,l}] = \delta_{i,j}h_{i,r+s}$.

Y6) Let $i \neq j \in I$ and let $M = 1 - d_i a_{ij}$. Then for all $k_1, \ldots, k_M \in \mathbb{Z}$ and $l \in \mathbb{Z}_+$
$\sum_{\pi\in\mathfrak{S}_M}[x^{\pm}_{i,r_{\pi(1)}}, [x^{\pm}_{i,r_{\pi(2)}}, \ldots, [x^{\pm}_{i,r_{\pi(M)}}, x^{\pm}_{j,s}]\ldots]] = 0$.

Definition

Y7) $[[x^{\pm}_{m,k}, x^{\pm}_{m+1,0}], [x^{\pm}_{m+1,0}, x^{\pm}_{m+2,t}]] = 0,\ k, t \in \mathbb{Z}$.
Let's note that $Y_\hbar(\mathfrak{g})$ be a $\mathbb{Z}_+$-graded superalgebra with grading defining by the following conditions on generators:
$deg(h_{i,r}) = deg(x^{\pm}_{i,r}) = r$ и $deg(\hbar) = 1$. Moreover, $p(h_{i,r}) = 0$ for $i \in I, r \in \mathbb{Z}_+$, and $p(x^{\pm}_{i,r}) = 0$ for $i \in I \backslash \{m+1\}$, and $p(x^{\pm}_{m+1,r}) = 1$ for $r \in \mathbb{Z}_+$. As above, this definition is suitable for the an arbitrary Dynkin diagram, but in this case $x^{\pm}_{m+1} = x^{\pm}_{\alpha_{m+1}}$ and α_{m+1}be an arbitrary simple odd root.
The same way we can define super Yangian for arbitrary simple root system.

Quantum Loop Superalgebra

Definition

Let $U_\hbar(L\mathfrak{g})$ be an associative superalgebra over $C[[\hbar]]$ generated by $\{E_{i,k}, F_{i,k}, H_{i,k}\}_{i\in I, k\in\mathbb{Z}}$, such that:

Q1) For all $i,j \in I$ and $r,s \in \mathbb{Z}$ $[H_{i,r}, H_{j,s}] = 0$.

Q2) For all $i,j \in I$ and $k \in \mathbb{Z}$ $[H_{i,0}, E_{j,k}] = a_{i,j}E_{j,k}$, $[H_{i,0}, F_{j,k}] = -a_{i,j}F_{j,k}$.

Q3) For all $i,j \in I$ and $r,s \in \mathbb{Z}\backslash\{0\}$
$[H_{i,r}, E_{j,k}] = \frac{[ra_{i,j}]_{q_i}}{r}E_{j,r+k}, \quad [H_{i,r}, F_{j,k}] = -\frac{[ra_{i,j}]_{q_i}}{r}F_{j,r+k}$.

Q4) For all $i,j \in I$ and $k,l \in \mathbb{Z}$
$E_{i,k+1}E_{j,l} - q_i^{a_{ij}}E_{j,l}E_{i,k+1} = q_i^{a_{ij}}E_{i,k}E_{j,l+1} - E_{j,l+1}E_{i,k}$,
$F_{i,k+1}F_{j,l} - q_i^{-a_{ij}}F_{j,l}F_{i,k+1} = q_i^{-a_{ij}}F_{i,k}F_{j,l+1} - F_{j,l+1}F_{i,k}$.

Q5) For all $i,j \in I$ and $k,l \in \mathbb{Z}$ $[E_{i,j}, F_{k,l}] = \delta_{i,j}\frac{\psi_{i,k+l}-\varphi_{i,k+l}}{q_i - q_i^{-1}}$.

