Commutators in C^*-algebras and traces. II

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Introduction

Dimension functions and traces on C^*-algebras are fundamental tools in the operator theory and its applications. Therefore, they have been actively studied in recent decades by large group of authors. For a C^*-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ put

$$\mathcal{A}_0 = \{X \in \mathcal{A} : \ X = \sum_{n \geq 1} [X_n, X_n^*] \ \text{ for } \ (X_n)_{n \geq 1} \subset \mathcal{A}\},$$

the series $\| \cdot \|$-converges. Then $\mathcal{A}_0$ coincides with the zero-space of all finite traces on $\mathcal{A}^{\mathrm{sa}}$.

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For a wide class of C^*-algebras that contains all von Neumann algebras we can consider only finite sums of the indicated form [Fack, T.: Finite sums of commutators in C^*-algebras. Ann. Inst. Fourier, Grenoble. 32 (1), 129–137 (1982)].

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Elements of unital C^*-algebras without tracial states can be represented as finite sums of commutators. Moreover, the number of commutators involved in these sums is bounded and depends only on the given C^*-algebra [Pop, C.: Finite sums of commutators, Proc. Amer. Math. Soc. 130 (10), 3039–3041 (2002)].

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An interesting problem is representation of elements of C^*-algebras via commutators of special form. So, every operator $A \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ is separable and infinite-dimensional, is a sum of 5 commutators of idempotents in $\mathcal{B}(\mathcal{H})$ (L.W. Marcoux, H. Radjavi, and Y. Zhang, J. Funct. Anal., 2023; a sum of 50 commutators of idempotents – Bikch. and Fawwas, Russian Math., 2021).

Definitions and notation

Let $\mathcal{A}$ be an algebra, $\mathcal{A}^{\mathrm{id}} = \{A \in \mathcal{A} : A^2 = A\}$ be the set of all idempotents in $\mathcal{A}$. An element $X \in \mathcal{A}$ is a *commutator*, if $X = [A, B] = AB - BA$ for some $A, B \in \mathcal{A}$.

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A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$ by $\mathcal{A}^{\mathrm{pr}}$, and $\mathcal{A}^+$ we denote its subsets of projections ($A = A^* = A^2$), and positive elements, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$.

Definitions

A mapping $\varphi : \mathcal{A}^+ \to [0, +\infty]$ is called *a trace* on a C^*-algebra $\mathcal{A}$, if

- $\varphi(X + Y) = \varphi(X) + \varphi(Y)$ all $X, Y \in \mathcal{A}^+$;
- $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{A}^+,\ \lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$);
- $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{A}$.

A trace φ is called *faithful*, if $\varphi(X) > 0$ for any nonzero $X \in \mathcal{A}^+$. For a trace φ define

$$\mathfrak{M}_\varphi^+ = \{X \in \mathcal{A}^+ \colon\ \varphi(X) < +\infty\}, \quad \mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}}\mathfrak{M}_\varphi^+.$$

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ can always be extended by linearity to a functional on $\mathfrak{M}_\varphi$, which we denote by the same letter φ.

Definitions

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$.
By Gelfand–Naimark Theorem every C^*-algebra is isometrically $*$-isomorphic to a concrete C^*-algebra of operators on a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is separable, then $\mathcal{H}$ may be chosen to be separable.

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Let $\dim \mathcal{H} = +\infty$. The algebra $\mathcal{B}(\mathcal{H})$ is known to contain a proper uniformly closed ideal $\mathcal{J}$ that contains all other proper uniformly closed ideals of $\mathcal{B}(\mathcal{H})$, see Section 6 in: Brown A., Pearcy C., Ann. Math. 1965. In case $\mathcal{H}$ is separable, $\mathcal{J}$ is the ideal of compact operators. Combining Theorems 3 and 4 in [BP65] we get the following assertion.

Brown–Pearcy Theorem. *An operator $X \in \mathcal{B}(\mathcal{H})$ is a non-commutator if and only if $X = xI + J$ for some $x \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$.*

Commutators for $n = \dim \mathcal{H} < \infty$

For $A \in \mathbb{M}_n(\mathbb{C})$ the following conditions are equivalent:

- A is a commutator;
- $\mathrm{tr}(A) = 0$;
- A is unitarily equivalent to a zero-diagonal matrix;
- $\mathrm{tr}(|I + zA|) \geq n$ for all $z \in \mathbb{C}$ [Bik2016, Proc. Steklov...].