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$\sum_{\pi \in \mathfrak{S}_M} \sum_{s=0}^{M} (-1)^s \begin{bmatrix} M \\ s \end{bmatrix}_{q_i} E_{i,k_{\pi(1)}} \cdot \ldots \cdot E_{i,k_{\pi(s)}} \cdot E_{j,l} \cdot E_{i,k_{\pi(s+1)}} \cdot \ldots \cdot E_{i,k_{\pi(M)}} = 0,$

$\sum_{\pi \in \mathfrak{S}_M} \sum_{s=0}^{M} (-1)^s \begin{bmatrix} M \\ s \end{bmatrix}_{q_i} F_{i,k_{\pi(1)}} \cdot \ldots \cdot F_{i,k_{\pi(s)}} \cdot F_{j,l} \cdot F_{i,k_{\pi(s+1)}} \cdot \ldots \cdot F_{i,k_{\pi(M)}} = 0,$

Q7) $[[E_{m,k}, E_{m+1,0}]_q, [E_{m+1,0}, E_{m+2,r}]_q]_q = 0,$
$[[F_{m,k}, F_{m+1,0}]_q, [F_{m+1,0}, F_{m+2,r}]_q]_q = 0,$

Here the elements $\psi_{i,r}, \varphi_{i,r}$ are defined by the following formulas:
$\psi_i(z) = \sum_{r \geq 0} \psi_{i,r} z^{-r} = \exp(\frac{\hbar d_i}{2} H_{i,0}) \exp((q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,s} z^{-s}),$
$\varphi_i(z) = \sum_{r \geq 0} \varphi_{i,r} z^{-r} = \exp(-\frac{\hbar d_i}{2} H_{i,0}) \exp(-(q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,-s} z^{s}),$

where $\psi_{i,-k} = \varphi_{i,k} = 0$ for $k \geq 1$. Here, $p(H_{i,r}) = 0$ for

$i \in I, r \in \mathbb{Z}_+$, and $p(E_{i,r}) = p(F_{i,r}) = 0$ for $i \in I \backslash \{m+1\}, r \in \mathbb{Z}$, and $p(E_{m+1,r}) = p(F_{m+1,r}) = 0$ for $r \in \mathbb{Z}$.
Let us note, that this definition is suitable for the an arbitrary Dynkin diagram, but in this case $E_{m+1} = E_{\alpha_{m+1}}, F_{m+1} = F_{\alpha_{m=1}}$, α_{m+1}be an arbitrary simple odd root.

Example

Current Lie superalgebra $\mathfrak{g}[\mathfrak{t}]$ quantization $\to Y_\hbar(\mathfrak{g})$, loop Lie superalgebra $\mathfrak{g}[\mathfrak{t},\mathfrak{t}^{-1}]$ quantization $U_q(L\mathfrak{g})$. Let's consider homomorphism defined on generators by the following formula $\exp^* : \mathfrak{g}[\mathfrak{t},\mathfrak{t}^{-1}] \to \hat{\mathfrak{g}}[s], \quad X \otimes t^n \mapsto X \otimes e^{ns}, n \in \mathbb{Z}$.We would to quantize the mapping $\exp^* : \mathfrak{g}[\mathfrak{t},\mathfrak{t}^{-1}] \to \hat{\mathfrak{g}}[s]$ and obtain $\Phi : \hat{U}_q(L\mathfrak{g}) \to \hat{Y}_\hbar(\mathfrak{g})$.

Example

S.Gautam and V. Toledano-Laredo geneeralize above meentioned result on the case Yangians and Quantum Loop Algebras.

Isomorphism

Let $\{E_{i,r}, F_{i,r}, H_{i,r}\}_{i\in I, r\in\mathbb{Z}}$ be current generators of Quantum Loop Superalgebra $U_\hbar(L\mathfrak{g})$, and $\{e_{i,k}, f_{i,k}, h_{i,k}\}_{i\in I, k\in\mathbb{Z}_+}$ be generators of Yangian $Y_\hbar(\mathfrak{g})$. Let's define the map ([10])

$$\Phi : U_\hbar((L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})} \tag{1}$$

on generators by the following formulas:$\Phi(H_{i,r}) = \dfrac{\hbar}{q_i - q_i^{-1}} \sum_{k\geq 0} t_{i,k} \frac{r^k}{k!}$, $\Phi(E_{i,r}) = e^{r\sigma_i^+} \sum_{m\geq 0} g_{i,m}^+ e_{i,m}$, $\Phi(F_{i,r}) = e^{r\sigma_i^-} \sum_{m\geq 0} g_{i,m}^- f_{i,m}$.

Here we use the notations: $q = e^{\hbar/2}$, $q_i = q^{d_i}$, d_i be elements of symmetrizable matrix $D = diag[d_1, \ldots, , d_{m+n+1}]$ of Cartan matrix $A = (a_{i,j})$ of Lie Superalgebra $\mathfrak{g} = A(m,n)(d_i = 1, i \in \{1, \ldots, m\}$, $d_i = -1, i \in \{m+1, \ldots, m+n+1\}$.

We'll use the logarithmic generators $\{t_{i,r}\}_{i\in I, r\in\mathbb{N}}$ of commutative subsuperalgebra $Y_\hbar(\mathfrak{h}) \subset Y_\hbar(\mathfrak{g})$:
$\hbar \sum_{r\geq 0} t_{i,r} u^{-r-1} = \log(1 + \sum_{r\geq 0} h_{i,r} u^{-r-1})$.The elements

$\{g^{\pm}_{i,m}\}_{i\in I, m\in Z_0}$ belongs to the completion $\widehat{Y^0}$ of superalgebra $Y^0 = Y_\hbar(\mathfrak{h})$ and defined as follows. Let's $G(v) = \log\left(\frac{v}{e^{v/2} - e^{-v/2}}\right) \in Q[[v]]$ and define $\gamma_i \in \hat{Y^0}[v]$ by formula: $\gamma_i(v) = \hbar \sum_{r\geq 0} \frac{t_{i,r}}{r!}\left(-\frac{d}{dv}\right)^{r+1} G(v)$. Then, $\sum_{m\geq 0} g^{\pm}_{i,m} v^m = \left(\frac{\hbar}{q_i - q_i^{-1}}\right)^{1/2} \exp\left(\frac{\gamma_i(v)}{2}\right)$. Finally, $\sigma_i^{\pm}$ are homomorphisms of subsuperalgebras $\sigma_i^{\pm} : Y_\hbar(\mathfrak{b}_\pm)(\subset Y_\hbar(\mathfrak{g})) \to Y_\hbar(\mathfrak{b}_\pm)$, which defined onto generators as follows $\sigma_i^{\pm} : h_{i,k} \to h_{i,k}$, $\sigma_i^{+} : x^{+}_{j,r} \to x^{+}_{j,r+\delta_{ij}}$,$\sigma_i^{-} : x^{-}_{j,r} \to x^{-}_{j,r+\delta_{ij}}$. These homomorphisms can be continued to homomorphism of associative superalgebras $\hat{Y}^{\pm} \to \hat{Y}^{\pm}$

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First basic result

Theorem

1) The above defined mapping $\Phi : U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$, is homomorphism of associative superalgebras.
2) The mapping Φ uniquely continued to homomorphism of topological completion : $\hat{\Phi} : \widehat{U_\hbar(L\mathfrak{g})} \to \widehat{Y_\hbar(\mathfrak{g})}$ of these superalgebras. Moreover, mapping $\hat{\Phi}$ be an isomorphism of topological superalgebras.

Drinfeld comultiplication

Now we consider so called Drinfeld comultiplication. Then Drinfeld comultiplicaton on super Yangian $Y_\hbar(\mathfrak{g})$, $\mathfrak{g} = A(m,n)$, is defined by the following formulas
$\Delta_s(h_i(u)) = h_i(u-s) \otimes h_i(u), \quad \Delta_s(x_i^+(u)) = x_i^+(u-s) \otimes 1 + \oint_{C_2} \frac{1}{u-v} h_i(v-s) \otimes x_i^+(v)dv, \quad \Delta_s(x_i^-(u)) = \oint_{C_1} \frac{1}{u-v} x_i^-(v-s) h_i(v)dv + 1 \otimes x_i^-(u).$
We also can defined the Drinfeld comultiplication on $U_q(L\mathfrak{g})$ by the formulas
$\Delta_u(\Psi_i(z)) = \Psi_i(u^{-1}z) \otimes \Psi_i(z), \quad \Delta_u(X_i^+(z)) = X_i^+(u^{-1}z) \otimes 1 + \oint_{C_2} \frac{zw^{-1}}{z-w} \Psi_i(u^{-1}w) \otimes X_i^+(w)dw, \quad \Delta_u(X_i^-(z)) = 1 \otimes X_i^-(u^{-1}z) + \oint_{C_1} \frac{zw^{-1}}{z-w} X_i^-(w) \otimes \Psi_i(u^{-1}w)dw.$
$X_i^\pm(z) = \sum_{k\in\mathbb{Z}} X_{i,k}^\pm z^k, \quad \Psi_i(z) = \psi_i^+(z) - \psi_i^-(z), \quad X_{i,k}^+ = E_{i,k}, X_{i,k}^- = F_{i,k}.$