Projection differences and commutators

An operator $T \in \mathcal{B}(\mathcal{H})$ for a separable space $\mathcal{H}$ is a commutator of projections if and only if $T^* = -T$, $\|T\| \leq \frac{1}{2}$ and T is unitary equivalent to T^*, see Li, Q.: Commutators of orthogonal projections. Nihonkai Math. J. 15 (1), 93–99 (2004).
Every skew-Hermitian element of any properly infinite von Neumann algebra $\mathcal{A}$ can be represented in the form of a finite sum of commutators of projections of the algebra $\mathcal{A}$ [Bikch., A.M.: On the representation ... of products of projections. III. Commutators in C^*-algebras. Sb. Math. 199 (3-4), 477–493 (2008).]

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Theorem 1. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. Then $|[P, Q]| = f(|P - Q|)$, for the real function $f(t) = t\sqrt{1 - t^2}$, $0 \leq t \leq 1$.*

Corollary 1. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ and $0 \leq t \leq \frac{1}{\sqrt{2}}$ be so that $\|P - Q\| = t$. Then $\|[P, Q]\| = f(t)$.*

Corollary & example

Corollary 2. *Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras and a mapping $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be so that $\mathcal{F}(\mathcal{A}^{\mathrm{pr}}) \subset \mathcal{B}^{\mathrm{pr}}$. If $\|\mathcal{F}(P) - \mathcal{F}(Q)\| = \|P - Q\|$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ with $\|P - Q\| \leq \frac{1}{\sqrt{2}}$ then $\|[\mathcal{F}(P), \mathcal{F}(Q)]\| = \|[P, Q]\|$ for all $P, Q \in \mathcal{A}^{\mathrm{pr}}$ with $\|P - Q\| \leq \frac{1}{\sqrt{2}}$.*

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Consider a unital C^*-algebra $\mathcal{A}$ and put $\mathcal{B} = \mathcal{A}$. Then 1) $\mathcal{F}(A) = I - A$ for all $A \in \mathcal{A}$ or 2) $\mathcal{F}(A) = UAU^*$ for all $A \in \mathcal{A}$ and a fixed isometry $U \in \mathcal{A}$ are examples of such mappings.

Example & Theorem

If $P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}$ are isoclinic with angle $\theta \in (0, \pi/2)$ (i.e., $P = \sin^2\theta\, PQP$ and $Q = \sin^2\theta\, QPQ$) then $|P - Q| = \sin\theta\, P \vee Q$ [item (iii) of Theorem 10.5], Sherstnev, A.N.: Methods of bilinear forms in non-commutative measure and integral theory (Russian), Fizmatlit, Moscow (2008)
and $|[P, Q]| = \cos\theta\, |P - Q|$. Thus $\|P - Q\| = \sin\theta$ and $\|[P, Q]\| = \sin\theta\, \cos\theta$. We have $\cos\theta\, |P - Q| = f(|P - Q|)$, for the real function $f(t) = t\sqrt{1 - t^2}$, $0 \le t \le 1$.

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Theorem 2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda, \mu, \nu \in \mathbb{C}$, $\lambda \ne 0$. Put $A_{\nu,\mu} = \nu I + \mu A$. If $A_{\nu,\mu} B A_{\nu,\mu} = \lambda A_{\nu,\mu}$ for some ν, μ, λ and $A_{\nu,\mu} B \ne \lambda I$ then the operator A possesses a non-trivial invariant subspace.*

Theorem 3 & example

Theorem 3. *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda, \mu, \nu \in \mathbb{C}$, $\lambda \neq 0$. Put $A_{\nu,\mu} = \nu I + \mu A$. If $A_{\nu,\mu} B A_{\nu,\mu} = \lambda A_{\nu,\mu}$ for some ν, μ, λ and A is non-commutator with $\nu + a\mu \neq 0$ (the number "a" from Brown–Pearcy Theorem) then B is non-commutator.*

Theorem 3 & example

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Consider the following complex 2×2 matrices

$$P = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ z & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix}.$$

Then $P, Q \in \mathbb{M}_2(\mathbb{C})^{\mathrm{id}}$ and $PAP = \lambda P$, $PQP = (1 + z^2)P$, $QPQ = (1 + z^2)Q$. We have $|P - Q| = |z|I$ for all $z \in \mathbb{C}$ and $|PQ - QP| = \sqrt{2}I$ for $z = 1$.