Theorem

Above defined mapping is isomorphism of Hopf superalgebra isomorphism

$$\hat{\Phi} : \widehat{U_\hbar(L\mathfrak{g})} \to \widehat{Y_\hbar(\mathfrak{g})}$$

is Hopf superalgebra isomorphism of completions of super Yangian and quantum loop superalgebra with above defined Drinfeld comultiplications.

Second theorem

Theorem

.Mapping
$\hat{\Phi} : \widehat{U_\hbar(L\mathfrak{g})} \to \widehat{Y_\hbar(\mathfrak{g})}$*is Hopf superalgebra isomorphism of completions of super Yangian and quantum loop superalgebra with above defined Drinfeld comultiplications and induced equivalence* Φ_{cat} *of monoidal categories of representations of super Yangian and Quantum Loop Superalgebra.*

Corollary

So we obtain exact faithful monoidal functor $\Phi_{cat} : \mathfrak{O}^{\Pi}(Y_\hbar(\mathfrak{g})) \to \mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$*which acts between some analogues of category* $\mathfrak{O}$*of representations. New line*

Second theorem

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1. What happens to the structure of the Hopf superalgebra when the isomorphism defined above is applied?

2. How to describe all possible structures of the Hopf superalgebra on the Yangian and the quantum loop superalgebra if we consider them only with a fixed structure of the associative superalgebra?

We define the Weyl groupoid as a some supercategory. Based on this abstract definition, we give an explicit realization of the Weyl quantum groupoid in terms of isomorphisms of quantum superalgebras generated by isomorphisms,
which are induced by reflections (relatively both even and odd roots).

Quantum Weyl groupoid for super Yangian.

Quantum Weyl groupoid for super Yangian can be defined for even reflections as .

$$T_i = \exp adx^+_{i,0} \exp adx^-_{i,0} \exp adx^+_{i,0}.$$

Even reflections

Theorem

We have following relations

$$T_i(x_{j,1}^+) = \begin{cases} -x_{i,1}^- + \frac{\hbar}{2}\{h_{i,0}, x_{i,0}^-\}, & if \quad i = j, \\ [x_{i,0}^+, x_{j,1}^+], & if \quad a_{ij} = \pm 1, \\ x_{j,1}^+, & if \quad a_{ij} = 0 \end{cases},$$

$$T_i(\tilde{h}_{j,1}) = \begin{cases} -\tilde{h}_{i,1} - \frac{\hbar}{2}\{x_{i,0}^+, x_{i,0}^-\}, & if \quad i = j, \\ \tilde{h}_{i,1} + \tilde{h}_{j,1} + \frac{\hbar}{2}\{x_{i,0}^+, x_{i,0}^-\}, & if \quad a_{ij} = \pm 1, \\ \tilde{h}_{j,1}, & if \quad a_{ij} = 0 \end{cases}.$$

$$T_i(x_{j,1}^-) = \begin{cases} -x_{i,1}^- + \frac{\hbar}{2}\{h_{i,0}, x_{i,0}^+\}, & \text{if} \quad i = j, \\ -[x_{i,0}^-, x_{j,1}^-], & \text{if} \quad a_{ij} = \pm 1, \\ x_{j,1}^-, & \text{if} \quad a_{ij} = 0 \end{cases},$$

Odd reflections

Now we define action of odd reflection. Let $|\alpha_i| = 1$, and let $s_i : \Pi \to \Pi'$ be the odd reflection. Let $\beta_i := s(\alpha_i)$. We also consider root system $\Pi' \subset E$ as geometrical object and define on E geometrical reflection r_i which acts on Π' as on root system of Lie algebra $A(m+n)$, $s_i : \Pi' \to \Pi'$. Then we define quantum odd reflection by following formulas

$T_i(h_{\alpha_i,0}) = h_{s_i(\beta_i),0}, \quad T_i(h_{\alpha_i,1}) = h_{s_i(\beta_i),1},$
$T_i(x^{\pm}_{\alpha_j,0}) = x^{\pm}_{s_i(\beta_j),0}, \quad T_i(x^{\pm}_{\alpha_j,1}) = x^{\pm}_{s_i(\beta_j),1}$. We can define new isomorphisms $\tilde{T}_i$induced by quantum reflections/

$$\tilde{T}_i(x^{+}_{\alpha_j,0}) = \begin{cases} -x^{-}_{s_i(\alpha_i),0}, & \text{if} \quad i = j, \\ [x^{+}_{s_i(\alpha_i),0}, x^{+}_{s_i(\alpha_j),0}], & \text{if} \quad a_{ij} = \pm 1, \\ x^{+}_{\alpha_j,0}, \text{if} & a_{ij} = 0, \end{cases} ,$$

$$
\tilde{T}_i(x^-_{\alpha_j,0}) = \begin{cases} -x^+_{s_i(\alpha_i,0)}, & \text{if} \quad i=j, \\ [x^-_{s_i(\alpha_i),0}, x^-_{s_i(\alpha_j),0}], & \text{if} \quad a_{ij} = \pm 1, \\ x^-_{s_i(\alpha_j),0}, & \text{if} \quad a_{ij} = 0, \end{cases}
$$

$$
\tilde{T}_i(h_{\alpha_j,0}) = \begin{cases} -h_{s_i(\alpha_i),0}, & \text{if} \quad i=j, \\ h_{s_i(\alpha_i),0} + h_{s_i(\alpha_j),0}, & \text{if} \quad a_{ij} = \pm 1, \\ h_{s_i(\alpha_j),0}, \text{if} & a_{ij} = 0, \end{cases}
$$

$$
\tilde{T}_i(x^+_{\alpha_j,1}) = \begin{cases} -x^-_{s_i(\alpha_i),1}, & \text{if} \quad i=j, \\ [x^+_{s_i(\alpha_i),0}, x^+_{s_i(\alpha_j),1}], & \text{if} \quad a_{ij} = \pm 1, \\ x^+_{s_i(\alpha_j),1}, & \text{if} \quad a_{ij} = 0, \end{cases}
$$

Odd reflections II

I

$$\tilde{T}_i(x^-_{\alpha_j,1}) = \begin{cases} -x^-_{s_i(\alpha_i),1}, & \text{if} \quad i=j, \\ -[x^-_{s_i(\alpha_i),0}, x^-_{s_i(\alpha_j),1}], & \text{if} \quad a_{ij}=\pm 1, \\ x^-_{s_i(\alpha_j),1}, & \text{if} \quad a_{ij}=0, \end{cases} \quad \tilde{T}_i(\tilde{h}_{\alpha_j,1}) = \begin{cases} -\tilde{h}_{s_i(\alpha_i),1}, & \text{if} \quad i=j, \\ \tilde{h}_{s_i(\alpha_i),1} + \tilde{h}_{s_i(\alpha_i),1}, & \text{if} \quad a_{ij}=\pm 1, \\ \tilde{h}_{s_i(\alpha_i),1}, & \text{if} \quad a_{ij}=0. \end{cases}$$

Theorem

The action of generators T_i is compatible with comultiplication and Drinfel'd comultiplication

$$\Delta(T_i) = (T_i \otimes T_i) \circ \Delta, \quad \Delta_s^D(T_i) = (T_i \otimes T_i) \circ \Delta_s^D.$$

Quantum Weyl groupoid for Quantum Loop Superalgebra

Quantum Weyl group action on quantum loop superalgebra. We introduce the element T_i^L which is natural analogue of Lusztig automorphisms.