Theorem 4

Theorem 4. *Let* $\theta \in (0, \pi/2)$ *and consider the real function* $g(t) = \cos^{-2}\theta\, t^2 - t$, $-1 \le t \le 2$. *Then for* $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ *the following conditions are equivalent:*

(i) P, Q *are isoclinic with angle* θ;

(ii) $P + Q = g(PQ + QP)$;

(iii) $\cos\theta(P + Q) = |PQ + QP|$;

(iv) $\cos\theta|P - Q| = |PQ - QP|$.

Corollaries

Corollary 3. *If* $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ *are isoclinic with angle* $\theta \in (0, \pi/2)$ *then* $|PQ + QP| = \cos\theta\, g(PQ + QP)$, *for the real function* $g(t) = \cos^{-2}\theta\, t^2 - t$, $-1 \le t \le 2$.

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Corollary 4. *Let* $\dim \mathcal{H} = n < \infty$ *and* $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ *be isoclinic with angle* $\theta \in (0, \pi/2)$, $\mathrm{i} \in \mathbb{C}$ *with* $\mathrm{i}^2 = -1$. *Then*
(i) $\det(PQ + QP) \in \{-\cos^n\theta \det(P + Q),\ \cos^n\theta \det(P + Q)\}$;
(ii) $\det(\mathrm{i}(PQ - QP)) \in \{-\cos^n\theta \det(P - Q),\ \cos^n\theta \det(P - Q)\}$
and $\det(P - Q) = 0$ *for all odd* $n \in \mathbb{N}$.

Theorem 5

Theorem 5. *Let* $\mathcal{A}$ *be a* C^**-algebra,* $P, Q \in \mathcal{A}^{\mathrm{pr}}$ *and* $PQP = \lambda P$ *with some* $0 < \lambda < 1$*. If* $\varphi(Q) \leq \varphi(P) < +\infty$ *for a faithful trace* φ *on* $\mathcal{A}$*, then* P *and* Q *are isoclinic with angle* $\theta \in (0, \pi/2)$*, such that* $\cos^2\theta = \lambda$*.*

Theorem 6 and Corollary

Theorem 6. *Let $\mathcal{A}$ be a unital algebra, let $P, Q \in \mathcal{A}^{\mathrm{id}}$ be such that $Q \neq I$ and $PQP = \lambda P$, $P^{\perp}QP^{\perp} = \lambda P^{\perp}$ for some $\lambda \in \mathbb{C}$. Then $\lambda = \frac{1}{2}$ and $Q^{\perp} = S_P Q S_P$. Moreover, $QPQ = QP^{\perp}Q = \frac{1}{2}Q$ and $Q^{\perp}PQ^{\perp} = Q^{\perp}P^{\perp}Q^{\perp} = \frac{1}{2}Q^{\perp}$, $P^{\perp} = S_Q P S_Q$.*

Recall that $S_P := 2P - I$ is a symmetry ($S_P^2 = I$) for $P \in \mathcal{A}^{\mathrm{id}}$;
$P^{\perp} := I - P$.

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Recall that $S_P := 2P - I$ is a symmetry ($S_P^2 = I$) for $P \in \mathcal{A}^{\mathrm{id}}$; $P^{\perp} := I - P$.

Corollary 5. *Let $\dim \mathcal{H} = n < \infty$ and $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$ be such that $Q^{\perp} = S_P Q S_P$. Then n is even and $\det(P - Q)^2 = 2^{-n}$.*

Theorem 7 and Corollary 6

Theorem 7. *Let* φ *be a faithful trace on a* C^**-algebra* $\mathcal{A}$*; let* $A, B \in \mathcal{A}^{\mathrm{id}} \setminus \{0\}$ *be such that* $ABA = \lambda A$ *and* $BAB = \lambda B$ *for some* $\lambda \in \mathbb{C} \setminus \{0, 1\}$*. Then* $[A, B]^n \neq 0$ *for all* $n \in \mathbb{N}$*.*

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Corollary 6. *Let φ be a faithful tracial state on a C^*-algebra $\mathcal{A}$, let $A, B \in \mathcal{A}^{\mathrm{id}} \setminus \{0\}$ be such that $ABA = \lambda A$ and $BAB = \lambda B$ for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Then the element $[A, B]^{2n}$ is a non-commutator for all $n \in \mathbb{N}$.*

Theorem 8 and Corollaries 7, 8, 9

Theorem 8. *Let* $\dim \mathcal{H} = \infty$*, and let an operator* $X \in \mathcal{B}(\mathcal{H})$ *be a non-commutator. If* $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{B}(\mathcal{H})$ *and* $A_1B_1, \ldots, A_nB_n$ *are non-commutators then the operator* $A_n \cdots A_1 X B_1 \cdots B_n$ *is a non-commutator.*

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7. If $\mathcal{H}$ is separable and an operator $X \in \mathcal{B}(\mathcal{H})$ admits a left (resp., right) essential inverse X_l^{-1} (resp., X_r^{-1}) then X_l^{-1} (resp., X_r^{-1}) is a non-commutator if and only if X is a non-commutator.