$$T_i^L(x) = U_i x U_i^{-1},$$

$$U_i = \exp_{q_i^{-1}}(-q_i^{-1} F_{i,0} q_i^{H_{i,0}}) \exp_{q_i^{-1}}(E_{i,0}) \exp_{q_i^{-1}}(-q_i^{-1} F_{i,0} q_i^{H_{i,0}}) q_i^{H_{i,0}(H_{i,0}+1}$$

$$\exp_p(x) = \sum_{n \geq 0} \frac{1}{[n]_p} p^{m(m-1)/2} x^n$$

Relation between actions of Quantum Weyl groupoids

Theorem

Isomorphism $\hat{\Phi}$ is compatible with action of quntum Weyl groups for super Yangian and Quantum Loop Superalgebra.

Outline

1. Motivation
   - Introduction
   - Super Yangian and Quantum Loop Superalgebra
2. Our Results/Contribution
   - Main Results
   - Quantum affine Weyl groupoid and classification of Hopf superalgebra structures
   - Main Results for representations
   - Affine super Yangian

Category representations of super Yangian and Quantum Loop Superalgebra

Let Π, Ω be some subsets of the complex plane are invariant with respect to the group of additive shifts on $\hbar$, respectively, multiplicative shifts by q.
We introduce the category $\mathfrak{O}^{\Pi}(Y_{\hbar}(\mathfrak{g}))$ as a full subcategory of category $\mathfrak{O}(Y_{\hbar}(\mathfrak{g}))$ consisting of the representations V such that, for every $(\lambda, \{P_i^d\}, Q_{m+1}^d) \in \Pi_+^Y$ for which $[V : L(\lambda, \{P_i^d\}, Q_{m+1}^d)] \neq 0$, the roots of $P_i^d, i \in I, Q_{m+1}^d$ lie in Π.
Let similarly $\Omega \in \mathbb{C}^*$ be a subset stable under multiplication by $q^{\pm}$.
We define $\mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ to be a the full subcategory of $\mathfrak{O}(U_q(L\mathfrak{g}))$ consisting those U such that for every $(\lambda, \{P_i^{\delta}\}, Q_{m+1}^{\delta}) \in \Pi_+^U$ for which $[U : L(\lambda, \{P_i^{\delta}\}, Q_{m+1}^{\delta})] \neq 0$, the roots of $P_i^{\delta}, i \in I, Q_{m+1}^{\delta}$ lie in Ω.

Theorem

i) $\mathfrak{O}^{\Pi}(Y_\hbar(\mathfrak{g}))$ *and* $\mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ *are Serre subcategories of* $\mathfrak{O}(Y_\hbar(\mathfrak{g}))$ *and* $\mathfrak{O}(U_q(L\mathfrak{g}))$*, respectively. Other words, they are closed under taking direct sum, subobjects, quotients and extensions.*
(ii) $\mathfrak{O}^{\Pi}(Y_\hbar(\mathfrak{g}))$ *and* $\mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ *are closed under tensor product.*
(iii) Categories $\mathfrak{O}^{\Pi}(Y_\hbar(\mathfrak{g}))$ *and* $\mathfrak{O}^{\Omega}(U_q(L\mathfrak{g}))$ *are equivalent.*
(iv) There is exist exact faithful monoidal functor

$$\Phi : \mathfrak{O}^{\Pi}(Y_\hbar(\mathfrak{g})) \to \mathfrak{O}^{\Omega}(U_q(L\mathfrak{g})) \tag{2}$$

which is category equivalence.

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Construction of affine super Yangian

We define affine Super Yangian $Y_\hbar(\hat{sl}(m|n), \Pi)$ for affine special linear superalgebra $\hat{sl}(m|n)$ and arbitrary system of simple roots Π.

Definition

The Yangian $Y_\hbar(\hat{sl}(m|n))$ is unital associative $\mathbb{C}[h]$-algebra generated by the elements $x^{\pm}_{\alpha_i,r}$, $h_{\alpha_i,r}$, for $i \in \{1, \ldots, m+n-1\}$ and $r \in \mathbb{Z}_{\geq 0}$, subject to the relations
$[h_{i,r}, h_{j,s}] = 0, \quad [h_{i,0}, x^{\pm}_{j,s}] = \pm a_{ij} x^{\pm}_{j,s} [x^{+}_{i,r}, x^{-}_{j,s}] =$
$\delta_{ij} h_{i,r+s}, \quad [h_{i,r+1}, x^{\pm}_{j,s}] - [h_{i,r}, x^{\pm}_{j,s+1}] = \pm \frac{\hbar a_{ij}}{2} \{h_{i,r}, x^{\pm}_{j,s}\},$
$[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] - [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}] = \pm \frac{\hbar a_{ij}}{2} \{x^{\pm}_{i,r}, x^{\pm}_{j,s}\},$
$\sum_{\sigma_i \in S_n} [x^{\pm}_{i,r_{\sigma(i)}}, [x^{\pm}_{i,r_{\sigma(2)}}, \ldots, [x^{\pm}_{i,r_{\sigma(m)}}, x^{\pm}_{j,s}] \ldots]] = 0 \quad for \quad i \neq$
$j \quad and \quad n = 1 - a_{ij}$
$[x_{i,r}, x_{i,s}] = 0 \quad (i = 0, m), \quad [[x^{\pm}_{i-1,0}, x^{\pm}_{i,0}], [x^{\pm}_{i,0}, x^{\pm}_{i+1,0}]] = 0 \quad (i =$
$0, m).$

Main theorem

Theorem

Suppose $m, n \geq 2$ and $m \neq n$. The affine super Yangian $Y_\hbar(\hat{sl}(m|n))$ is isomorphic to associative superalgebra generated by $x_{i,r}^{\pm}$,
$h_{i,r}$, for $i \in \{1, \dots, m+n-1\}$ and $r \in \{0, 1\}$, subject to the relations:

$[h_{i,r}, h_{j,s}] = 0, \quad [x_{i,0}^{+}, x_{j,0}^{-}] = \delta_{ij} h_{i,0},$

$[x_{i,1}^{+}, x_{j,0}^{-}] = \delta_{ij} h_{i,1} = [x_{i,0}^{+}, x_{j,1}^{-}], \quad [h_{i,0}, x_{i,r}^{\pm}] = \pm a_{ij} x_{j,r}^{\pm},$

$[x_{i,1}^{\pm}, x_{j,0}^{\pm}] - [x_{i,0}^{\pm}, x_{j,1}^{\pm}] = \pm \frac{\hbar a_{ij}}{2} \{x_{i,0}^{\pm}, x_{j,0}^{\pm}\},$

$[\tilde{h}_{i,1}, x_{j,0}^{\pm}] = \pm a_{ij} x_{j,1}^{\pm}, \quad (ad x_{i,0}^{\pm})^{(1+|a_{ij}|)}(x_{j,0}^{\pm}) = 0, \quad (i \neq j),$

$[x_{i,0}^{\pm}, x_{i,0}^{\pm}] = 0, \quad if \quad i = (0, m), \quad [[x_{i-1,0}^{\pm}, x_{i,0}^{\pm}][x_{i,0}^{\pm}, x_{i+1,0}^{\pm}]] = 0, \quad if \quad i = (0, m).$

Summary

- The first main message of my talk is theorem about existence isomorphism between completions of super Yangian and Quantum Loop Superalgebra in category of Hopf superalgebras.
- The second main message of my talk is interpretation of above mentioned isomorphism and different Hopf superalgebra structures on super Yangian in terms of Weyl groupoid actions.
- Perhaps a third message, is an applicatication to representation theories of super Yangian and Quantum Loop Superalggebra and description of construction of affine super Yangian.
- Outlook
  - What we have not done yet? Explicite description of automorphisms and isomorphisms of Affine super Yangian as

For Further Reading I

- Stukopin V. Quasi-triangular structures on the super-Yangian and quantum loop superalgebra and difference equations. – Theoretical and mathematical physics, v.213(2022), no 1, p. 1423–1440.
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For Further Reading II

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For Further Reading III

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