8. Let $\lambda \in \mathbb{C}$ be a regular point of $X \in \mathcal{B}(\mathcal{H})$ and $R_\lambda = (X - \lambda I)^{-1}$ be the resolvent of X. If X is a non-commutator then R_λ is a non-commutator.

9. Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $ABA = \lambda A + J$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and an operator $J \in \mathcal{J}$. If A is a non-commutator then B is also a non-commutator.

Lemma

Lemma. *Let* $\mathcal{J}$ *be a proper uniformly closed ideal in a unital* C^**-algebra* $\mathcal{A}$*. Let a Hermitian element* $X \in \mathcal{A}$ *be of the form* $X = xI + J_1$*, where* $x \in \mathbb{R}$ *and* $J_1 \in \mathcal{J}$*. The equality* $f(X) = f(x)I + J$ *holds for any continuous real-valued function* f *on the spectrum* $\sigma(X)$*, here* $J \in \mathcal{J}$*.*

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In particular, an operator $X \in \mathcal{B}(\mathcal{H})^+$ is a non-commutator if and only if an operator X^q is a non-commutator for some (consequently, for all) $q > 0$.

Theorem 9, Corollary 10

Theorem 9. *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let $X = U|X|$ be the polar decomposition of an operator $X \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*
(i) *X is a non-commutator;*
(ii) *U and $|X|$ are non-commutators.*

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Corollary 10. *Let $\dim \mathcal{H} = \infty$, and let $T = U|T|$ be the polar decomposition of an operator $T \in \mathcal{B}(\mathcal{H})$.*
(i) *If T is a non-commutator then for any real number $\lambda \in [0, 1]$, the λ-Aluthge transformation $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$ is a non-commutator.*
(ii) *If $|T|$ and $\Delta_\lambda(T)$ for some number $\lambda \in [0, 1]$ are non-commutators then T is a non-commutator.*

Theorems 10, 11

Theorem 10. *Let* $\dim \mathcal{H} = \infty$. *Then for* $X \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ *the following conditions are equivalent:*
(i) X *is a commutator;*
(ii) *the Cayley transform* $\mathcal{K}(X) = (X + \mathrm{i}I)(X - \mathrm{i}I)^{-1}$ *is a commutator.*

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Theorem 11. *Let* $\mathcal{H}$ *be a Hilbert space and* $\dim \mathcal{H} \leq +\infty$*,* $A, B \in \mathcal{B}(\mathcal{H})$ *and* $P \in \mathcal{B}(\mathcal{H})^{\rm id}$*.*
(i) *If* $AB = \lambda BA$ *for some* $\lambda \in \mathbb{C} \setminus \{1\}$ *then the operator* AB *is a commutator.*
(ii) *If* $\dim \mathcal{H} < +\infty$ *then* AB *is a commutator if and only if* BA *is a commutator.*
(iii) *The operator* AP *is a commutator if and only if* PA *is a commutator.*

Examples to (i) of Theorem 11

In $\mathbb{M}_2(\mathbb{C})$ for matrices $A = \mathrm{diag}(1, -1)$ and $B = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$ we have $AB = -BA$.

Consider the primitive cubic roots of 1: $\omega_1 = 1$, $\omega_2 = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$, $\omega_3 = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$. In $\mathbb{M}_3(\mathbb{C})$ for the matrices

$$A = \begin{pmatrix} 0 & 0 & \omega_1 \\ \omega_2 & 0 & 0 \\ 0 & \omega_3 & 0 \end{pmatrix}$$

and $B = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$ we have $AB = \omega_3 BA$.

Further results

Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$. If the operator XY is a non-commutator, then $X^p Y X^{1-p}$ is a non-commutator for every $0 < p < 1$.
Let $A \in \mathcal{B}(\mathcal{H})$ be p-hyponormal for some $0 < p \leq 1$. If $|A^*|^r$ is a non-commutator for some $r > 0$ then $|A|^q$ is a non-commutator for every $q > 0$.
Let $\mathcal{H}$ be separable and $A \in \mathcal{B}(\mathcal{H})$ be a non-commutator. If A is hyponormal (or cohyponormal) then A is normal.

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THANK YOU FOR ATTENTION!